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FORMULATION OF A BI-ALGORITHM TO DETERMINE MEASURES OF NONLINEARITY IN NLR-MODELS WITH TWO PARAMETERS

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ABSTRACT

This paper introduces a novel computational algorithm for determining the root mean square (RMS) curvature and the values of the Hougaard measure of skewness simultaneously as nonlinearity measure in a nonlinear regression model (NLR-model) with two parameters. This suggested Bi-algorithm is suited for implementation using Computer Algebra Systems (CAS). We apply this algorithm throughout two illustrative examples.

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Key words: *Bias, CAS, Curvature, Measures of nonlinearity, Nonlinear regression models, Skewness.*

1. INTRODUCTION

The more general normal NLR-model is given by the following form:

$$y_i = f(x_i, \Theta) + \varepsilon_i,\ i = 1, 2, \ldots, m \tag{1.1}$$

where, (x_i, Θ) depends on a vector x_i of predictors for the *i*th of m observations and a $P\times 1$ vector $\Theta = (\theta_1, \theta_2, \ldots, \theta_P)$ of unknown parameters. The response function f is a known, scalar-valued function that is twice continuously differentiable in Θ and y_i denotes the *i*th response. The errors $\varepsilon_i, i = 1, 2, \ldots, m$ are usually assumed to be independent and identically normal distributed random variables with mean zero and constant variance σ^2 [16-18].

One of the methods used to analyze the nonlinear behavior of a model data set combination is the calculation of the so called measures of nonlinearity [1-4, 6-19]. Previously, a number of measures and procedures of studying the estimation behavior of NLR-models have been described. These include the curvature measures of intrinsic and parameter-effects nonlinearity, the bias measure of Box and the asymmetry measure of bias of Lowry. Moreover, the Hougaard measure, which is best to use a direct measure of skewness, was derived by Hougaard. Computer programs for calculating these measures of nonlinearity are presented in several literatures.

In this study the RMS curvature and the Hougaard measure of skewness will be considered. We describe these measures and show how this measure is used to indicate a degree of nonlinear behavior for each parameter in an NLR-model. A new Bi-algorithm with its corresponding procedure for calculating these mentioned measures of a two parameters NLR-model will be presented.

2. DESCRIPTION OF RMS CURVATURE MEASURES AND HOUGAARD MEASURE OF SKEWNESS

In this section, we describe the RMS curvature measures and the Hougaard measure of skewness with the used notations.

2.1 RMS curvature measures

In this subsection, we give a general description of the RMS curvature measures in an NLR model (1.1). The nonlinear behavior of a model data set combination can be analyzed by calculating the measures of nonlinearity [1-4, 9, 10, 13, 17-19]. Using concepts from differential geometry, two aspects of curvature as useful measures of nonlinearity were

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developed and this development is based on the NLR-model (1.1). The first aspect is the intrinsic curvature which has relation of the expectation surface, and the other aspect is the parameter-effects curvature which depends on the method of parameterization. These measures can be used to compare different models with different parameterizations combined with different data sets. Using some algebraic steps, the curvature and its components were converted to dimension-less relative curvatures to remove the dependence of the curvature-values on the scaling of the data. The curvature measure is the square root of the average over all directions of the squared curvature. The final forms of the two components of RMS, the parameter-effects RMS curvature " c^{PE} " and the intrinsic RMS curvature " c^{IN} ", are given in the following two equations respectively:

$$[c^{PE}]^2 = \frac{1}{P(P+2)} \sum_{n=1}^{P} [2 \sum_{p=1}^{P}\sum_{q=1}^{P} c_{npq}^2 + [\sum_{p=1}^{P} c_{npp}]^2] \tag{2.1}$$

and

$$[c^{IN}]^2 = \frac{1}{P(P+2)} \sum_{n=P+1}^{k} [2 \sum_{p=1}^{P}\sum_{q=1}^{P} c_{npq}^2 + [\sum_{p=1}^{P} c_{npp}]^2] \tag{2.2}$$

