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FUZZY BANACH MANIFOLD

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ABSTRACT

In this paper we introduce the notion of fuzzy Banach chart, fuzzy Banach atlas, fuzzy Banach manifold and discuss some of the fuzzy topological properties of fuzzy Banach manifold.

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Key words: *Fuzzy Banach chart, Fuzzy Banach atlas, fuzzy transition maps, Fuzzy Banach Manifold.*

1. INTRODUCTION

In consequence to the development in Fuzzy set theory introduced by L. A. Zadeh [6], many mathematicians introduced fuzzy norm and fuzzy metric in different perception and proved various results on fuzzy normed space and fuzzy metric space.

In this paper we introduce the concept of fuzzy Banach manifold by introducing fuzzy topological vector space on fuzzy Banach space $(X, N,*)$ introduced by R. S. Saadati and S. M. Vaezpour [9].

The concept of C^1fuzzy manifold was introduced by M. Ferraro and D. H. Foster [7], we use their approach to develop the fuzzy Banach manifold. Our results and our approach are based on fuzzy topology on fuzzy Banach space$(X, N,*)$.

2. PRELIMINARIES

Some of the basic definitions referred to introduce fuzzy Banach manifold are as follows.

The concept of Fuzzy set follow L. A. Zadeh[6], fuzzy points and neighbourhood follow Pu and Liu[2]. The fuzzy topology we consider is due to Chang [3], the fuzzy normed linear space is due to R. S. Saadati and S. M. Vaezpour[9].

Definition 2.1: Let X be a set. A fuzzy subset A of X is defined as, $A = \{(x, \mu_A): \forall\, x \in A\} = \mu_A$ where $\mu_A: A \to [0,1]$.

Definition 2.2: A fuzzy subset in X is called as a fuzzy point iff it takes the value 0 for all $y \in X$ except at one point say, $x \in X$ and if its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its support.

Remark: A fuzzy set with constant membership function $\mu_{k_c}(x) = c$ for all $x \in X$ is denoted by k_c.

Definition 2.3: A fuzzy topology on a set X is a family τ of fuzzy subsets in X which satisfies the following conditions:

i) $k_0, k_1 \in \tau$
ii) If $A, B \in \tau$ then $A \cap B \in \tau$
iii) If $A_j \in \tau, \forall\, j \in J$ (J some index set) then $\bigcup_{j \in J} A_j \in \tau$

The pair (X, τ) is called a fuzzy topological space and the members of τ are called open fuzzy subsets.

If a fuzzy topology defined above satisfies Lowen's definition [8], then we refer to it as a proper fuzzy topology.

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Definition 2.4: Let τ be a fuzzy topology on a set X. A subfamily $\mathfrak{B}$ of τ is called a base if each member of τ can be expressed as the union of members of $\mathfrak{B}$.

Proposition 2.1: A family $\mathfrak{B}$ of fuzzy sets in X is a base for a proper fuzzy topology on X if it satisfies following conditions:

i) $sup_{B\in\mathfrak{B}}\{\mu_B(x)\} = 1, \forall\, x \in X$
ii) If $B_1, B_2 \in \mathfrak{B}$ then $B_1 \cap B_2 \in \mathfrak{B}$
iii) For every $0 \le c < 1$ and every $B \in \mathfrak{B}, k_c \cap B \in \mathfrak{B}$.

Remark: If a base for an improper fuzzy topology is considered then condition (iii) is unnecessary.

Definition 2.5: A fuzzy topological vector space is a vector space X over the field K of real or complex numbers, X equipped with fuzzy topology τ and K equipped with the usual topology κ, such that the mappings

i) $(x, y) \to (x + y)$ of $(X, \tau) \times (X, \tau)$ onto (X, τ)
ii) $(\alpha, x) \to (\alpha x)$ of $(K, \kappa) \times (K, \kappa)$ onto (K, κ) are continuous.

