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OSCILLATORY FLOW OF A GENERALISED OLDROYED-B FLUID THROUGH A POROUS MEDIUM IN A ROTATING SYSTEM IN PRESENCE OF A TRANSVERSE MAGNETIC FIELD

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ABSTRACT

An exact solution for the oscillatory flow of a generalised Oldroyed-B fluid in a horizontal channel embedded in a porous medium is obtained in this paper. A uniform magnetic field is applied along the axis perpendicular to the plane of the plates about which the entire system rotates with angular velocity Ω. A fractional calculus approached has been used in which the time derivative term of integer order is replaced by the Caputo fractional calculus operator in the constitutive equation to obtain the velocity field in series form in terms of Mittage-Leffler function by utilizing the integral transform technique. The dependence of the velocity field on the pertinent parameters is illustrated graphically.

Keywords: *Oldroyed-B fluid, Porous Medium, Transverse Magnetic Field, Fractional derivative, Mittage-Leffler function*

1. INTRODUCTION

The study of non-Newtonian fluid flow through porous medium between two infinite parallel plates has much importance in nature. It has wide applications in several physical problems such as filtration and purification processes of crude oil, polymer technology, petroleum industry etc. The characteristics of non-Newtonian fluids can not be described by Newtonian model. For this reason various constitutive equations have been proposed by the scientists. In formulation and solution of such flow problems fractional calculus approach has been extensively utilized in the last few decades. The time derivative of integer order in the constitutive equation is replaced by so called Caputo operator. Bose *et al.* [1] have studied unsteady incompressible flow of a generalized Oldroyed-B fluid between two infinite parallel plates. Chand *et al.* [2] have discussed the hydrodynamic oscillatory flow through a porous channel in the presence of Hall current with variable suction and permeability. Das *et al.* [3] have studied unsteady Couette flow in a rotating system. Guria *et al.* [4] have investigated unsteady Couette flow in a rotating system. Jana and Datta [5] have made an analysis on Couette flow and heat transfer in a rotating system. Mazumdar [6] have studied an exact solution of oscillatory Couette flow in a rotating system. Prasad and Kumar [7] have made an analysis on unsteady hydrodynamic Couette flow through a porous medium in a rotating system. Singh and Sharma [8] have discussed three dimensional Couette flow through a porous medium with heat transfer.

In the present work, oscillatory flow of a generalized Oldroyed-B fluid within a horizontal channel through porous medium in a rotating system has been studied. The flow is generated in the presence of transverse magnetic field acting in the direction perpendicular to the plates. In all the works mentioned above Navier-Stokes equation has been considered as basic equations. But in our problem we have used generalized calculus by replacing the time derivative of integer order by so called Caputo operator in the constitutive equation and it is different from approach of aforesaid problems. The dependence of the velocity field on the fractional calculus, rotation and material parameters has been illustrated graphically.

2. CONSTITUTIVE FLUID MODEL

The constitutive relation involving the Cauchy Stress tensor $\boldsymbol{T}$ in a homogeneous and incompressible Oldroyed-B fluid with fractional calculus model can be proposed as

$$\boldsymbol{T} = -\bar{p}\boldsymbol{I} + \boldsymbol{S}, \boldsymbol{S} + \lambda \frac{D^{\alpha}\boldsymbol{S}}{Dt^{\alpha}} = \mu\left(1 + \lambda_r \frac{D^{\beta}}{Dt^{\beta}}\right)\boldsymbol{A} \quad (1)$$

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where $\bar{p}$ is the hydrostatic pressure, $\boldsymbol{I}$ is the identity tensor, λ is the time of relaxation, $\boldsymbol{\lambda_r}$ is the time of retardation, μ is the coefficient of viscosity of the fluid, $\boldsymbol{S}$ is the extra stress tensor, α and β are the fractional calculus parameter, V is the fluid velocity, $\boldsymbol{A} = \boldsymbol{\nabla V} + (\boldsymbol{\nabla V})^{\boldsymbol{T}}$ is the Rivlin-Erickson tensor and

$$\frac{\boldsymbol{D^{\alpha} S}}{\boldsymbol{D t^{\alpha}}} = \boldsymbol{D_t^{\alpha} S} + \boldsymbol{V.\nabla S} - (\boldsymbol{\nabla V})\boldsymbol{S} - \boldsymbol{S}(\boldsymbol{\nabla V})^{\boldsymbol{T}} \tag{2}$$

$$\frac{\boldsymbol{D^{\beta} A}}{\boldsymbol{D t^{\beta}}} = \boldsymbol{D_t^{\beta} A} + \boldsymbol{V.\nabla A} - (\boldsymbol{\nabla V})\boldsymbol{A} - \boldsymbol{A}\,(\boldsymbol{\nabla V})^{\boldsymbol{T}} \tag{3}$$

$\boldsymbol{D_t^{\alpha}}$ and $\boldsymbol{D_t^{\beta}}$ are Caputo fractional calculus operators of order α and β respectively defined by

$$D_t^{p_1} g(t) = \frac{1}{\Gamma(1-p_1)} \int_0^t g^{'}(t)(t-\tau)^{-m} d\tau, 0 \le p_1 < 1 \tag{4}$$

3. MATHEMATICAL ANALYSIS

We consider the unsteady flow of a viscoelastic fluid between two infinite parallel plates at a distance d apart embedded in a porous medium. The fluids and the plates are initially at rest and at time $t \to 0^+$ the upper plate begins to oscillate with a velocity $U(t)$ about a non-zero mean velocity U_0 together with the entire system rotating with angular velocity Ω about an axis perpendicular to the plane of the plates. A uniform magnetic field of strength B_0 is applied along the axis of rotation of the entire system. The x-and y-axes are taken along the direction and perpendicular to the direction of the plane of the horizontal plates respectively and z –axis perpendicular to the xy-plane. The lower plate is on the xz-plane and the origin is on the lower plate. Since the plates are of infinite dimensions in the x-z direction we may assume that all the flow variables as functions of y and t only. We assume the velocity of the form $\vec{\boldsymbol{u}} = \big(u(y,t), 0, w(y,t)\big)$ and the stress of the form

$$S = S(y,t) \tag{5}$$

with initial condition $S(y,0) = 0{,}0 \le y \le d$ and $S_{xx} = S_{yy} = S_{zz} = S_{zx} = 0, S_{yz} = S_{zy}\,, S_{xy} = S_{yx}$.