where c_{npq} and c_{npp} denote to the elements in the relative curvature array and moreover k is an integer number which at most takes the value $P(P+3)/2$, see [3, 18]. RMS curvature measures are not very meaningful as they stand, because its user does not know what constitutes a "large" value. A convenient scale of reference can be established by comparing the RMS curvature with that of the confidence disk at specified level, $100\,(1-\alpha)\%$. Thus, a RMS curvature will be considered small if it is much less than the curvature of the $100\,(1-\alpha)\%$ confidence disk, that is, if (RMS curvature) $\sqrt{F} < 1$, where $F = F\,(P, m-P; \alpha)$ is the upper α quantile for F-distribution with the usual value $\alpha = 0.05$. The results of the suggested algorithm provide two values " $c^{PE}\sqrt{F}$ " and " $c^{IN}\sqrt{F}$ ". For more details, see [1, 3, 8, 16-18].

2.2 Hougaard measure of skewness

Now, we describe the Hougaard measure of skewness and explain how to use this measure to indicate a degree of the nonlinear behavior in NLR-models [1, 3, 7, 12, 13, 16-19]. Consider a NLR-model (1.1) with p parameters. For a nonlinear model:

$$\mathbf{E}\Big[\hat{\theta}_p - \mathbf{E}(\hat{\theta}_p)\Big]^3 = -(s^2)^2 \sum_j \sum_l \sum_u L_{pj} L_{pl} L_{pu} (W_{jlu} + W_{lju} + W_{ulj}) \tag{2.3}$$

the $m \times p$ Jacobian matrix with respect to the parameters is written as $\mathbf{J}(\hat{\Theta})$ with typical elements " $J_{ip} = \dfrac{\partial f(x_i, \Theta)}{\partial \theta_p}$ " evaluated at $\hat{\Theta}$. Also, the $m \times P \times P$ Hessian array of $\mathbf{f}(\mathbf{x}, \Theta)$ with respect to the parameters is written as $\mathbf{H}(\hat{\Theta})$ with typical elements " $H_{ipq} = \dfrac{\partial^2 f(x_i, \Theta)}{\partial \theta_p \partial \theta_q}$ " evaluated at $\hat{\Theta}$. Here, i runs from 1 to m while p and q run from 1 to P. Now, let L_{pq}, p, q = 1, 2,..., P denote the elements of $\mathbf{L} = \Big[\mathbf{J}^T(\hat{\Theta})\mathbf{J}(\hat{\Theta})\Big]^{-1}$ where $\mathbf{J}^T(\hat{\Theta})$ is the transpose of $\mathbf{J}(\hat{\Theta})$. If $W_{jlu} = \sum_{n=1}^{m} J_{nj} H_{nlu}$, then an estimate of the third moment of $\hat{\theta}_p$ is given by

$$\mathbf{E}\Big[\hat{\theta}_p - \mathbf{E}(\hat{\theta}_p)\Big]^3 = -(s^2)^2 \sum_j \sum_l \sum_u L_{pj} L_{pl} L_{pu} (W_{jlu} + W_{lju} + W_{ulj})$$

with the indices j, l and u each ranging from 1 to P. The estimate of the residual variance is $S^2 = RSS(\hat{\Theta})/(m-P)$, which is based on the residual sum of squares " $RSS(\hat{\Theta}) = \sum_{i=1}^{m} \Big[y_i - f(x_i, \hat{\Theta})\Big]^2$ " at $\hat{\Theta}$ and the degree of freedom $(m-P)$. The standardized third moment can be given as:

$$\text{SKEWNESS}_{\theta_p} = \mathbf{E}\Big[\hat{\theta}_p - \mathbf{E}(\hat{\theta}_p)\Big]^3 / (s^2 L_{pp})^{3/2} \tag{2.4}$$

which provides a direct measure of the skewness of $\hat{\theta}_p$.

