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# USING DELBOSCO'S SET TO PROVE UNIFIED FIXED POINT THEOREMS IN THE SETTING OF 2-METRIC SPACES

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## ABSTRACT

In this paper, we present some common fixed point theorems by using Delbosco's set for two pairs of weakly compatible mappings in the setting of a 2-metric spaces. As an application of our main result, a generalized common fixed point theorem is also derived besides furnishing two illustrative examples.

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Key Words and Phrases: Delbosco's set, fixed point, 2-metric spaces, and weakly compatible mappings.

## 1. INTRODUCTION

The concept of 2-metric spaces was initiated and developed (to a considerable extent) by Gahler in a series of papers [7-8] and by now there exists considerable literature on this topic. In this course of development, a number of authors have studied various aspects of metric fixed point theory in the setting of 2-metric spaces which are generally motivated by the corresponding existing concepts already known for ordinary metric spaces. Iseki [13] (also see [14), appears to be the first mathematician who studied fixed point theorems in the setting of 2-metric spaces.

The authors of the articles [3,11,20,22,25,29,30] also utilized the concepts of weakly commuting mappings, compatible mappings, compatible mappings of type (A), compatible mappings of type (P) and weakly compatible mappings of type (A) to prove fixed point theorems in the setting of 2-metric spaces.

Jungck [17] introduced the notion of weakly compatible mappings in ordinary metric spaces which is proving handy to prove common fixed point theorems with minimal commutativity requirement. Recently, Popa [26] utilized implicit relations to prove results on common fixed points which are proving fruitful as they cover several definitions in one go.

The purpose of this paper is two fold which can be described as follows.

- 1. Motivated by D. Delbosco's [1] and V. Popa's [33] results, we prove some common fixed point theorems for two pairs of weakly compatible mappings by using Delbosco's set in the setting of 2-metric spaces. Also, we present on example to illustrate the effectiveness of our results.
- 2. As an application of our main result, a generalized common fixed point theorem has been proved besides deriving related results and furnishing illustrate examples.

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## 2. NOTATIONS AND DEFINITIONS

Throughout this paper, we will adopt the following notations:  $\mathbb{N}$  is the set of all natural numbers,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .  $\mathbb{R}^+$  is the set of all non-negative real numbers, i.e.  $\mathbb{R}^+ = [0, +\infty)$ . In [1], D. Delbosco introduced a unified approach for contraction mapping. D. Delbosco considered the set  $\mathcal{F}$  of all continuous function g:  $(\mathbb{R}^+)^3 \to \mathbb{R}^+$  satisfying the following conditions:

(g-1): g(1,1,1) = h < 1;

(g-2): If  $u, v \in \mathbb{R}^+$  are such that  $u \le g(v, v, u)$  or  $u \le g(u, v, v)$  or  $u \le g(v, u, v)$ , then  $u \le hv$ .

And prove the following:

**Theorem: 2.1** (see [1]) Let (X, d) be a complete metric space. If S and T are two mappings from X into itself, satisfying the following conditions:

 $d(Sx, Ty) \le g(d(x, y), d(x, Sx), d(y, Ty))$ 

for all x, y  $\in$  X, where g  $\in$  F. Then S and T have a unique common fixed point in X.

Some authors proved many kinds of fixed point theorems for contractive type mappings by using Delbosco's set in metric spaces (See [31-32]).

We recall the following definitions and results:

**Definition: 2.1** (see [7]) Let X be a nonempty set. A real valued function d on X<sup>3</sup> is said to a 2-metric if,

(M1) To each pair of distinct points x, y in X, there exists a point  $z \in X$  such that  $d(x, y, z) \neq 0$ ,

(M2) d(x, y, z) = 0 when at least two of x, y, z are equal,

(M3) d(x, y, z) = d(x, z, y) = d(y, z, x),

 $(M4) \quad d(x, y, z) \le d(x, y, u) + d(x, u, z) + d(u, y, z) \text{ for all } x, y, z, u \in X.$ 

The function d is called a 2-metric on the set X whereas the pair (X, d) stands for 2-metric space. Geometrically a 2-metric d(x, y, z) represents the area of a triangle with vertices x, y and z.

If has been know since Gahler [7] that a 2-metric d is a non-negative continuous function in any of its three arguments. A 2-metric d is said to be continuous. If it is continuous in all of its arguments. Throughout this paper d stands for a continuous 2-metric.

**Definition: 2.2** (see [7]) A sequence  $\{x_n\}$  in a 2-metric space (X, d) is said to be convergent to a point  $x \in X$ , if  $\lim_{n\to\infty} d(x_n, x, z) = 0$  for all  $z \in X$ .

**Definitions: 2.3** (see [7]) A sequence  $\{x_n\}$  in a 2-metric space (X, d) is said to be Cauchy sequence if  $\lim_{n\to\infty} d(x_n, x_m, z) = 0$  for all  $z \in X$ .

**Definitions: 2.4** (see [7]) A sequence  $\{x_n\}$  in a 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

**Definitions: 2.5** (see [20]) Let S and T be mappings from a 2-metric space (X, d) into itself. The pair (S, T) is said to be compatible pair (co. p.) if  $\lim_{n\to\infty} d(STx_n, TSx_n, z) = 0$  for all  $z \in X$ , whenever  $\{x_n\}$  is a sequence in X such that  $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$  for some  $t \in X$ .

**Definitions: 2.5** (see [20]) Let S and T be mappings from a 2-metric space (X, d) into itself. The pair (S, T) is said to be weakly compatible pair (w. co. p.) if Sx = Tx (for some  $x \in X$ ) implies STx = TSx.