Using the forms of the velocity and the stress fields given by Equation (5) we have from the Equation (1) the x- and y-components of the equation of momentum as

$$\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial y} S_{xy} - \sigma B_0^2 u + 2\rho\Omega w - \frac{\mu u}{K} \tag{6}$$

$$\rho \frac{\partial w}{\partial t} = \frac{\partial}{\partial y} S_{yz} - \sigma B_0^2 w - 2\rho\Omega u - \frac{\mu w}{K} \tag{7}$$

where $\vec{\boldsymbol{B}} = (0, B_0, 0)$ is the uniform magnetic field vector acted in the direction perpendicular to the plane of the plates, K is the permeability of the porous medium, ρ is the fluid density, μ is the coefficient of viscosity. Applying the assuming form of the stress field we get from Equation (1)

$$(1 + \lambda D_t^{\alpha}) S_{xy} = \mu\big(1 + \lambda_r D_t^{\beta}\big) \frac{\partial u(y,t)}{\partial y} \tag{8}$$

$$(1 + \lambda D_t^{\alpha}) S_{yz} = \mu\big(1 + \lambda_r D_t^{\beta}\big) \frac{\partial w(y,t)}{\partial y} \tag{9}$$

Eliminating $\mathbf{S_{xy}}$ between the Equations (6) and (8) we get the governing equation as

$$\rho(1 + \lambda D_t^{\alpha}) \frac{\partial u}{\partial t} = \mu\big(1 + \lambda_r D_t^{\beta}\big) \frac{\partial^2 u}{\partial y^2} - (1 + \lambda D_t^{\alpha}) \left(\sigma B_0^2 u - 2\rho\Omega w + \frac{\mu u}{K}\right) \tag{10}$$

Eliminating $\boldsymbol{S_{yz}}$ between Equations (7) and (9) we get the governing equation as

$$\rho(1 + \lambda D_t^{\alpha}) \frac{\partial w}{\partial t} = \mu\big(1 + \lambda_r D_t^{\beta}\big) \frac{\partial^2 w}{\partial y^2} - (1 + \lambda D_t^{\alpha}) \left(\sigma B_0^2 w + 2\rho\Omega u + \frac{\mu w}{K}\right) \tag{11}$$

The initial conditions can be stated as $\boldsymbol{u(y,0) = 0, w(y,0) = 0,\ \ 0 \le y \le d}$ And the boundary conditions are given by

$$u(0,t) = 0, u(d,t) = U(t) = U_0(1 + \varepsilon\, cos\omega t), t > 0 \tag{12}$$

and $w(0,t)=0, w(d,t)=0, t>0$ where U_0 is the mean velocity, ε is some constant and ω is the frequency of the oscillation. Combining Equations (10) and (11) we get the governing equation in terms of complex variable q as

$$\rho(1+\lambda D_t^\alpha)\frac{\partial q}{\partial t}=\mu\left(1+\lambda_r D_t^\beta\right)\frac{\partial^2 q}{\partial y^2}-(1+\lambda D_t^\alpha)\left(\sigma B_0^2+2i\rho\Omega+\frac{\mu}{K}\right)q \tag{13}$$

where $q(y,t)=u(y,t)+iw(y,t)$ is the complex variable and i $=\sqrt{-1}$.

The Equation (13) can be rewritten as

$$(1+\lambda D_t^\alpha)\frac{\partial q}{\partial t}=\nu\left(1+\lambda_r D_t^\beta\right)\frac{\partial^2 q}{\partial y^2}-(1+\lambda D_t^\alpha)\left(\frac{\sigma B_0^2}{\rho}+2i\Omega+\frac{\mu}{\rho K}\right)q \tag{14}$$

where $\nu=\frac{\mu}{\rho}$ is kinematic viscosity.

We consider the non-dimensional variables $y^*=\frac{y}{d}, q^*=\frac{q}{U_0}, t^*=t\omega$

Then the governing equation in terms of non-dimensional variables can be written as

$$(1+\lambda D_t^\alpha)\frac{\partial q}{\partial t}=\nu\left(1+\lambda_r D_t^\beta\right)\frac{\partial^2 q}{\partial y^2}-M(1+\lambda D_t^\alpha)q \tag{15}$$

where $M=\left(\frac{\sigma B_0^2}{\rho}+2i\Omega+\frac{\mu}{\rho K}\right)\frac{1}{\omega}$