According to [7, 12, 17], the following table indicates a degree of a nonlinear behavior of the estimator $\hat{\theta}_p$ of the parameter θ_p in an NLR-model (1.1).

Table-1: The standard absolute values of Hougaard measure $\text{SKEWNESS}_{\theta_p}$ for θ_p

$0.00 < \left|\text{SKEWNESS}_{\theta_p}\right| < 0.10$	: The estimator is very close-to-linear in behavior.
$0.10 \le \left|\text{SKEWNESS}_{\theta_p}\right| < 0.25$	: The estimator is reasonably close-to-linear in behavior.
$0.25 \le \left|\text{SKEWNESS}_{\theta_p}\right| < 1.00$	: The skewness is very apparent.
$1.00 \le \left|\text{SKEWNESS}_{\theta_p}\right| < \infty$	: These values indicate considerable nonlinear behavior.

3. BI-ALGORITHM TO COMPUTE THE MEASURES OF NONLINEARITY IN THE NLR-MODEL

Some calculations of each of the measures of nonlinearity are provided separately for some different NLR models with two or more parameters [3, 9, 10, 13, 16, 17]. Recently, a formulation of a method for calculating only the two components of the RMS curvature is provided in [8]. Moreover, the directed calculation-method of the Hougaard measure of skewness is described in [7]. Here, we will concern with the calculations of the considered measures of nonlinearity simultaneously especially for the NLR model with two parameters in which there is a conditionally linear parameter. To verify this aim we will apply the new suggested algorithm which will be explained with two illustrative examples.

Now, we indicate to the model and the data, which will be considered in the calculations with respect to our suggested Bi-algorithm and its MAPLE-procedure for determining the mentioned measures of nonlinearity. We consider a model with two parameters to explain the application of this proposed algorithm directly by the MAPLE program. We consider the Michaelis-Menten model for enzyme kinetics which relates the initial "velocity" of an enzymatic reaction to the substrate concentration x [7, 8, 18]. This model is given by

$$\mathbf{f}(x,\Theta) = \theta_1 x/(\theta_2 + x) \tag{3.1}$$

where **f** is predicated velocity, $\Theta = (\theta_1, \theta_2)$ is a vector of parameters (i.e., $P = 2$). This is a model in which there is conditionally linear parameter θ_1. The Bi-algorithm to calculate the RMS curvature measure and the Hougaard measure of skewness for the model (3.1) is explained. The corresponding procedure is formulated using the MAPLE program. Moreover, we apply this Bi-algorithm for two different data sets with the same number of observations ($m = 12$, say).

3.1 A Bi-algorithm Explanation

In this subsection we give an explanation of our suggested algorithm to calculate the considered measures of nonlinearity for the two parameters in the model (3.1) through the following steps:

Step-1: Describe the parameter space as two dimensional spaces.
Step-2: Define the considering P-parameter model as model equation (P=2).
Step-3: Define the expectation surface as a (1, m)-vector through calculation of the model function for each mass-point (m=12).
Step-4: Define the Jacobian-matrix "JAC" (i.e., $\mathbf{J}(\hat{\Theta})$) of the model function with respect to the parameter-vector and calculate this matrix for the estimated parameters.
Step-5: Define "**L**" as the inverse matrix of the multiplication [$\mathbf{J}^{\mathbf{T}}(\hat{\Theta})\, \mathbf{J}(\hat{\Theta})$].
Step-6: Define a (m, P, P)-tensor "HES" (i.e., $\mathbf{H}(\hat{\Theta})$) and calculate this matrix for the same estimated parameters.
Step-7: Formulate the terms "W [j, l, u]"(i.e., W_{jlu}) as a multiplication form of JAC and HES.
Step-8: Formulate the third moment "TM[i]" (i.e., $\mathbf{E}\left[\hat{\theta}_i - \mathbf{E}(\hat{\theta}_i)\right]^3$) of θ_i .
Step-9: Formulate the Hougaard measure "SK[i]" (i.e., $\text{SKEWNESS}_{\theta_i}$) of the parameter θ_i .