**Definitions: 2.6** (see [20]) Let S and T be mappings from a 2-metric space (X, d) into itself. The pair (S, T) is said to be compatible of type (A) if  $\lim_{n\to\infty} d(TSx_n, SSx_n, z) = \lim_{n\to\infty} d(STx_n, TTx_n, z) = 0$  for all  $z \in X$ , whenever  $\{x_n\}$  is a sequence in X such that  $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$  for some  $t \in X$ .

**Definitions: 2.7** (see [20]) Let S and T be mappings from a 2-metric space (X, d) into itself. The pair (S, T) is said to be weakly compatible of type (A) if

$$\begin{split} &\lim_{n\to\infty} d(STx_n,TTx_n,z)\leq \lim_{n\to\infty} d(TSx_n,TTx_n,z)\\ &\text{and}\quad &\lim d(TSx_n,SSx_n,z)\leq \lim_{n\to\infty} d(STx_n,SSx_n,z) \end{split}$$

for all  $z \in X$ , whenever  $\{x_n\}$  is a sequence in X such that  $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$  for some  $t \in X$ .

In view of Proposition 2.4 of [25], every pair of compatible mappings of type (A) is weakly compatible mappings of type (A) whereas in view of Proposition 2.9 of [25], every pair of compatible mappings of type (A) is weakly compatible pair.

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#### **3. MAIN RESULTS**

The following proposition notes that in the following specific setting the common fixed point of the involved four self mappings is always unique provided it exists.

**Proposition 3.1:** Let (X, d) be a 2-metric space and let A, B, S, T:  $X \to X$  be four mappings satisfying the condition:  $d(Ax, By, a) \le g(d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a))$  (3.1)

for all  $x, y \in X$  and for all  $a \in X$ , where  $g \in \mathcal{F}$ . Then A, B, S and T have at most one common fixed point.

**Proof:** Assume, to the contrary, that A, B, S and T have two common fixed points u and v such that  $u \neq v$ . By (3.1), we obtain;

 $\begin{aligned} d(u, v, a) &= d(Au, Bv, a) \leq g(d(Su, Tv, a), d(Su, Au, a), d(Tv, Bv, a)) \\ &= g(d(u, v, a), d(u, u, a), d(v, v, a)) \\ &= g(d(u, v, a), 0, 0) \end{aligned}$ 

Thus, by (g-2), we have d(u, v, a) = 0 for all  $a \in X$ .

This is a contradiction. Thus, our assumption that A, B, S and T have two common fixed points u and v such that  $u \neq v$  was wrong. This finishes the proof.

Let A, B, S and T be mappings from a 2-metric space into itself satisfying the following condition: A(X)  $\subseteq$  T(X) and B(X)  $\subseteq$  S(X).

Since  $A(X) \subseteq T(X)$ , for arbitrary point  $x_0 \in X$ , there exists a point  $x_1 \in X$  such that  $Ax_0 = Tx_1$ . Since  $B(X) \subseteq S(X)$ , for the point  $x_1$ , we choose a point  $x_2 \in X$  such that  $Bx_1 = Sx_2$  and so on. Inductively, we can define a sequence  $\{y_n\}$  in X such that

$$\begin{cases} y_{2n} = Ax_{2n} = Tx_{2n+1,} \\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2,} \end{cases} \forall n \in \mathbb{N}_0.$$
(3.3)

**Lemma 3.1** Let (X, d) be a 2-metric space and let A, B, S, T: X  $\rightarrow$  X be four mappings which satisfy conditions (3.1) and (3.2), then

- (A)  $d(y_n, y_{n+1}, y_{n+2}) = 0, \forall n \in \mathbb{N}_0.$
- (B) Sequence  $\{ d_n(a) \}$  is a non-decreasing sequence in  $\mathbb{R}^+$ .

**Proof:** (A) from (3.1), we obtain;

 $\begin{aligned} d(y_{2n+2}, y_{2n+1}, y_{2n}) &= d(Ax_{2n+2}, Bx_{2n+1}, y_{2n}) \\ &\leq g \begin{pmatrix} d(Sx_{2n+2}, Tx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Ax_{2n+2}, y_{2n}), \\ d(Tx_{2n+1}, Bx_{2n+1}, y_{2n}) \end{pmatrix} \\ &= g \begin{pmatrix} d(y_{2n+1}, y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n+2}, y_{2n}), \\ d(y_{2n}, y_{2n+1}, y_{2n}) \end{pmatrix} \\ &= g(0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0) \end{aligned}$ 

yielding thereby

 $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$  (due to (g-2)).

Again, from (3.1), we have

 $\begin{aligned} d(y_{2n+1}, y_{2n}, y_{2n-1}) &= d(Ax_{2n+1}, Bx_{2n}, y_{2n-1}) \\ &\leq g \binom{d(Sx_{2n+1}, Tx_{2n}, y_{2n-1}), d(Sx_{2n+1}, Ax_{2n+1}, y_{2n-1}),}{d(Tx_{2n}, Bx_{2n}, y_{2n-1})} \\ &= g \binom{d(y_{2n}, y_{2n-1}, y_{2n-1}), d(y_{2n}, y_{2n+1}, y_{2n-1}),}{d(y_{2n-1}, y_{2n}, y_{2n-1})} \\ &= g(0, d(y_{2n}, y_{2n+1}, y_{2n-1}), 0) \end{aligned}$ 

By using (g-2), we obtain  $d(y_{2n+1}, y_{2n}, y_{2n-1}) = 0.$ 

Thus, from (3.4) and (3.5), it follows that  $d(y_n, y_{n+1}, y_{n+2}) = 0$ ,  $\forall n \in \mathbb{N}_0$ .