The initial and the boundary conditions in terms of non-dimensional variables can be respectively given by

$$q(y,0)=0, 0\le y\le 1$$

$$\text{and } q(0,t)=0, q(1,t)=1+\varepsilon cost \quad t>0 \tag{16}$$

Multiplying both sides of Equation (15) by $sin\, n\pi y$ and then integrating with respect to y from 0 to 1 and utilizing (16 b) we get

$$(1+\lambda D_t^\alpha)\frac{d}{dt}q_s(n,t)=\nu\left(1+\lambda_r D_t^\beta\right)[(-1)^{n+1}n\pi(1+\varepsilon cost)-(n\pi)^2 q_s(n,t)]-M(1+\lambda\, D_t^\alpha)q_s(n,t) \tag{17}$$

where $q_s(n,t)$ is the finite Fourier sine transformation of $q(y,t)$ defined by

$$q_s(n,t)=\int_0^1 q(y,t)\, sin\, n\pi y\, dy \quad n=1,2,3,\dots\dots\dots\dots$$

Taking Laplace transformation of both sides of the Equation (17) and $\bar{q_s}(n,0)=0$ we get

$$\begin{aligned}\bar{q_s}(n,p)=\nu(-1)^{n+1}n\pi&\frac{1}{p\{(p+M)(1+\lambda p^\alpha)+(n\pi)^2\nu(1+\lambda_r p^\beta)\}}\\&+\nu(-1)^{n+1}n\pi\varepsilon\frac{p}{(p^2+1)\{(p+M)(1+\lambda p^\alpha)+(n\pi)^2\nu(1+\lambda_r p^\beta)\}}\\&-\varepsilon\lambda_r\nu(-1)^{n+1}n\pi\frac{p^{\beta-1}}{(p^2+1)\{(p+M)(1+\lambda p^\alpha)+(n\pi)^2\nu\left(1+\lambda_r p^\beta\right)\}}\end{aligned} \tag{18}$$

where $\bar{q_s}(n,p)=\int_0^\infty e^{-pt}q_s(n,t)dt$ is the Laplace transformation of $q_s(n,t)$ and 'p' is the Laplace transform parameter.

In order to avoid the lengthy procedure of residues and contour integral we rewrite the Equation (18) in series form as

$$\begin{aligned}\bar{q_s}(n,p)=&\frac{(-1)^{n+1}}{n\pi}[\frac{1}{p}-\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\frac{p^{\alpha s+k-m}}{(p^\beta+\lambda_r^{-1})^{k+1}}\\&-\lambda_r\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{k!}{s!\,(k-s)!}\lambda^s\frac{p^{\alpha s+k-m+\beta-1}}{(p^\beta+\lambda_r^{-1})^{k+1}}]\\&+\frac{(-1)^{n+1}\varepsilon}{n\pi}[\frac{p}{p^2+1}-\frac{p}{p^2+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\end{aligned}$$

$$\times \lambda^s \frac{p^{\alpha s+k+1-m}}{(p^\beta+\lambda_r^{-1})^{k+1}} - \frac{p}{p^2+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\frac{p^{\alpha s+k-m+\beta}}{(p^\beta+\lambda_r^{-1})^{k+1}}]$$
$$-\frac{(-1)^{n+1}\varepsilon\lambda_r}{n\pi}[\frac{p^{\beta-1}}{p^2+1} - \frac{1}{p^2+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}$$
$$\times \lambda^s\frac{p^{\alpha s+k-m+\beta}}{(p^\beta+\lambda_r^{-1})^{k+1}} - \frac{1}{p^2+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}\lambda^s\frac{p^{\alpha s+k-m+2\beta-1}}{(p^\beta+\lambda_r^{-1})^{k+1}}] \tag{19}$$

Now we have an important Laplace transformation of Mittage-Leffler function

$$\int_0^\infty e^{-pt}\,t^{\alpha n+\lambda-1}E_{\alpha,\lambda}^{(n)}(-at^\alpha)dt = \frac{n!\,p^{\alpha-\lambda}}{(p^\alpha+a)^{n+1}}$$
where
$$E_{\alpha,\lambda}^{(n)}(z) \equiv \frac{d^n}{dz^n}E_{\alpha,\lambda}(z) = \sum_{j=0}^{\infty}\frac{(j+n)!\,z^j}{j!\,\Gamma(\alpha j+\alpha n+\lambda)}$$

Taking inverse Laplace transformation we get from the Equation (19)

$$q_s(n,t) = \frac{(-1)^{n+1}}{n\pi}[1 - \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}$$
$$\times E_{\beta,\beta-\alpha s-k+m}^{(k)}(-\lambda_r^{-1}t^\beta) - \lambda_r\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}\lambda^s\frac{t^{\beta k-\alpha s-k+m}}{k!}$$
$$\times E_{\beta,1-\alpha s-k+m}^{(k)}(-\lambda_r^{-1}t^\beta)] + \frac{(-1)^{n+1}\varepsilon}{n\pi}[cost - \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}$$
$$\times\sum_{s=0}^{k+1}\frac{(k+1)!}{s!(k+1-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau - \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau]$$
$$-\frac{(-1)^{n+1}\varepsilon\lambda_r}{n\pi}[\int_0^t\sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau - \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}$$
$$\times\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau - \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau] \tag{20}$$

Taking finite Fourier sine inverse we get from Equation (20)

$$q(y,t) = y + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$
$$\times\frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^\beta\right) + 2\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}$$
$$\times\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^\beta\right) + \varepsilon y cost + 2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}$$
$$\times E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau + 2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}$$
$$\times\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right)d\tau - \varepsilon\lambda_r y\int_0^t\sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau$$
$$-2\varepsilon\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$

$$\times \int_0^t \sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau - 2\varepsilon\lambda_r \sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times \sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^k}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t \sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \qquad (21)$$