Step-10: Define the velocity (tangent) vectors (columns I the velocity matrix "Vdot", i.e $\dot{V}$).

Step-11: Define the acceleration vectors in a matrix "Vddot", i.e $\ddot{V}$.

Step-12: Define a matrix "Wddot" (i.e $\ddot{W}$) through columns-arrangement of " $P(P+1)/2$ " non-redundant acceleration vectors such that these columns (in the order) are linear independent on the tangent vectors.

Step-13: Construct "Dmatrix" through $\dot{V}$ and $\ddot{W}$

Step-14: Perform a QR decomposition "Q" on the matrix "Dmatrix".

Step-15: Determine the (12, 3)-submatrix "Q1" of the matrix "Q" and its transpose matrix "Q1transp" (i.e. $Q1^T$).

Step-16: Calculate the (3, 2, 2)-acceleration-tensor "ACCddot" (i.e $\ddot{A}$) through the multi-plication of " $\ddot{V}$ " from the left with the transpose matrix " $Q1^T$ ".

Step-17: Define the (2, 2)-submatrix "R11" which consists of the first two rows of the QR decomposition matrix of the matrix "Dmatrix" and calculate the inverse matrix "IR11" of "R11".

Step-18: Define the faces "C[j]" of the (3, 2, 2) tensors for the relative curvature, where the first two faces describe the parameter-effects relative curvature tensor and the rest face gives the intrinsic relative curvature tensor. The calculation was carried out through the multiplication of " $\ddot{A}$ " from the left and the right by the transpose matrix of "IR11" and the matrix "IR11" respectively, and the result is multiplied with "Roh" (i.e. ρ) which is the square root of the multiplication of the residual mean square (the variance estimate) "ssq" and "P".

Step-19: Using the (2.1) and (2.2) to calculate the square of the parameter-effects RMS curvature "csPE" and the square of the intrinsic RMS curvature "csIN", respectively.

Step-20: Find c^{PE} and c^{IN} by calculating the root square of csPE and csIN respectively.

Step-21: Determine "FD" as the upper α quantile for F-distribution $F = F(P, m-P; \alpha)$.

Step-22: Define the two components of the RMS curvature measure $c^{PE}\sqrt{F}$ and $c^{IN}\sqrt{F}$ with the symbols "PEmeasure and INmeasure" respectively and also define the Hougaard measure for the two parameters θ_1 and θ_2 with the symbols " $SKEWNESS_{\theta_1}$ and $SKEWNESS_{\theta_2}$ " respectively.

Step-23: Give the using variables in the procedure " MESNLR(X,par,ssq,alpha,N, P, PD)" (say the mass-points x , the initial values of the model-parameters $\hat{\Theta}$ (i.e., "Par"), the variance estimate s^2 (i.e., "ssq"), the usual value of α and so on) to proceed the calculation for each element in the previous steps with respect to the considering model.

Step-24: Evaluate PEmeasure, INmeasure, $SKEWNESS_{\theta_1}$ and $SKEWNESS_{\theta_2}$ simultaneously.

3.2 Applications

We consider the model (3.1) to apply our Bi-algorithm for calculating the values of the measures of nonlinearity. We will use two different working data-sets with response space of 12-dimentional.

Application-1

The model (3.1) is selected to apply the suggested computational Bi-algorithm through its proposed procedure for calculating the values of the RMS curvature measures and the Hougaard measure of skewness simultaneously. The corresponding used procedure is formulated by using the MAPLE program and it is given in Appendix. As a Maple programming guide, see [5].

Here, we consider the following Puromycin-data on the velocity of an enzymatic reaction [3, 8]. This data is shown in Table - 2.