(3.5)

(3.4)

(3.2)

**Proof:** (B) For simplicity, we denote  $d(y_n, y_{n+1}, a)$  by  $d_n(a) \forall n \in \mathbb{N}_0$  and for all  $a \in X$ . First, we shall show that  $\{ d_n(a) \}$  is a non-decreasing sequence in  $\mathbb{R}^+$ . By (3.1), we have;  $d_{2n}(a) = d(y_{2n}, y_{2n+1}, a) = d(Ax_{2n}, Bx_{2n+1}, a)$ 

(3.6)

(3.7)

(3.8)

(3.11)

 $\leq g(d(Sx_{2n}, Tx_{2n+1}, a), d(Sx_{2n}, Ax_{2n}, a), d(Tx_{2n+1}, Bx_{2n+1}, a))$ =  $g(d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a))$ =  $g(d_{2n-1}(a), d_{2n-1}(a), d_{2n}(a))$ 

Implying thereby,  $d_{2n}(a) \le hd_{2n-1}(a) < d_{2n-1}(a) \forall n \in \mathbb{N}_0.$ 

Again, from (3.1), we have

 $\begin{aligned} d_{2n+1}(a) &= d(y_{2n+1}, y_{2n+2}, a) \\ &= d(Ax_{2n+1}, Bx_{2n+2}, a) \\ &\leq g(d(Sx_{2n+1}, Tx_{2n+2}, a), d(Sx_{2n+1}, Ax_{2n+1}, a), d(Tx_{2n+2}, Bx_{2n+2}, a)) \\ &= g(d(y_{2n}, y_{2n+1}, a), d(y_{2n}, y_{2n+1}, a), d(y_{2n+1}, y_{2n+2}, a)) \\ &= g(d_{2n}(a), d_{2n}(a), d_{2n+1}(a)) \end{aligned}$ 

Implying there by  $d_{2n+1}(a) \le hd_{2n}(a) < d_{2n}(a) \text{ (due to (g-2))}$ 

Therefore, by (3.6) and (3.7),  $d_{n+1}(a) < d_n(a) \quad \forall n \in \mathbb{N}_0.$ 

Hence  $\{ d_n(a) \}$  is a non-decreasing sequence in  $\mathbb{R}^+$ .

**Lemma: 3.2** Let  $\{y_n\}$  be a sequence in a 2-metric space (X, d) described by (3.3), then (C)  $\lim_{n\to\infty} d(y_n, y_{n+1}, a) = \lim_{n\to\infty} d_n(a) = 0 \quad \forall a \in X.$ (D)  $d(y_i, y_i, y_k) = 0$ , for i, j,  $k \in \mathbb{N}_0$ , where  $\{y_n\}$  is a sequence described by (3.3).

**Proof:** (C) As in lemma 3.1, we have;  $d_{2n+1}(a) \le hd_{2n}(a)$  and  $d_{2n}(a) \le hd_{2n-1}(a)$ .

Thus  $d_n(a) \le hd_{n-1}(a) \forall n \in \mathbb{N}, \forall a \in X$ . By simple induction, we obtain:  $d_n(a) \le h^n d_0$ .

Hence  $\lim_{n\to\infty} d(y_n, y_{n+1}, a) = \lim_{n\to\infty} d_n(a) = 0$ .

**Proof:** (**D**) By axiom (M<sub>4</sub>), we have;  $d(y_i, y_j, y_k) \le d(y_i, y_j, a) + d(y_i, a, y_k) + d(a, y_j, y_k)$ (3.9)

Suppose that i < j, then again by axiom (M<sub>4</sub>), we get; for all  $a \in X$ ,  $d(y_i, y_j, a) \le d_i(y_{i+2}) + d_{i+1}(y_{i+3}) + d_{i+2}(y_{i+4}) + \dots + d_{j-2}(y_j) + d_{j-1}(a)$ (3.10)

Using (A), on taking i,  $j \to \infty$  in the above inequality and using (C), we get;  $\lim_{i,j\to\infty} d(y_i, y_j, a) = 0.$ 

Similarly, we can show that;

 $\lim_{j,k\to\infty} d(y_j, y_k, a) = 0 \text{ and } \lim_{i,j\to\infty} d(y_i, y_k, a) = 0$ (3.12)

On taking i, j,  $k \to \infty$  in (3.9) and using (3.11) and (3.12), we obtain that;  $\lim_{i,j,k\to\infty} d(y_i, y_j, y_k) = 0 \text{ for } i, j, k \in \mathbb{N}_0.$ (3.13)

**Lemma: 3.3** Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying (3.1) and (3.2). Then the sequence  $\{y_n\}$  decribed by (3.3) is a Cauchy sequence.

**Proof:** Since  $\lim_{n\to\infty} d_n(a) = 0$ , for all  $a \in X$ , by Lemma 3.2, it is sufficient to show that a subsequence  $\{y_{2n}\}$  of  $\{y_n\}$  is a Cauchy sequence in X. Suppose to the contrary, that  $\{y_{2n}\}$  is not a Cauchy sequence in X. Then for every  $\varepsilon > 0$ , there exists  $a \in X$  and strictly increasing sequences  $\{m_k\}, \{n_k\}$  of positive integers such that  $m_k > n_k \ge k$  with  $d(y_{2m_k}, y_{2n_k}, a) \ge \varepsilon$  (3.14)