Inserting $M = (\frac{\sigma B_0^2}{\rho} + 2i\Omega + \frac{\mu}{\rho K})\frac{1}{\omega}$ in Equation (21) and comparing the real and imaginary parts of both sides of the resulting equation we get,

$$u(y,t) = y + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k+1}\omega^m}$$
$$\times\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$
$$\times\frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + 2\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}$$
$$\times\frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s$$
$$\times\frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + \varepsilon y cost + 2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}$$
$$\times\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r$$
$$\times\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau + 2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}$$
$$\times\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^k\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}$$
$$\times\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau$$
$$-\varepsilon\lambda_r y\int_0^t\sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau - 2\varepsilon\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}$$
$$\times\frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}$$
$$\times\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau - 2\varepsilon\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^k\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r$$
$$\times\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \qquad (22)$$

and

$$w(y,t) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}sin\,n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k+1}\omega^m}$$
$$\times\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^r\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$
$$\times\frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + 2\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}$$

$$\times \frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s$$
$$\times \frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right)+\varepsilon y cost+2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}$$
$$\times \sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}$$
$$\times \sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau+2\varepsilon\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}$$
$$\times \sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}$$
$$\times \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau$$
$$-\varepsilon\lambda_r y\int_0^t\sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau-2\varepsilon\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}$$
$$\times \frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}$$
$$\times \lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau-2\varepsilon\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times \sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}$$
$$\times \sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \qquad (23)$$

The non-dimensional shear stresses at the stationary plate(y=0) due to the primary and secondary flows is given by

$$\tau_x+i\tau_y=\left(\frac{\partial q}{\partial y}\right)_{y=0}=1+2\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$
$$\times \frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right)+2\lambda_r\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin n\pi y\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k+1}}$$
$$\times \sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right)+\varepsilon cost+2\varepsilon\sum_{n=1}^{\infty}(-1)^n$$
$$\times \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}$$
$$\times E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau+2\varepsilon\sum_{n=1}^{\infty}(-1)^n\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k}}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}$$
$$\times \lambda^s\int_0^t\cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau-\varepsilon\lambda_r\int_0^t\sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau$$
$$-2\varepsilon\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{M^m}{\lambda_r^{k+1}}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s$$
$$\times \int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau-2\varepsilon\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times \sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{M^m}{\lambda_r^{k}}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t\sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \qquad (24)$$

Inserting $M = (\frac{\sigma B_0^2}{\rho} + 2i\Omega + \frac{\mu}{\rho K})\frac{1}{\omega}$ in Equation (24) and comparing the real and imaginary parts of both sides of the resulting equation we get the shear stresses due to the primary and secondary flows as

$$\begin{aligned}
\tau_x &= 1 + 2\sum_{n=1}^{\infty}(-1)^n \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k+1}\omega^m} \\
&\times \sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s \\
&\times \frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + 2\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!} \\
&\times \frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k}\frac{k!}{s!(k-s)!}\lambda^s \\
&\times \frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + \varepsilon cost + 2\varepsilon\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}} \\
&\times \sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!}\frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r} \\
&\times \sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s\int_0^t \cos(t-\tau)\frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!}E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau + 2\varepsilon\sum_{n=1}^{\infty}(-1)^n \\
&\times \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!} \\
&\times \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t \cos(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \\
&-\varepsilon\lambda_r\int_0^t \sin(t-\tau)\frac{\tau^{-\beta}}{\Gamma(1-\beta)}d\tau - 2\varepsilon\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k+1}\frac{(k+1)!}{m!\,(k+1-m)!} \\
&\times \frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!} \\
&\times \lambda^s\int_0^t \sin(t-\tau)\frac{\tau^{\beta k+m-\alpha s-k-1}}{k!}E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau - 2\varepsilon\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k} \\
&\times \sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\cos(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r} \\
&\times \sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s\int_0^t \sin(t-\tau)\frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!}E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^{\beta}\right)d\tau \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
\tau_y &= 2\sum_{n=1}^{\infty}(-1)^n \sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{1}{\lambda_r^{k+1}\omega^m} \\
&\times \sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k+1}\frac{(k+1)!}{s!\,(k+1-s)!}\lambda^s \\
&\times \frac{t^{\beta k+\beta-\alpha s-k+m-1}}{k!}E_{\beta,\beta-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + 2\lambda_r\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2k}\nu^k}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!} \\
&\times \frac{1}{\lambda_r^{k+1}\omega^m}\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}(2\Omega)^{m-j}\sin(m-j)\frac{\pi}{2}\sum_{r=0}^{j}\frac{j!}{r!\,(j-r)!}\left(\frac{\sigma B_0^2}{\rho}\right)^{j-r}\left(\frac{\mu}{\rho K}\right)^{r}\sum_{s=0}^{k}\frac{k!}{s!\,(k-s)!}\lambda^s \\
&\times \frac{t^{\beta k-\alpha s-k+m}}{k!}E_{\beta,1-\alpha s-k+m}^{(k)}\left(-\lambda_r^{-1}t^{\beta}\right) + \varepsilon cost + 2\varepsilon\sum_{n=1}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}}
\end{aligned}$$