Definition 2.6: Let X, Y be fuzzy topological spaces. A bijection $f$of X onto Y is said to be a fuzzy continuous if for each open fuzzy set A in Y the inverse image $f^{-1}(A)$ is in X.

Definition 2.7: Let X, Y be fuzzy topological spaces. A bijection f of X onto Y is said to be a fuzzy open map if for each open fuzzy subset A in X the image $f(A)$ is in Y.

Definition 2.8: Let X, Y be fuzzy topological spaces. A bijection $f$of X onto Y is said to be a fuzzy homeomorphism if f is fuzzy continuous and fuzzy open map.

Definition 2.9: Let X, Y be fuzzy topological spaces. A bijection $f$of X onto Y is said to be a fuzzy diffeomorphism of class C^k if f and f^{-1} are fuzzy differentiable of class C^k.

Definition 2.10: The triplet $(X, N, *)$ is said to be fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying following conditions for every $x, y \in X$ and $t, s > 0$:

i) $N(x, y) > 0$,
ii) $N(x, t) = 1$ iff $x = 0$,
iii) $N(\alpha x, t) = (x, t/|\alpha|)$ for all $\alpha \neq 0$,
iv) $N(x, t) * N(y, s) \le N(x + y, t + s)$,
v) $N(x, .): (0, \infty) \to [0, 1]$ is continuous,
vi) $\lim_{t\to\infty} N(x, t) = 1$.

Lemma 2.1: Let $(X, N, *)$ is a fuzzy normed linear space. If we define $M(x, y, t) = N(x - y, t)$, then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

Lemma 2.2: If $(X, N, *)$ is a fuzzy normed linear space, then the mappings

i) $(x, y) \to (x + y)$ of $(X, \tau) \times (X, \tau)$ onto (X, τ)
ii) $(\alpha, x) \to (\alpha x)$ of $(K, \kappa) \times (K, \kappa)$ onto (K, κ) are continuous.

Definition 2.11: Let $(X, N, *)$ be fuzzy normed space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if $\exists\, x \in X$ such that, $\lim_{t\to\infty} N(x_{n+p} - x, t) = 1, \forall\, t > 0$, then x is called as the limit of the sequence $\{x_n\}$ and we denote it by $\lim x_n$.

Definition 2.12: A sequence $\{x_n\}$ in X is said to be Cauchy sequence if
$\lim_{t\to\infty} N(x_{n+p} - x_n, t) = 1, \forall\, t > 0, p = 1,2,3$ ....

Definition 2.13: The fuzzy normed space $(X, N, *)$ is said to be a fuzzy Banach space whenever X is complete with respect to the fuzzy metric induced by fuzzy norm.

3. FUZZY BANACH SPACE

In this section we consider fuzzy Banach space $(X, N, *)$ defined by R. S. Saadati and S. M. Vaezpour[9] and show that fuzzy Banach space induces fuzzy topological vector space defined in[1].

We prove this result by employing fuzzy closed set on $(X, N, *)$ so, let us define a fuzzy closed set on fuzzy Banach space.

Definition 3.1: A fuzzy set A in fuzzy Banach space $(X, N,*)$ is said to be closed if the limit of any Cauchy sequence in A is in A.

A fuzzy set A in $(X, N,*)$ is said to be open if A' is a fuzzy closed set, where A' is defined by $A'(x) = 1 - A(x)$ for any $x \in X$.