Without loss of generality, we can suppose that also  $m_k > n_k \ge k$ ,  $d(y_{2m_k}, y_{2n_k}, a) \ge \epsilon$ ,  $d(y_{2n_k}, y_{2m_{k-2}}, a) < \epsilon$  (3.15) © 2015, IJMA. All Rights Reserved 193 From (3.15) and the tetrahedral inequality (that holds for a 2-metric space), we have  $c < d(u_1 - u_2 - a)$ 

$$\leq d(y_{2m_{k}}, y_{2n_{k}}, a)$$

$$\leq d(y_{2m_{k}-2}, y_{2n_{k}}, a) + d(y_{2m_{k}}, y_{2m_{k}-2}, a) + d(y_{2m_{k}}, y_{2n_{k}}, y_{2m_{k}-2})$$

$$\leq d(y_{2m_{k}-2}, y_{2n_{k}}, a) + d(y_{2m_{k}-1}, y_{2m_{k}-2}, a) + d(y_{2m_{k}}, y_{2m_{k}-1}, a)$$

$$+ d(y_{2m_{k}}, y_{2m_{k}-2}, y_{2m_{k}-1}) + d(y_{2m_{k}}, y_{2n_{k}}, y_{2m_{k}-2})$$

$$\leq \varepsilon + d_{2m_{k}-2}(a) + d_{2m_{k}-1}(a) + d_{2m_{k}-2}(y_{2m_{k}}) + d(y_{2m_{k}}, y_{2n_{k}}, y_{2m_{k}-2})$$

$$(3.16)$$

On letting  $k \to +\infty$  in (3.8) and using Lemma (3.1) and Lemma (3.2), we get  $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}, a) = \epsilon$ 

It follows from (3.14) that  $0 < d(y_{2n_k}, y_{2m_k}, a) - d(y_{2n_k}, y_{2m_k-2}, a)$   $\leq d(y_{2n_k}, y_{2m_k-2}, a) + d(y_{2m_k-2}, y_{2m_k}, a) + d(y_{2n_k}, y_{2m_k}, y_{2m_k-2}) - d(y_{2n_k}, y_{2m_k-2}, a)$   $\leq d_{2m_k-2}(y_{2m_k}) + d_{2m_k-2}(a) + d_{2m_k-1}(a) + d(y_{2n_k}, y_{2m_k}, y_{2m_k-2})$ 

In view of lemma (3.1) and Lemma (3.2) and using (3.14), on making  $k \to +\infty$ , we immediately obtain that:  $\lim_{k\to+\infty} d(y_{2n_k}, y_{2m_k-2}, a) = \epsilon$ (3.18)

Note that

 $\begin{aligned} \left| d\big(y_{2n_k}, y_{2m_k-1}, a\big) - d\big(y_{2n_k}, y_{2m_k}, a\big) \right| &\leq d_{2m_k-1}(a) + d_{2m_k-1}\big(y_{2n_k}\big) \\ \left| d\big(y_{2n_k+1}, y_{2m_k}, a\big) - d\big(y_{2n_k}, y_{2m_k}, a\big) \right| &\leq d_{2n_k}(a) + d_{2n_k}\big(y_{2m_k}\big) \end{aligned}$ 

On letting  $k \to +\infty$ , in these inequalities and by using Lemma 3.1, inequalities (3.14) and (3.15), we obtain;  $\lim_{k\to+\infty} d(y_{2n_k}, y_{2m_k-1}, a) = \varepsilon,$   $\lim_{k\to+\infty} d(y_{2n_k+1}, y_{2m_k}, a) = \varepsilon.$ (3.19)

Now, from (2.1) with 
$$x = x_{2m_k}$$
 and  $y = 2n_k + 1$ , we get, for all  $a \in X$ .  

$$d(y_{2m_k}, y_{2n_k+1}, a) = d(Ax_{2m_k}, Bx_{2n_k+1}, a)$$

$$\leq g(d(Sx_{2m_k}, Tx_{2n_k+1}, a), d(Sx_{2m_k}, Tx_{2m_k}, a), d(Tx_{2n_k+1}, Bx_{2n_k+1}, a))$$

$$= g(d(y_{2m_k-1}, y_{2n_k}, a), d(y_{2m_k-1}, y_{2m_k}, a), d(y_{2n_k}, y_{2n_k+1}, a))$$
(3.20)

On letting limit  $n \to \infty$  in (3.20), using the fact that g is continuous and by (3.19), we obtain;  $\varepsilon \le g(\varepsilon, 0, 0)$ , implying there by  $\varepsilon \le h$ . 0 = 0 (due to (g-2)). This is a contradiction. Thus, our supposition that  $\{y_{2n}\}$  is not a Cauchy sequence was wrong. Hence  $\{y_{2n}\}$  is a Cauchy sequence.

**Theorem: 3.1** Let A, B, S and T be four self-mappings of a 2-metric space (X, d) satisfying the conditions (3.1) and (3.2). If one of A(X), B(X), S(X) or T(X) is a complete subspace of X, then

- (E) The pair (A,S) has a point of coincidence,
- (F) The pair (B,T) has a point of coincidence,

Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

**Proof:** Suppose that  $\{y_n\}$  be the defined by (3.3). By Lemma 3.3,  $\{y_n\}$  is a Cauchy sequence in X. By additional assumption, if S(X) is a complete subspace of X, then the subsequence  $\{y_{2n+1}\}$  which is contained in S(X). As  $\{y_n\}$  is a Cauchy sequence containing a convergent subsequence  $\{y_{2n+1}\}$ , therefore  $\{y_n\}$  is also convergence of the subsequence  $\{y_{2n}\}$ , that is,

$$\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Tx_{2n+1} = z,$$
  

$$\lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n+2} = z.$$
(3.21)

Assume that  $u \in S^{-1}(z)$ , then Su = z. If  $Au \neq z$ , By using (3.1), we obtain;  $d(Au, Bx_{2n-1}, a) \leq g(d(su, Tx_{2n-1}, a), d(Su, Au, a), d(Tx_{2n-1}, Bx_{2n-1}, a))$ (3.22)

On letting limit  $n \to \infty$  and using the fact that d is continuous reduces to,  $d(Au, z, a) \le g(d(Su, z, a), d(Su, Au, a), d(z, z, a))$  = g(d(z, z, a), d(z, Au, a), d(z, z, a))= g(0, d(z, Au, a), 0)

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(3.17)

yielding thereby d(z, Au, a) = 0 for all  $a \in X$  (due to (g-2)). Therefore Au = z. Hence z = Su = Au.