Table-2: The used data set (Observations)

x	0.02	0.02	0.06	0.06	0.11	0.11	0.22	0.22	0.56	0.56	1.10	1.10
y	76	47	97	107	123	139	159	152	191	201	207	200

Here, y is the treated reaction velocity for Puromycin-experiment. This data is considered with the initial values of the parameters $\Theta^0 = (\theta_1^0, \theta_2^0) = (205, 0.08)$ to verify the aim of the calculation for the measure of nonlinearity for (3.1). Convergence for the Puromycin data set in Table 2 was declared at the estimation-values of parameters $\hat{\Theta} = (\hat{\theta}_1, \hat{\theta}_2) = (212.683, 0.06412)$ with the residual sum of squares $RSS(\hat{\Theta}) = 1195.45$ (i.e. with the residual mean square or variance estimate $s^2 = RSS(\hat{\Theta})/(m-P) = 119.545$). The program produced the values of $c^{PE}\sqrt{F}$ and $c^{IN}\sqrt{F}$ as in Table - 3:

Table-3: The computed RMS curvature measures for the model (3.1) and the used data-set

RMS curvature measure	$c^{PE}\sqrt{F}$	$c^{IN}\sqrt{F}$
Values	0.2121	0.092

Moreover, we obtain on the following computed values of the Hougaard measure of skewness:

Table-4: The computed Hougaard measure of skewness for the parameter in the model (3.1) and the used data-set

Parameter	θ_1	θ_2
$SKEWNESS_{\theta_p}$	0.09601173417	0.320596099

These results are identical with the previous results obtained in literatures. Moreover, the MAPLE- procedure is equivalent to the other procedures with other programs, which are presented in literature, for calculating the RMS curvature measures.

Application-2

Here, we consider the same model (3.1) again to apply the Bi-algorithm through its formulated MAPLE-procedure, which is given in Appendix, for repeating the calculation of the values of the measures of nonlinearity. To verify the aim of the calculation, we consider the data in Table-5 with all corresponding possible used values of $\hat{\Theta} = (0.1056427059222, 1.702689979095)$ with $s^2 = 0.00002010567$, see [1, 17].

Table-5: The used data set (Observations)

x	2.0000	2.0000	0.6670	0.6670	0.4000	0.4000	0.2860	0.2860	0.2220	0.2220	0.2000	0.2000
y	0.0615	0.0527	0.0334	0.0258	0.0138	0.0258	0.0129	0.0183	0.0083	0.0169	0.0129	0.0087

By using our suggested computational Bi-algorithm with its corresponding MAPLE-procedure, the values of $c^{PE}\sqrt{F}$ and $c^{IN}\sqrt{F}$ are obtained as in the following table:

Table-6: The computed RMS curvature measures for the model (3.1) and the used data-set

RMS curvature measure	$c^{PE}\sqrt{F}$	$c^{IN}\sqrt{F}$
Values	0.7841	0.08055

Moreover, we obtain also on the values of the Hougaard measure of skewness for the parameters in the used model and for the considered data. The results are tabulated in the following table:

Table-7: The computed Hougaard measure of skewness for the parameter in the model (3.1) and the used data-set

Parameter	θ_1	θ_2
$SKEWNESS_{\theta_p}$	0.9887703882	1.180088401

These results are identical to the previous values obtained by using a listing of a FORTRAN-subroutine for computing the Hougaard measure $SKEWNESS_{\theta r}$ for $\theta_p, p = 1, 2$ in the same model (3.1) and for the same data. With regard to Ratkowsky's results in Table - 7, the skewness for the estimator of θ_1 is very apparent and the value of the skewness for the estimator of θ_2 indicates considerable nonlinear behavior. This means that the estimators of the parameters in this example are far-from-linear in their estimation behavior.

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4. CONCLUSION

In this study, we presented a Bi-algorithm for calculating the RMS curvature measures and the Hougaard measure of skewness simultaneously. This algorithm was applied using the MAPLE program. The corresponding presented procedure is very effective by using optimal accuracy. We provided a method for calculating the mentioned measures in a two parameter NLR-model, in which there is a conditionally linear parameter. The results indicated a degree of a nonlinear behavior in the model-data-set and in the estimator of the parameter in the studied model. The proposed calculations method is competitive to other methods appeared in previous literature in which the calculation is processed for each measure separately. The suggested algorithm is available for other NLR-models with two parameters when the corresponding used procedure is changed in some suitable functions. The results of the considered applications are identical to the corresponding previous results which are obtained separately in literature.