$$\times \sum_{m=0}^{k+1} \frac{(k+1)!}{m!\,(k+1-m)!} \frac{1}{\lambda_r^{k+1}\omega^m} \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} (2\Omega)^{m-j} \sin(m-j)\frac{\pi}{2} \sum_{r=0}^{j} \frac{j!}{r!\,(j-r)!} \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r} \left(\frac{\mu}{\rho K}\right)^r$$
$$\times \sum_{s=0}^{k+1} \frac{(k+1)!}{s!\,(k+1-s)!} \lambda^s \int_0^t \cos(t-\tau) \frac{\tau^{\beta k+\beta-\alpha s-k-2+m}}{k!} E_{\beta,\beta-\alpha s-k-1+m}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right) d\tau + 2\varepsilon \sum_{n=1}^{\infty} (-1)^n$$
$$\times \sum_{k=0}^{\infty} \frac{(-1)^k}{(n\pi)^{2k}\nu^k} \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{1}{\lambda_r^k \omega^m} \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} (2\Omega)^{m-j} \sin(m-j)\frac{\pi}{2} \sum_{r=0}^{j} \frac{j!}{r!\,(j-r)!}$$
$$\times \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r} \left(\frac{\mu}{\rho K}\right)^r \sum_{s=0}^{k} \frac{k!}{s!\,(k-s)!} \lambda^s \int_0^t \cos(t-\tau) \frac{\tau^{\beta k+m-\alpha s-k-1}}{k!} E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right) d\tau$$
$$-\varepsilon\lambda_r \int_0^t \sin(t-\tau) \frac{\tau^{-\beta}}{\Gamma(1-\beta)} d\tau - 2\varepsilon\lambda_r \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n\pi)^{2(k+1)}\nu^{k+1}} \sum_{m=0}^{k+1} \frac{(k+1)!}{m!\,(k+1-m)!}$$
$$\times \frac{1}{\lambda_r^{k+1}\omega^m} \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} (2\Omega)^{m-j} \sin(m-j)\frac{\pi}{2} \sum_{r=0}^{j} \frac{j!}{r!\,(j-r)!} \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r} \left(\frac{\mu}{\rho K}\right)^r \sum_{s=0}^{k+1} \frac{(k+1)!}{s!\,(k+1-s)!}$$
$$\times \lambda^s \int_0^t \sin(t-\tau) \frac{\tau^{\beta k+m-\alpha s-k-1}}{k!} E_{\beta,m-\alpha s-k}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right) d\tau - 2\varepsilon\lambda_r \sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n\pi)^{2k}\nu^k}$$
$$\times \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{1}{\lambda_r^k \omega^m} \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} (2\Omega)^{m-j} \sin(m-j)\frac{\pi}{2} \sum_{r=0}^{j} \frac{j!}{r!\,(j-r)!} \left(\frac{\sigma B_0^2}{\rho}\right)^{j-r} \left(\frac{\mu}{\rho K}\right)^r$$
$$\times \sum_{s=0}^{k} \frac{k!}{s!\,(k-s)!} \lambda^s \int_0^t \sin(t-\tau) \frac{\tau^{\beta k+m-\beta-\alpha s-k}}{k!} E_{\beta,m-\alpha s-k-\beta+1}^{(k)}\left(-\lambda_r^{-1}\tau^\beta\right) d\tau \qquad (26)$$

4. CONCLUSION AND NUMERICAL RESULTS

The oscillatory flow of a generalized Oldroyed-B fluid through a porous medium between two infinite parallel plates in the presence of a transverse magnetic field has been studied. The effects of the permeability of the porous medium, the fractional calculus parameters and material parameters on the velocity parameter have been investigated and analyzed with the help of graphical representations. The fractional calculus approach has been utilized in formulating the velocity field by replacing the time derivative of integer order with Caputo operator.

In Fig. 1 the effect of the fractional calculus parameter α on the primary velocity is shown. As α takes higher values the velocity u decreases. From Fig.2 it is observed that the primary velocity $u(y)$ increases with an increase in the fractional calculus parameter β which is different from the case in Fig. 1. The effect of permeability of the porous medium K on $u(y)$ can be observed in Fig.3. As K increases the velocity deceases that is the presence of the porous medium resists the primary flow. The secondary flow velocity w is depicted in Fig.4 against the distance from the stationary plate for different values of the parameter α. It is seen that the secondary flow decreases for higher values of α. In Fig.5 the scenario is different. The secondary flow increases as the parameter β increases. The secondary velocity is depicted against time t for different values of the rotation parameter **Ω** in Fig. 6 and from there it can be observed that as **Ω** takes higher values the secondary flow increases. In Fig.7 the secondary flow increases with increase in the distance from the stationary plate at $y = 0$. The shear stresses τ_x and τ_y due to the primary and secondary flows are depicted against the kinematic viscosity υ in Fig.8 and Fig.10 respectively for different values of the fractional calculus parameter α. τ_x decreases with increasing values of the parameter α in Fig.8 while it increases with increasing values of the parameter β in Fig. 9. In Fig.10 the stress τ_y due the secondary velocity decreases with the increase in parameter α.

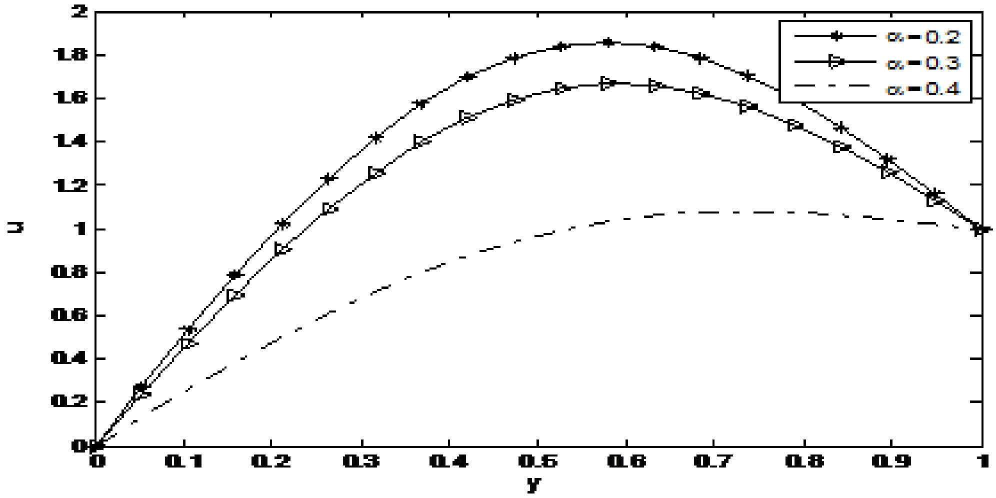

Fig. 1: The primary velocity u is depicted against the distance from the lower plate for different values of fractional calculus parameter α. $\upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \ \lambda = 2, \varepsilon = .002,$ $\beta = 0.8, t = 1, K = 1$