Theorem 3.1: Suppose $(X, N,*)$ is a fuzzy Banach space. Then (X, τ_F) is a fuzzy topological space, where τ_F is defined as follows

$$\tau_F = \{ A \subset (X, N,*): A \text{ is fuzzy closed set in } (X, N,*)\}$$

Proof: It suffices to prove that τ_F satisfies the three conditions of fuzzy topology defined on the collection of fuzzy closed sets.

i) It is obvious that k_1 and k_0 are fuzzy closed sets.
ii) For any $\{A, B\} \subset \tau_F$ we prove that $(A \cup B) \in \tau_F$.
Let $\{x_n\}$ be a Cauchy sequence in $(A \cup B)$ that is, $\{x_n\}$ is in A or B, say A, since $(X, N,*)$is fuzzy Banach space every Cauchy sequence in $(X, N,*)$ is convergent in $(X, N,*)$, therefore $\{x_n\}$ has a limit point. Since A is a closed fuzzy set, the limit of $\{x_n\}$ is in A. In consequence, the limit of $\{x_n\}$ is in $(A \cup B)$, which implies that $(A \cup B) \in \tau_F$.
iii) For any $\{A_i\}_{i\in I} \subset \tau_F$, where I is an arbitrary index set, it only need to be proved that $\bigcap_{i\in I} A_i \in \tau_F$.

For any Cauchy sequence $\{x_n\}$ in $\bigcap_{i\in I} A_i$, we have $\{x_n\} \subset A_i$ for each $i \in I$. Since every A_i a closed fuzzy set, the limit of $\{x_n\}$ is in A_i for each $i \in I$. Hence it follows that $\bigcap_{i\in I} A_i$ is a closed fuzzy set.

Therefore $\{A_i\}_{i\in I} \subset \tau_F$.

Hence we can say that every fuzzy Banach space $(X, N,*)$ induces fuzzy topological space (X, τ_F).

Corollary 3.1: Suppose $(X, N,*)$ is fuzzy Banach space and (X, τ_F) is a fuzzy topological space induced by fuzzy Banach space $(X, N,*)$. Then (X, τ_F) is fuzzy topological vector space.

Proof: Let $(X, N,*)$is fuzzy Banach space, by definition of fuzzy Banach space it is clear that X is a vector space over the field K of real or complex numbers, by theorem 3.1 we know that the fuzzy Banach space $(X, N,*)$induces a fuzzy topology (X, τ_F), by Lemma 2.2 we can say that the mappings

i) $(x, y) \to (x + y)$ of $(X, \tau) \times (X, \tau)$ onto (X, τ)
ii) $(\alpha, x) \to (\alpha x)$ of $(K, \kappa) \times (K, \kappa)$ onto (K, κ) are continuous on $(X, N,*)$.

Therefore by definition 2.5 of fuzzy topological vector space it is clear that fuzzy topology (X, τ_F) induced by fuzzy Banach space $(X, N,*)$ is fuzzy topological vector space.

If we model a non-empty set on such fuzzy Banach space then it is called as fuzzy Banach manifold.

4. FUZZY BANACH MANIFOLD

In this section we introduce the definition of fuzzy Banach chart, fuzzy Banach atlas and fuzzy Banach manifold with reference to the definition of C^1 fuzzy manifold [7]. We also show that fuzzy Banach manifold induces fuzzy topological structure and prove some related results.

The following definitions and results are based on [5].

Definition 4.1: Let M be any fuzzy topological space, U is a fuzzy subset of M such that $\sup\{\mu_U(x)\} = 1$, $\forall\, x \in M$ and φ is a fuzzy homeomorphism defined on the support of $U = \{ x \in M : \mu_U(x) > 0 \}$, which maps U onto an open fuzzy subset $\varphi(U)$ in some fuzzy Banach space E_i. Then the pair (U, φ) is called as fuzzy Banach chart.

Definition 4.2: A fuzzy Banach atlas A of class C^k on M is a collection of pairs (U_i, φ_i) (i $\in I$) subjected to the following conditions:

i) $\bigcup_{i\in I} U_i = M$ that is the domain of fuzzy Banach charts in A cover M.
ii) Each fuzzy homeomorphism φ_i, defined on the support of $U_i = \{ x \in M : \mu_U(x) > 0 \}$ which maps U_i onto an open fuzzy subset $\varphi_i(U_i)$ in some fuzzy Banach space E_i, and for each $i, j \in I, \varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are open fuzzy subset in E_i
iii) The maps $\varphi_i \circ \varphi_j^{-1}$ which maps $\varphi_j(U_i \cap U_j)$ onto $\varphi_i(U_i \cap U_j)$ is fuzzy diffeomorphism of class C^k $(k \geq 1)$ for each pair of indices i, j.