Since  $A(X) \subseteq T(X)$ , there exists  $v \in T^{-1}(z)$ . Then Tv = z. By (2.1), we obtain;  $d(Au, Bv, a) \leq g(d(Su, Tv, a), d(Su, Au, a), d(Tv, Bv, a))$ 

That is,  $d(z, Bv, a) \le g(d(z, z, a), d(z, z, a), d(z, Bv, a))$ = g(0, 0, d(z, Bv, a))

Thus, by (g-2), we obtain; d(z, Bv, a) = 0 for all  $a \in X$ . Therefore, z = Bv. Hence z = Tv = Bv. This establishes (c) and (d).

If one assumes that T(X) is a complete subspace of X, then analogous arguments establish (E) and (F). The remaining two cases also pertain essentially to the previous cases. Indeed, if A(X) is complete, then  $z \in A(X) \subseteq T(X)$ . Similarly if B(X) is complete, then  $z \in B(X) \subseteq S(X)$ . Thus in all cases (E) and (F) are completely established.

Since the pair (A, S) is w. co. p. and Au = Su = z, then ASu = SAu which implies Az = Sz. By (3.1), we obtain;  $d(Az, Bv, a) \le g(d(Sz, Tv, a), d(Sz, Az, a), d(Tv, Bv, a))$ 

Or  $d(Az, z, a) \leq g(d(Az, z, a), d(Az, Az, a), d(z, z, a))$ = g(d(Az, z, a), 0, 0)

Thus, by (g-2), we get; d(Az, z, a) = 0 for all  $a \in X$ .

Therefore, Az = z and hence Az = Sz = z. Also, (B, T) is w. co. p. and Bv = Tv = z, then BTv = TBv which implies Bz = Tz. By (3.1), we get;

 $d(Az, Bz, a) \leq g(d(Sz, Tz, a), d(Sz, Az, a), d(Tz, Bz, a))$ 

Or  $d(z, Bz, a) \leq g(d(z, Bz, a), d(z, z, a), d(Bz, Bz, a))$ = g(d(z, Bz, a), 0, 0)

Thus, by (g-2), we get;

d(z, Bz, a) = 0 for all  $a \in X$ .

Hence z = Bz = Tz. Therefore z = Az = Sz = Bz = Tz, this shows that the point  $z \in X$  is a common fixed point of the mappings A, B, S and T. In view of proposition 3.1, the point z is the unique common fixed point of the mappings A, B, S and T.

Corollary: 3.1 Let A, B, S and T be self-mappings of a 2-metric space (X, d) satisfying the conditions

 $d(ax, By, a) \le k \max\{d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a)\}$  (3.23)

for all  $x, y \in X$  and for all  $a \in X$ , where  $k \in (0,1)$  and  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ . If one of A(X), B(X), S(X) and T(X) is a complete subspace of X, Then

- (G) The pair (A,S) has a point of coincidence,
- (H) The pair (B,T) has a point of coincidence,

Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

**Proof:** Define a function  $g: [0, +\infty)^3 \rightarrow [0, +\infty)$  by  $g(u, v, w) = k \max\{u, v, w\}$  for all  $u, v, w \in [0, +\infty)$ , where  $k \in [0, +\infty)$ (0,1). It is easy to see that  $g \in \mathcal{F}$  and by theorem 3.1, the corollary follows.

**Corollary:** 3.2 The conclusions of theorem 3.1 remain true if (for all x, y,  $a \in X$ ) Delbosco's contractive condition (3.1) is replaced by any one of the following.

- (I)  $d(Ax, By, a) \le a_1 d(Sx, Ty, a) + a_2 d(Sx, Ax, a) + a_3 d(Ty, By, a)$  where,  $\sum_{i=1}^3 a_i < 1$ .
- $(\mathbf{J}) \quad d(Ax, By, a) \leq \alpha \begin{bmatrix} \beta \max\{d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a)\} \\ +(1 \beta)[\max\{d^2(Sx, Ax, a), d(Sx, Ax, a), d(Ty, By, a)\}]^{1/2} \end{bmatrix}$ where  $\alpha \in (0, 1)$  and  $0 \leq \beta \leq 1$ .  $(\mathbf{K}) \quad d^r(Ax, By, a) \leq a \cdot d^r(Sx, By, a) + b \cdot d^r(Sx, Ax, a) + c \cdot d^r(Ty, By, a)$ where  $a, b, c \geq 0$  and a + b + c < 1, with r a positive integer.

**Proof:** (I) Define a function g:  $(\mathbb{R}^+)^3 \to \mathbb{R}^+$  by g(u, v, w) =  $a_1u + a_2v + a_3w$ , for all u, v,  $w \in \mathbb{R}^+$  where,  $\sum_{i=1}^3 a_i < 1$ , Then  $g \in \mathcal{F}$  and by Theorem 3.1, Corollary 3.2 (I) follows.