In the future, we intend to modify the presented algorithms and their procedures to be suitable for multi-parameters models. Moreover, we plan to provide a Multi-algorithm with its procedure for calculating more than two measures of nonlinearity simultaneously.

APPENDIX:

The used MAPLE-procedure corresponding to the suggested Bi-algorithm is formulated for the considered model (3.1) and also for the two given data sets as follows:

A MAPLE Bi-procedure for calculating the RMS curvature measures and the Hougaard measure of skewness

```
>   restart: Digits:=15: with(linalg) :with(stats):
MESNLR:=proc(X::vector,par::vector,ssq::float,alpha::float,N::nonnegint,P::nonnegint, PD::nonnegint)
local i,j,k,p,s,u,c,l,r,x,J,H,JAC,HES,HESS,TM,L,z,Wddot,U,W,q,Q1,g1,g2,g3,IR11,csIN;
global Theta,m,a,f,F,SK,C,Dmatrix,cPE,csPE,cIN,R11 ,Vdot,Vddot,Q,RANK,ACCddot,R,PE,FD,Addot,Roh,Q1t;
m:=vectdim(X):a:=2:Theta:=array(1..a):
f:=(x,Theta)->Theta[1]*x/(Theta[2]+x);
F:=array(1..m): J:=array(1..m):H:=array(1..m):
for i from 1 to m do
   F[i]:=f(X[i],Theta);
od:
for c from 1 to m do
   J[c]:=jacobian(vector([F[c]]),[Theta[1],Theta[2]]);  H[c]:=hessian(F[c],[Theta[1],Theta[2]]);
od:
JAC:=subs(Theta[1]=par[1],Theta[2]=par[2], stackmatrix(J[1],J[2],J[3],J[4],J[5],J[6],J[7],J[8],J[9],J[10],J[11], J[12]));
L:=inverse(evalm(transpose(JAC)&*JAC)):
HES:=subs(Theta[1]=par[1],Theta[2]=par[2],stackmatrix (H[1],H[2],H[3],H[4],H[5],H[6],H[7], H[8],H[9], H[10],
H[11],H[12]));
for s from a by a to a*m do
   HESS[s/2]:=submatrix(HES,s-1..s,1..a);
od:
for l from 1 to a do
   W[1,1,l]:=sum(JAC[r,1]*HESS[r][1,l],r=1..m);  W[1,2,l]:=sum(JAC[r,1]*HESS[r][2,l],r=1..m);
   W[2,1,l]:=sum(JAC[r,2]*HESS[r][1,l],r=1..m);  W[2,2,l]:=sum(JAC[r,2]*HESS[r][2,l],r=1..m);
od:
i:='i': j:='j':
for i from 1 to a do
   TM[i]:=-((ssq)^2)*sum(sum(sum(L[i,j]*L[i,u]* L[i,p]*(W[j,u,p]+W[u,j,p]+W[p,u,j]), j=1..a), u=1..a), p=1..a);
od:
i:='i':
for i from 1 to a do
   SK[i]:=TM[i]/((ssq*L[i,i])^(3/2)):
od;
Vdot:=subs(Theta[1]=par[1],Theta[2]=par[2],JAC):  Vddot:=array(1..m,1..P,1..P):
for i from 1 to m do
   for j from 1 to P  do
      for k from 1 to P  do
Vddot[i,j,u]:=subs(Theta[1]=par[1],Theta[2]=par[2], diff(F[i],Theta[j],Theta[u])):
end do:end do:end do:
Wddot:=array(1..m,1..P*(P+1)/2):
for i from 1 to m do
Wddot[i,1]:=subs(Theta[1]=par[1],Theta[2]=par[2], diff(F[i],Theta[1],Theta[1])):
```