The maps $\varphi_i \circ {\varphi_j}^{-1}$ and $\varphi_j \circ {\varphi_i}^{-1}$ for $i, j \in I$ are called fuzzy transition maps.

Remark: A fuzzy Banach atlas is said to be of class C^1 if the fuzzy transition maps are fuzzy diffeomorphism of class C^1 and is said to be of class C^∞ if fuzzy transition maps are fuzzy diffeomorphism of class C^k for every positive integer $k \geq 1$.

Proposition 4.1: Let A be a fuzzy Banach atlas with fuzzy Banach charts (U_i, φ_i). The collection of fuzzy subsets of M defined as $\mathfrak{B} = \{V \subset M : V \subset U_i, (U_i, \varphi_i) \in A$ and $\varphi_i(V)$ is an open fuzzy subset in $E_i\}$ is a base for fuzzy topology on M.

Proof: Let $\mathfrak{B} = \{V \subset M : V \subset U_i, (U_i, \varphi_i) \in A$ and $\varphi_i(V)$ is an open fuzzy subset in $E_i\}$. It is clear that domains of each fuzzy Banach charts U_i is a member of $\mathfrak{B}$.

Hence the members of $\mathfrak{B}$ cover M. By definition of fuzzy Banach charts and fuzzy Banach atlas it is clear that $sup_{V \in \mathfrak{B}}\{\mu_U(x)\} = 1, \forall\, x \in M$.

Next if $V, W \in \mathfrak{B}$ we show that $V \cap W \in \mathfrak{B}$.

Since $V, W \in \mathfrak{B}$ there exist fuzzy Banach charts (U_i, φ_i) and $\left(U_j, \varphi_j\right)$ in A such that $V \subset U_i$, $W \subset V_i$ and $\varphi_i(V)$, $\varphi_j(W)$ are open fuzzy subsets of E_i, E_j respectively.

Now $V \cap W \subset U_i$ and $V \cap W \subset U_j$

$$\begin{aligned} \Rightarrow \varphi_i(V \cap W) &= \varphi_i\left(V \cap W \cap U_i \cap U_j\right) \\ &= \varphi_i(V) \cap \varphi_i\left(W \cap U_i \cap U_j\right) \\ &= \varphi_i(V) \cap \varphi_i \circ {\varphi_j}^{-1}\left(\varphi_j(W) \cap \varphi_j\left(U_i \cap U_j\right)\right) \end{aligned}$$

$\Rightarrow \varphi_i(V \cap W)$ is an open fuzzy subset in E_i, hence $V \cap W \in \mathfrak{B}$.

Finally, for each j and c, $0 \leq c < 1$ and $V_j \subset U_j$ such that $\left(U_j, \varphi_j\right) \in A$ and $\varphi_j\left(V_j\right)$ is an open fuzzy subset in E_j.

If we define $V_j' = k_c \cap V_j$ such that $V_j' \subset M$

Clearly, $V_j' \subset V_j$ and $V_j \subset U_j$

$\Rightarrow V_j' \subset U_j$ such that $\left(U_j, \varphi_j\right) \in A$ and $\varphi_j\left(V_j\right)$ is an open fuzzy subset in E_j

and
$$\begin{aligned} \varphi_j\left(V_j'\right) &= \varphi_j\left(V_j \cap k_c\right) \\ &= \varphi_j\left(V_j \cap k_c \cap U_j\right) \\ &= \varphi_j\left(V_j\right) \cap \varphi_j\left(k_c \cap U_j\right) \\ &= \varphi_j\left(V_j\right) \cap \varphi_j \circ {\varphi_l}^{-1}\left(\varphi_l(k_c) \cap \varphi_l\left(U_j\right)\right) \end{aligned}$$

$\Rightarrow \varphi_j\left(V_j'\right)$ is an open fuzzy subset in E_j.