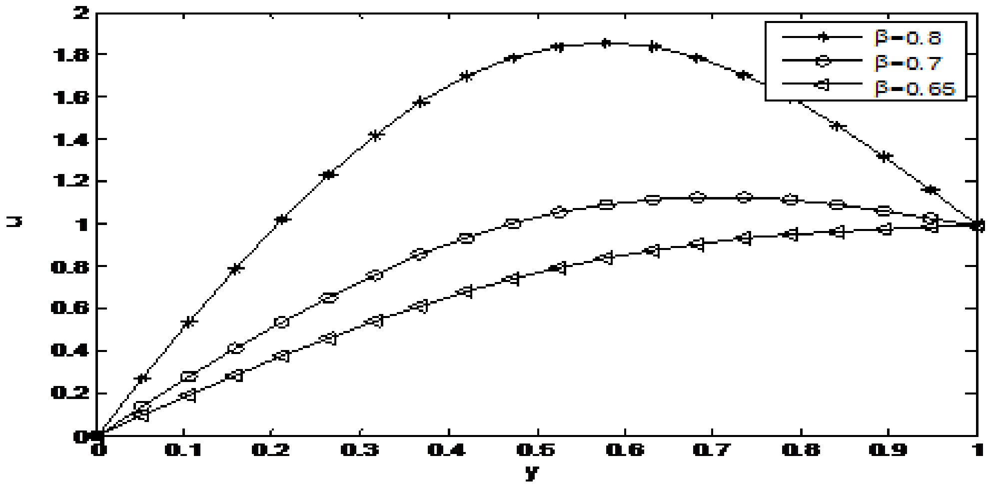

Fig. 2: The primary velocity u is depicted against the distance from the lower plate for different values of fractional calculus parameter β $\upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \varepsilon = 0.02, \lambda = 2,$ $\alpha = 0.2, t = 1, K = 1$

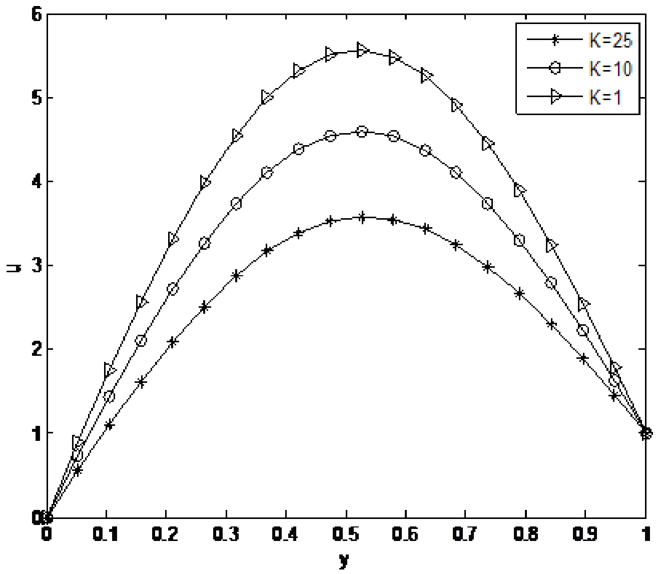

Fig. 3: The primary velocity u is depicted against the distance from the lower plate for different values of the parameter K $\varepsilon = 0.02, \nu = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \ \lambda = 2, \ \alpha = 0.2, \beta = 0.8, t = 1$

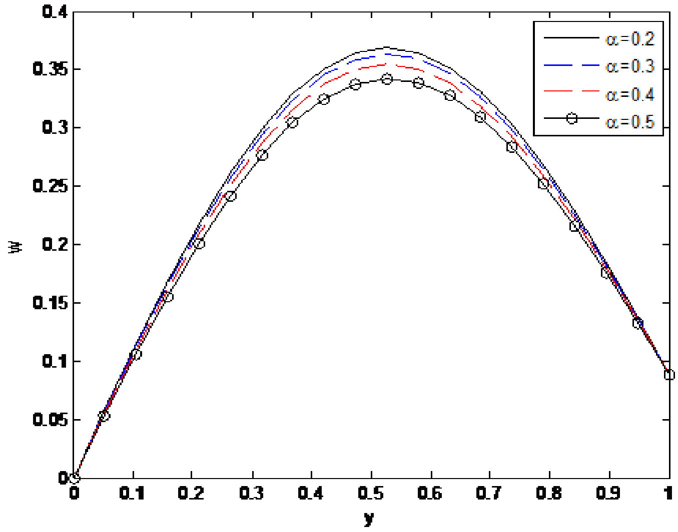

Fig. 4: The secondary velocity is depicted against the distance from the lower Stationary plate for different values of the parameter α. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \ \beta = 0.8, t = 1, K = 1$