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```
end do:
for i from 1 to m do
Wddot[i,2]:=subs(Theta[1]=par[1],Theta[2]=par[2], diff(F[i],Theta[2],Theta[1])):
end do:
for i from 1 to m do
Wddot[i,3]:=subs(Theta[1]=par[1],Theta[2]=par[2], diff(F[i],Theta[2],Theta[2])):
end do:
for s from 1 to P*(P+1)/2 do
   U[s] := L -> Wddot[L,s]:W[s]:=vector(m,U[s]):
end do:
for s from 1 to P*(P+1)/2 do
   U[s] := L -> Wddot[L,s]:W[s]:=vector(m,U[s]):
end do:
for i from 1 to m do
if W[1,i,1]<>0 then z:=1; break; else
z:=0;
fi
end do;
if z=0 then
Dmatrix:=augment(Vdot,W[1],W[2],W[3]);
else
Dmatrix:=augment(Vdot,W[3],W[2],W[1]);
fi;
R:=QRdecomp(Dmatrix,Q='q',rank='r',fullspan=true): Q:=evalm(q):
Q1:=submatrix(Q, 1..m,1..P+PD):Q1t:=transpose(%):
i:='i':j:='j':k:='k':
for g1 from 1 to P+PD do
for g2 from 1 to P do
      for g3 from 1 to P do    ACCddot[g1,g2,g3]:=sum(Q1t[g1,i]*Vddot[i,g2,g3], i=1..m);
od: od: od:
for k from 1 to P+PD do
ACCddot[k]:=matrix(P,P,[ACCddot[k,1,1], ACCddot[k,1,2],ACCddot[k,2,1],ACCddot[k,2,2]]);
end do:
Addot:=augment(ACCddot[1],ACCddot[2], ACCddot[3]):
R11:=submatrix(R, 1..P, 1..P):IR11:=inverse(%):Roh:=evalf(sqrt(ssq*P)):
for j from 1 to P+PD do
C[j]:=simplify(evalm(transpose(IR11)&*ACCddot[j]&*IR11*Roh)):
end do:
Digits:=4:
csPE:=(1/(P*(P+2)))*sum(2*(sum(sum((C[n][p,q])^2, q=1..P),p=1..P))+(sum(C[n][p,p],p=1..P))^2,n=1..P):
csIN:=(1/(P*(P+2)))*sum(2*(sum(sum((C[n][p,q])^2, q=1..P),p=1..P))+(sum(C[n][p,p],p=1..2))^2, n=P+1..P+PD):
cPE:=sqrt(csPE);cIN:=sqrt(csIN);
FD:=statevalf[icdf,fratio[P,m-P]](1-alpha); 1/sqrt(FD);sqrt(FD):
print(PEmeasure=cPE*sqrt(FD));      print(INmeasure=cIN*sqrt(FD));
print(SKEWNESS[theta[1]]=SK[1]);  print(SKEWNESS[theta[2]]=SK[2]);
end:
>  X:=vector([0.02,0.02,0.06,0.06,0.11,0.11,0.22,0.22, 0.56,0.56,1.10,1.10] ):
par:=vector([212.7,0.0641]):ssq:=119.5:alpha:=0.05: N:=6:P:=2:PD:=1: MESNLR(X,par,ssq,alpha,N,P,PD);
```

$PEmeasure=0.2121$, $INmeasure=0.9200$

$SKEWNESS_{\theta 1}=0.096011734165989$, $SKEWNESS_{\theta 2}=0.320596098871428$