**Proof:** (**J**) Define a function  $g: (\mathbb{R}^+)^3 \to \mathbb{R}^+$  by

 $g(u, v, w) = \alpha \left[\beta \max\{u, v, w\} + (1 - \beta) [\max\{u^2, vw\}]^{1/2}\right]$ 

for all  $u, v, w \in \mathbb{R}^+$ ,  $\alpha \in (0,1)$  and  $\beta \in [0,1]$ . Then  $g \in \mathcal{F}$  and by Theorem 3.1, Corollary 3.2 (J) follows.

**Proof:** (K) Define a function g:  $(\mathbb{R}^+)^3 \to \mathbb{R}^+$  by

 $g(u, v, w) = (a. u^{r} + b. v^{r} + c. w^{r})^{1/r}$ 

for all u, v,  $w \in \mathbb{R}^+$ , where a, b,  $c \ge 0$  and a + b + c < 1, with r a positive integer. Then  $g \in \mathcal{F}$  and Thus, by theorem 3.1, corollary 3.2 (K) follows.

For a mapping T: X  $\rightarrow$  X, we denote  $\mathcal{F}(T) = \{x \in X: Tx = x\}$ .

Now, we will prove the following.

**Theorem: 3.2** Let A, B, S and T be four self-mappings of a 2-metric space (X, d) into itself. If inequality (3.1) holds for all  $x, y \in X$  and for all  $a \in X$ , then

$$\mathcal{F}(A) \cap \mathcal{F}(S) \cap \mathcal{F}(T) = \mathcal{F}(B) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$$

**Proof:** Let z be an arbitrary point in  $\mathcal{F}(A) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ . By (3.1), we have  $d(Az, Bz, a) \leq g(d(Sz, Tz, a), d(Sz, Az, a), d(Tz, Bz, a))$ 

Or  $d(z, Bz, a) \leq g(d(z, z, a), d(z, z, a), d(z, Bz, a))$ 

Thus, by (g-2), we get; d(z, Bz, a) = 0 all  $a \in X$ .

Therefore, Bz = z and then  $z \in \mathcal{F}(B) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ .

Thus,  $\mathcal{F}(A) \cap \mathcal{F}(S) \cap \mathcal{F}(T) \subset \mathcal{F}(B) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ . Now, suppose that  $z^* \in \mathcal{F}(B) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ 

By, using (3.1), we have, for all  $a \in X$ ,  $d(Az^*, Bz^*, a) \le g(d(Sz^*, Tz^*, a), d(Sz^*, Az^*, a), d(Tz^*, Bz^*, a))$ 

Or  $d(Az^*, z^*, a) \le g(d(z^*, z^*, a), d(z^*, Az^*, a), d(z^*, z^*, a))$ =  $g(0, d(z^*, Az^*, a), 0)$ 

yielding thereby  $d(Az^*, z^*, a) = 0$  (due to (g-2)). Hence  $Az^* = z^*$  and  $z^* \in \mathcal{F}(A) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ . Thus  $\mathcal{F}(B) \cap \mathcal{F}(S) \cap \mathcal{F}(T) \subset \mathcal{F}(A) \cap \mathcal{F}(S) \cap \mathcal{F}(T)$ . This finishes the proof.

Theorems 3.1 and 3.2 imply the following one.

**Theorem: 3.3** Let A, B and  $\{T_i\}_{i \in \mathbb{N} \cup \{0\}}$  be mappings of a 2-metric space (X, d) into itself satisfying the following conditions:

(L)  $T_0(X) \subseteq A(X)$  and  $T_i(X) \subseteq B(X)$ ,  $i \in \mathbb{N}$ ,

(**M**) The inequality

 $d(T_0x, T_iy, a) \le g(d(Ax, By, a), d(Ax, T_0x, a), d(By, T_iy, a))$ 

for each x, y  $\in$  X and for each a  $\in$  X,  $\forall$  i  $\in$  N, where g  $\in$   $\mathcal{F}$ .

(N) The pairs  $(T_0, B)$  and  $(T_i, A)$   $(i \in \mathbb{N})$  are weakly compatible,

(O) If one of A(X), B(X) or  $T_0(X)$  is a complete subspace of X.

Then A, B and  $\{T_i\}_{i \in \mathbb{N}_0}$  have a unique common fixed point in X.

The following examples illustrates theorem 3.1.

**Example:** 3.1 Let  $X = \{(0,0), (4,0), (8,0), (0,1)\}$  be a finite subset of  $\mathbb{R}^2$  equipped with natural area function on  $X^3$ , then it is easy to see that (X, d) is a 2-metric space. Define four self mappings A, B, C and T on X as follows:

		1	11 0
A(0,0) = (0,0),	B(0,0) = (0,0),	S(0,0) = (0,0),	T(0,0) = (0,0),
A(4,0) = (0,0),	B(4,0) = (0,0),	S(4,0) = (0,0),	T(4,0) = (0,0),
A(8,0) = (4,0),	B(8,0) = (0,0),	S(8,0) = (8,0),	T(8,0) = (4,0),
A(0,1) = (0,0),	B(0,1) = (4,0),	S(0,1) = (4,0),	T(0,1) = (8,0).