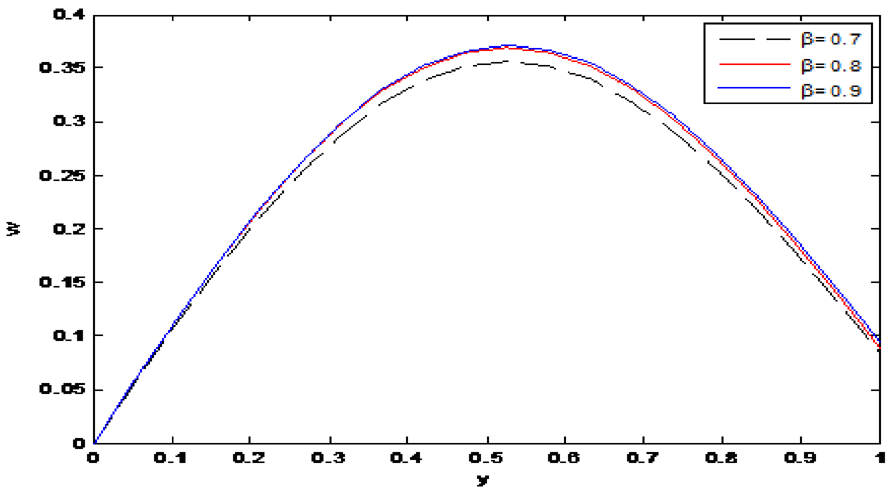

Fig. 5: The secondary velocity is depicted against the distance from the lower stationary plate for different values of the parameter β. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \ \lambda = 2, \alpha = 0.2, t = 1, K = 1$

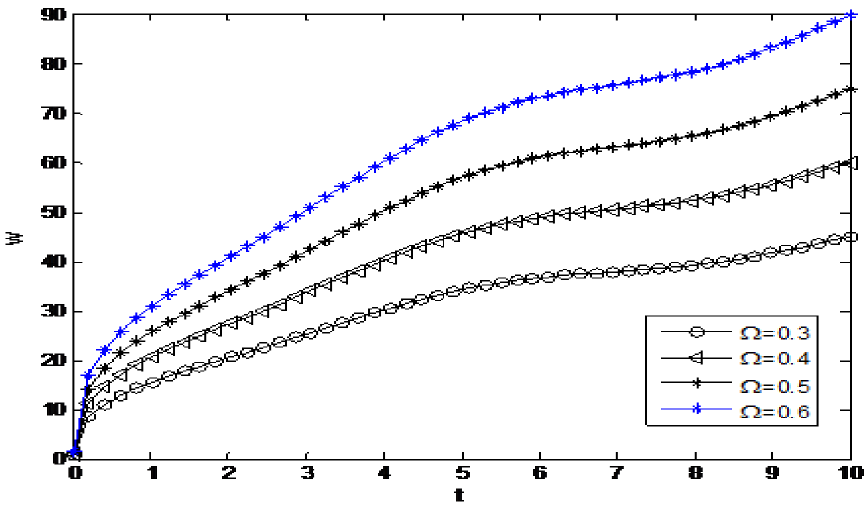

Fig. 6: The secondary velocity is depicted against time for different values of the parameter Ω. $\varepsilon = 0.02, \ \upsilon = 0.03, \ \lambda_r = 3, \omega = 0.1, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \alpha = 0.2, \ \beta = 0.8, K = 1, y = 0.2$

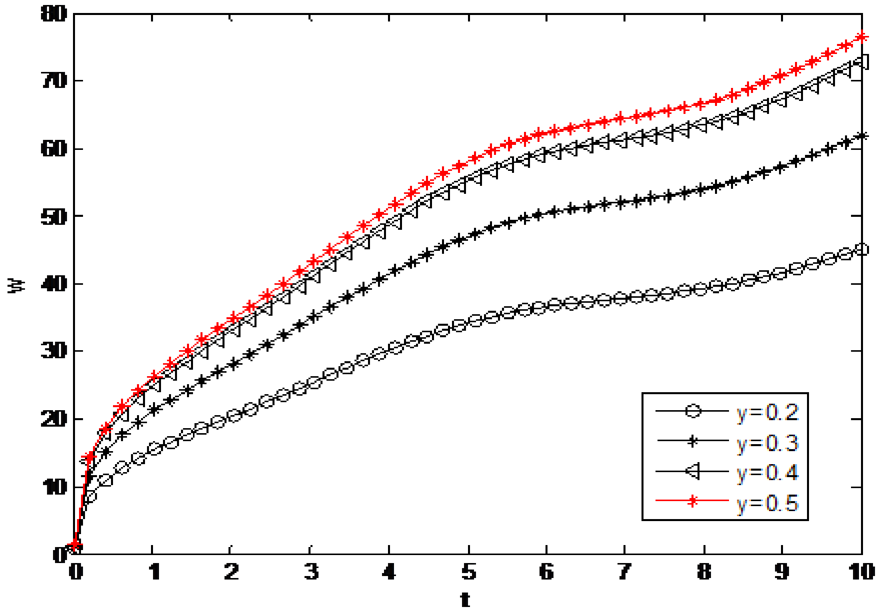

Fig. 7: The secondary velocity is depicted against time for different values of the distance from the stationary plate y. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \alpha = 0.2, \beta = 0.8, K = 1$