```
>  X:=vector([2,2,0.667,0.667,0.4,0.4,0.286,0.286,0.222,0.222,0.2,0.2]):
par:=vector([0.1056427059222,1.702689979095]):ssq:=0.00002010567: alpha:=0.05: N:=6:P:=2:PD:=1:
MESNLR(X,par,ssq,alpha,N,P,PD);
```

$PEmeasure=0.7841$, $INmeasure=0.08055$,

$SKEWNESS_{\theta 1}=0.988770300786167$, $SKEWNESS_{\theta 2}=1.18008828310710$

REFERENCES

1. Bates, D.M. and Watts, D.G. (1980). Relative curvature measures of nonlinearity (with discussion), *Journal of the Royal Statistical Society. Series B*, 42, 1-25.
2. Bates, D. M and Watts, D.G. (1981). Parameter transformations for improved approximate confidence region in nonlinear least squares, *Annals of Statistics*, 9 (6), 1152 – 1167.

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3. Bates, D. M. and Watts, D.G. (1988). Nonlinear Regression Analysis and its Applications, John Wiley and Sons, Inc., New York.
4. Bates, D. M., Hamilton, D.C. and Watts, D.G. (1983). Calculation of intrinsic and parameter effects curvatures for nonlinear regression models, *Communications in Statistics-simulation and Computation*, 12 (4), 469- 477.
5. Bernardin, L., Chin, P., DeMarco, P., Geddes, K. O., Hare, D. E. G., Head, K. M., Labahn, G., May J. P., McCarron, J., Monagan, M. B., Ohashi, D. and Vorkoetter, S. M. (2011). Maple Programming Guide, *Copyright © Maple Soft, a Dividion of Waterloo Maple Inc*
6. Box, M. J. (1971). Bias in nonlinear estimation, *Journal of the Royal Statistical Society. Series B.*, 33 (2), 171-201.
7. El-Shehawy, S. A. (2009). On calculating the Hougaard measure of Skewness in a nonlinear regression model with two parameters, *Journal of Mathematics and Statistics*, 5 (4), 360-364.
8. El-Shehawy, S. A. and Karawia, A.A. (2007). An alternative computational algorithm for calculating the nonlinearity of regression models with two parameters, *Applied Mathematics and Computation*, 188 (1), 686-692.
9. Guay, M. (1995). Curvature measures for multiresponse regression models, *Biometrika*, 82 (2), 411-417.
10. Haines, L. M., 'Brien, T.E.O. and Clarke, G.P.Y. (2004). Kurtosis and curvature measures for nonlinear regression models, *Statistica Sincia*, 14 (2), 547-570.
11. Hamilton, D. C., Watts, D.G. and Bates, D.M. (1982). Accounting for intrinsic nonlinearity in nonlinear regression parameter inference regions", *Annals of Statistics*, 10, 386-393.
12. Hougaard P. (1985), "The appropriateness of the asymptotic distribution in a nonlinear regression model in relation to curvature", *Journal of the Royal Statistical Society. Series B*, 47, 1 (1985) 103-114.
13. Lennartz, F., Müller, H., Nollau, V., Schmitz, G. H. and El-Shehawy, S. A. (2008). Statistical evaluation and choice of soil water retention models, *Water Resour. Res.*, 44, W12424, doi: 10. 1029/2007WR006138, pp:14
14. Lowry, R. K. and Morton, R. (1983). An asymmetry measure for estimators in non-linear regression models, *In Proc. 44th Session Int. Statist. Inst., Madrid, Contrib. Papers*, 1, 351-354.
15. Morton, R. (1987). Asymmetry of estimators in nonlinear regreesion, *Biometrika*, 74 (4), 679-685.
16. Ratkowsky, D. A. (1983). Nonlinear Regression Modeling, Marcel Dekker, New York.
17. Ratkowsky, D. A. (1990). Handbook of Nonlinear Regression Models, Marcel Dekker, New York.
18. Seber, G.A.F. and Wild, C.J. (2005). Nonlinear Regression, John Wiley and Sons, New York.
19. Ueda, C. M., Yamamoto, A. Y., Nunes, W. M. de C., Scapim, C. A. and Guedes, T. A. (2010). Non-linear models for describing the Citrus Variegated Chlorosis in groves of two counties at northwestern Paraná state, Brazil, *Acta Scientiarum - Agronomy*, 32 (4), 603-611.

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