Clearly,

$$A(X) = \{(0,0), (4,0)\}, B(X) = \{(0,0), (4,0)\}$$
  
S(X) = {(0,0), (4,0), (8,0)}, T(X) = {(0,0), (4,0), (8,0)}

Notice that  $A(X) \subset T(X)$  and  $B(X) \subset S(X)$  and also A(X), B(X), S(X) and T(X) are complete subspace of X. The pair (A, S) is weakly compatible but not commuting as  $AS(8,0) \neq SA(8,0)$  whereas the pair (B, T) is commuting and hence weakly compatible. Define a function g:  $(\mathbb{R}^+)^3 \to \mathbb{R}^+$  by

 $g(u, v, w) = k \max\{u, v, w\} \forall u, v, w \in \mathbb{R}^+$  where  $k \in [0, 1[$ .

Obviously, g is continuous function and g(1,1,1) = k < 1. Also, the condition (g-2) is satisfied. Thus  $g \in \mathcal{F}$ .

Thus, condition (3.1) holds for x = (0,0) = y and for all  $a = (a_1, a_2) \in X$ .

If we take x = (4,0), y = (8,0), a = (0,1)  

$$d(A(4,0), B(8,0), (0,1)) = d((0,0), (0,0), (0,1))$$

$$= 0 < 2k \text{, where } k \in [0,1[$$

$$\leq k \max\{2,0,2\}$$

$$d((0,0), (4,0), (0,1)),$$

$$d((0,0), (0,0), (0,1)),$$

$$d((4,0), (0,0), (0,1)),$$

$$d((4,0), (0,0), (0,1)),$$

$$d(S(4,0), A(4,0), (0,1)),$$

$$d(S(4,0), B(8,0), (0,1)),$$

Hence, for x = (4,0), y = (8,0) and a = (0,1), the condition (3.1) is satisfied. Then by a routine calculation, one can verify that the condition is satisfied with  $k \in [0, 1[$ . Thus, all the conditions of Theorem 3.1 are satisfied and hence in view of Theorem 3.1, the mappings A, B, S and T have a unique common fixed point in X. Obviously, this common fixed point is  $(0,0) \in X$ . Here one may notice that both the pairs (A, S) and (B, T) have two points of coincidence, namely (0,0) and (4,0).

Next, as an application of theorem 3.1, we give a generalized common fixed point theorem for four finite families of self mappings which runs as follows.

**Theorem: 3.4** Let  $\{A_i\}_{1 \le i \le m}$ ,  $\{B_j\}_{1 \le i \le n}$ ,  $\{S_k\}_{1 \le k \le p}$  and  $\{T_l\}_{1 \le i \le q}$  be four finite families of self-mappings on a 2-metric space (X, d) with  $A = \prod_{i=1}^{m} A_i$ ,  $B = \prod_{j=1}^{n} B_j$ ,  $S = \prod_{k=1}^{p} S_k$  and  $T = \prod_{l=1}^{q} T_l$  such that A, B, S and T satisfying the following conditions:

(P)  $d(Ax, By, a) \le g(d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a))$  for all  $x, y \in X$  and for all  $a \in X$ , where  $g \in \mathcal{F}$ . (Q)  $A(X) \subset T(X)$  and  $B(X) \subset S(X)$ . P. L. Sanodia<sup>1</sup>, Arvind Gupta<sup>2</sup>, Amit Kumar Pandey<sup>\*3</sup>, Manoj Ughade<sup>3</sup> / Using Delbosco's Set To Prove Unified Fixed Point Theorems in The Setting of 2-Metric Spaces / IJMA- 6(4), April-2015.

- (**R**) If one of A(X), B(X), S(X) and T(X) is a complete subspace of X. Then
- (S) The pair (A,S) has a point of coincidence,
- (T) The pair (B,T) has a point of coincidence,

Moreover, if  $A_oA_r = A_rA_o$ ,  $S_uS_v = S_vS_u$ ,  $B_sB_t = B_tB_s$ ,  $T_eT_h = T_hT_e$ ,  $A_oS_u = S_uA_o$  and  $B_sT_e = B_sT_e$  for all  $o, r \in I_1$ ,  $u, v \in I_2$ ,  $s, t \in I_3$ , and  $e, h \in I_4$ , where  $I_1 = \{1, 2, \dots, m\}, I_2 = \{1, 2, \dots, p\}, I_3 = \{1, 2, \dots, n\}$  and  $I_4 = \{1, 2, \dots, q\}$ , then for all  $o \in I_1$ ,  $u \in I_2$ ,  $S \in I_3$  and  $e \in I_4$ ,  $A_o, B_s, S_u$  and  $T_e$  have a common fixed point.

**Proof:** The conclusions (S) and (T) are immediate as A, B, S and T satisfy all the conditions of theorem 2.1. In view of pairwise commutativity of various pairs of the families (A, S) and (B, T) the weak compatibility of pairs (A, S) and (B, T) are immediate. Thus all the conditions of theorem 4.1 (for mappings A, B, S and T) are satisfied ensuring the existence of a unique common fixed point, Say z. Now, one needs to show that z remains the fixed point of all the component maps. For this consider

Similarly, one can show that,

$$\begin{split} A(S_u z) &= S_u(Az) = S_u z, \\ S(S_u z) &= S_u(Sz) = S_u z, \\ S(A_0 z) &= A_o(Sz) = A_o z, \\ B(B_s z) &= B_s(Bz) = B_s z, \\ B(T_e z) &= T_e(Bz) = T_e z, \\ T(T_e z) &= T_e(Tz) = T_e z, \\ B(T_e z) &= T_e(Bz) = T_e z. \end{split}$$

Which show that (for all  $o \in I_1$ ,  $u \in I_2$ ,  $S \in I_3$  and  $e \in I_4$ )  $A_o z$  and  $S_u z$  are other fixed points of the pair (A, S) whereas  $B_s z$  and  $T_e z$  are other fixed points of the pair(B, T).