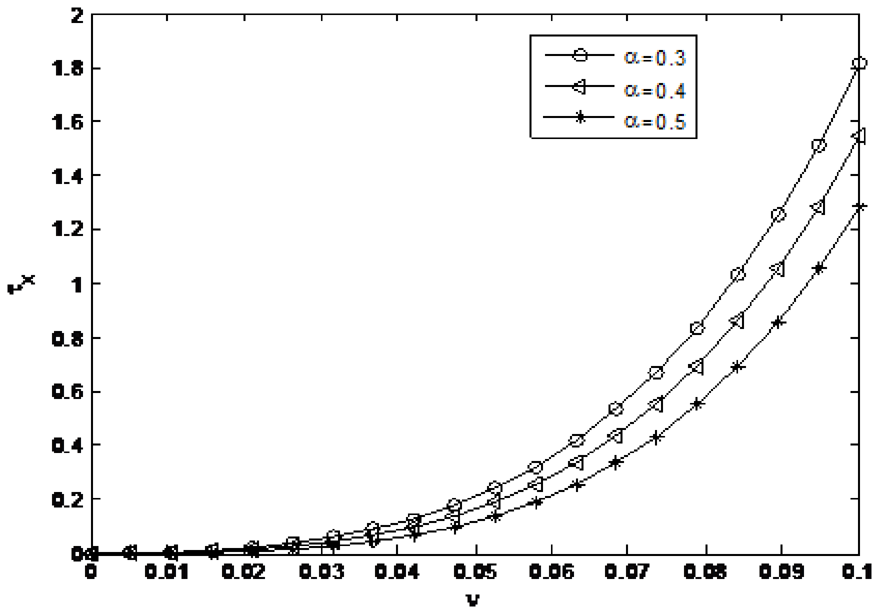

Fig. 8: The shear stress due to the primary flow is depicted against the kinematic viscosity for different values of the parameter α. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \beta = 0.8, t = 1, K = 1$

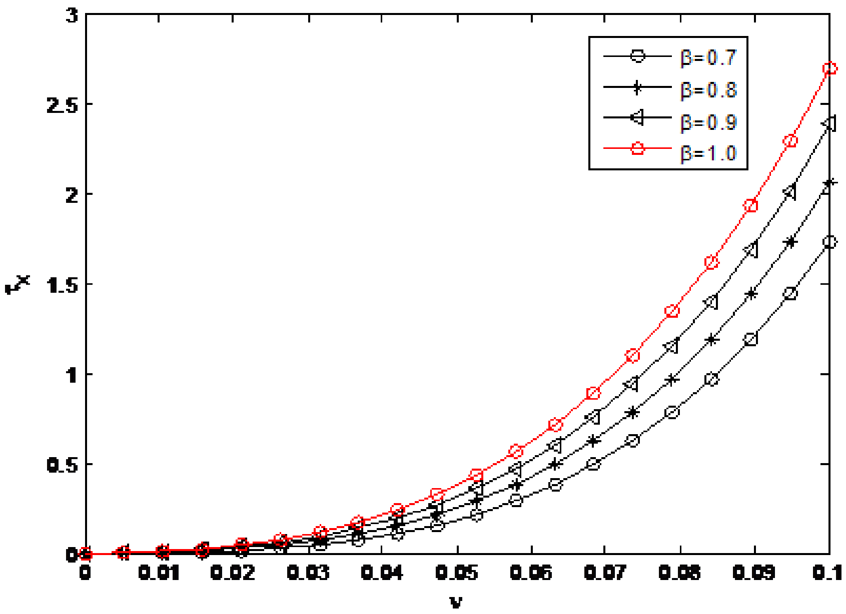

Fig. 9: The shear stress due to the primary flow is depicted against the kinematic viscosity for different values of the parameter β. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \alpha = 0.2,$ $t = 1, K = 1$

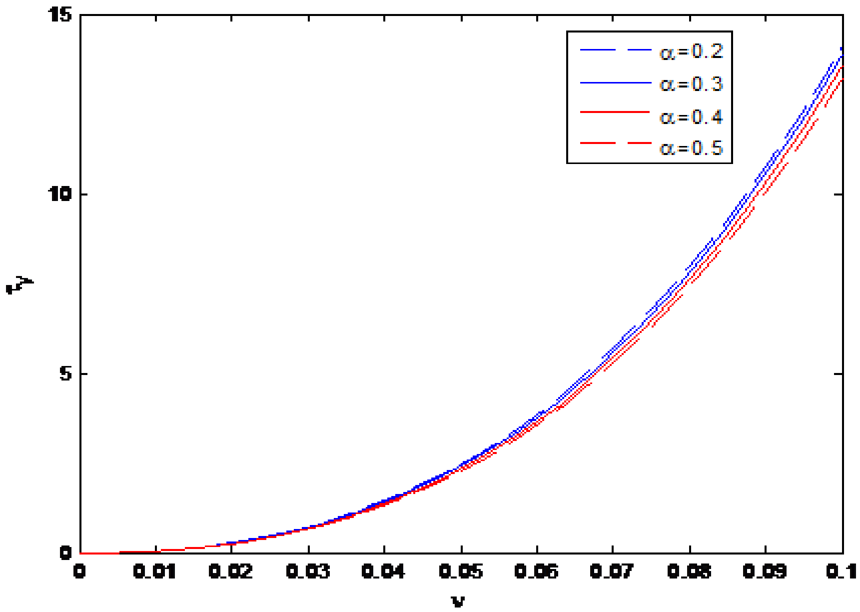

Fig. 10: The shear stress due to the secondary flow is depicted against the kinematic viscosity for different values of the parameter α. $\varepsilon = 0.02, \upsilon = 0.03, \lambda_r = 3, \omega = 0.1, \Omega = 0.3, \sigma = 1, B_0 = 5, \rho = 0.05, \mu = 0.04, \lambda = 2, \beta = 0.8,$ $t = 1, K = 1$

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