Now in view of uniqueness of the fixed point A, B, S and T (for all  $o \in I_1$ ,  $u \in I_2$ ,  $s \in I_3$  and  $e \in I_4$ ), one can write  $A_o z = S_u z = B_s z = T_e z = z$ .

This means that the point z is a common fixed point of  $A_0$ ,  $S_u$ ,  $B_s$  and  $T_e$ . for all  $o \in I_1$ ,  $u \in I_2$ ,  $s \in I_3$  and  $e \in I_4$ . By setting

 $\begin{array}{l} A_1 = A_2 = \cdots \dots \dots = A_m = A \ , \\ B_1 = B_2 = \cdots \dots \dots = B_n = B \ , \\ S_1 = S_2 = \cdots \dots \dots = S_p = S \ , \\ T_1 = T_2 = \cdots \dots \dots = T_q = T \ , \end{array}$ 

One deduces the following corollary for various iterates of A, B, S and T, which can also be viewed as partial generalization of theorem 3.4.

Corollary: 3.3 Let (A, S) and (B, T) be two commuting pairs of self-mappings of 2-metric space (X, d) such that

(U)  $d(A^m x, B^n y, a) \le g(d(S^p x, T^q y, a), d(S^p x, A^m x, a), d(T^q y, B^n y, a))$  for all x, y,  $a \in X$ , where  $g \in \mathcal{F}$ .

(V)  $A^m(X) \subseteq T^q(X)$  and  $B^n(X) \subseteq S^p(X)$ .

(W) If one of the  $A^{m}(X)$ ,  $B^{n}(X)$ ,  $S^{p}(X)$  or  $T^{q}(X)$  is a complete subspace of X.

Then

- (X) The pair (A,S) has a point of coincidence,
- (Y) The pair (B,T) has a point of coincidence,
- (Z) A, B, S and T have a unique common fixed point.

**Remark: 3.1** A result similar to corollary 3.1 and corollary 3.2 involving various iterates of mappings corresponding to corollary 3.3 can also be derived. Due to repetition, the details are avoided.

Next, we furnish an example which establishes the utility of Corollary3.3 over the Theorem 3.1.

**Example:** 3.1 Let  $X = \{((0,0), (1,0), (2,0), (0,1)\}$  is a finite subset of  $\mathbb{R}^+ \times \mathbb{R}^+$  equipped with natural area function d on  $X^3$ . Define self-mapping A, B, S, T:  $X \to X$  by

A(0,0) = (0,0),	B(0,0) = (0,0),	S(0,0) = (0,0),	T(0,0) = (0,0),
A(1,0) = (0,0)	B(1,0) = (0,0)	S(1,0) = (0,0)	T(1,0) = (0,0)
A(2,0) = (1,0),	B(2,0) = (0,0),	S(2,0) = (1,0),	T(2,0) = (0,0),
A(0,1) = (0,0),	B(0,1) = (2,0),	S(0,1) = (1,0),	T(0,1) = (0,0).

Then we have,

 $\begin{aligned} A^2(X) &= \{(0,0)\} = A^m(X), \forall m \in N - \{1\}, \\ S^2(X) &= \{(0,0)\} = S^p(X), \forall p \in N - \{1\}, \\ T^1(X) &= \{(0,0)\} = T^q(X), \forall q \in N. \end{aligned}$ 

Obviously,

 $A^{2}(X) = A^{m}(X) = T^{1}(X) = T^{q}(X)$  and  $B^{2}(X) = B^{n}(X) = S^{2}(X) = S^{p}(X)$ .

Also, the pairs (A, S) and (B, T) are c. p.

Also,  $A^m(X)$ ,  $B^n(X)$ ,  $S^p(X)$  or  $T^q(X)$  are complete subspace of X for each m, n, p  $\in N - \{1\}$  and  $q \in N$ . The pairs (A, S) and (B, T) are commuting and hence weakly compatible. Define a function g:  $(\mathbb{R}^+)^3 \to \mathbb{R}^+$  by

g(u, v, w) = au + bv + cwfor all u, v, w  $\in \mathbb{R}^+$ , where a, b, c  $\ge 0$  and a + b + c < 1, obviously g  $\in \mathcal{F}$ .

Then, it is straight forward to verify that Delbosco's contraction condition (p) is satisfied for  $A^m$ ,  $B^n$ ,  $S^p$  or  $T^q$  as  $d(A^m x, B^n y, a) = d((0,0), (0,0), z) = 0$  for all x, y, z  $\in$  X. and for all m, n, p  $\in$  N – {1}; q  $\in$  N. Thus all the conditions of Corollary 3.3, are satisfied for  $A^m$ ,  $B^n$ ,  $S^p$  or  $T^q$  and hence in view of Corollary 3.3, the mappings A, B, S and T have a unique common fixed point (obviously, this common fixed point is  $(0,0) \in X$ ). Here one may notice that both the pairs have two points of coincidence, namely (0,0) and (1,0).

However, Theorem 3.1 is not applicable in the context of this example, as  $A(X) = \{(0,0), (1,0)\} \notin \{(0,0)\} = T(X)$  and  $B(X) = \{(0,0), (2,0)\} \notin \{(0,0), (1,0)\} = S(X)$ . Moreover, Delbosco's contraction condition (3.1) is not satisfied for A, B, S and T. To substantiate this, consider the case when x = (2,0) and y = (0,0), then one gets

$$1 \le a.1 + b.0 + c.0 = a$$

Which is a contradiction to the fact that  $a, b, c \ge 0$  and a + b + c < 1, that is a < 1. Thus, in all Corollary 3.3 is genuinely different to Theorem 3.1.

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