Therefore $V_j' \in \mathfrak{B}$, the membership function of V_j' is given by
$\mu_{V_j'}(x) = min\left\{\mu_{k_c}(x), \mu_{V_j}(x)\right\} \forall\, x \in M$.

Thus for every c, $0 \leq c < 1$, $k_c \cap V_j \in \mathfrak{B}$.

Therefore $\mathfrak{B}$ satisfies all the condition of base for a fuzzy topology on M. Thus by proposition 2.1 $\mathfrak{B}$ is base for fuzzy topology on M.

The fuzzy topology on M specified by the base $\mathfrak{B}$ is called the fuzzy topology induced by the atlas A.

The non-empty set M endowed with fuzzy topological structure is called Fuzzy Banach topological manifold. The open subsets of E_i are compliments of the closed sets containing the convergent points of Cauchy sequences defined by the fuzzy norm [9].

Proposition 4.2: Let $A = \{(U_i, \varphi_i)\}$ be a fuzzy Banach atlas on M. A subset U of M is an open fuzzy subset in the induced fuzzy topology if and only if U intersects with some fuzzy Banach chart $(U_i, \varphi_i) \in A$, and $\varphi_i(U \cap U_i)$ is an open fuzzy subset in E_i.

Proof: Suppose U is an open fuzzy subset in the induced fuzzy topology on M and (U_i, φ_i) be any fuzzy Banach chart in A. Then $U \cap U_i$ is an open fuzzy subset in the induced fuzzy topology, since φ_i is a fuzzy open map of U_i onto $\varphi_i(U_i)$, $\varphi_i(U \cap U_i)$ is an open fuzzy subset in E_i.

On the other hand, suppose U is a subset of M such that $\varphi_i(U \cap U_i)$ is an open fuzzy subset in E_i for every fuzzy Banach chart $(U_i, \varphi_i) \in A$. Since the domain of the charts cover M, we have $U = \bigcup_{i\in I}(U \cap U_i)$

For every fuzzy Banach chart (U_i, φ_i), $U \cap U_i \subset U_i$ and by hypothesis $\varphi_i(U \cap U_i)$ is an open fuzzy subset in E_i.

Hence by proposition 4.1 it follows that $U \cap U_i$ is an open fuzzy set in the induced fuzzy topology on M. U being union of open fuzzy subsets is open fuzzy subset.

5. DIFFERENTIABLE FUZZY BANACH MANIFOLD

Let us define equivalence relation $\sim$ on fuzzy Banach manifold.

Let $A^k(M)$ denote the set of all fuzzy Banach atlases of class C^k. We say that two charts (U_i, φ_i), $(U_j, \varphi_j) \in A$ are compatible if the fuzzy transition maps $\varphi_i \circ {\varphi_j}^{-1}$ and $\varphi_j \circ {\varphi_i}^{-1}$ for $i, j \in I$ are of class C^k. Also if $A_1, A_2 \in A^k(M)$ generates an equivalence relation on fuzzy Banach manifold M, if $(U_i, \varphi_i) \in A_1 \cup A_2$, then either $(U_i, \varphi_i) \in A_1$ or A_2. Since the fuzzy Banach charts of A_1 or A_2 are compatible the charts of $A_1 \cup A_2$ are also compatible. Hence the fuzzy Banach charts of $A^k(M)$ are compatible inducing global equivalence relation leading to differentiable structure on fuzzy Banach manifold.

The fuzzy Banach manifold with such equivalence relation is called as a differentiable fuzzy Banach manifold.

Thus a non-empty set M endowed with fuzzy topological structure of fuzzy norm and differentiable structure, induces a new space called differentiable fuzzy Banach manifold. On such differentiable fuzzy Banach manifold we study some fuzzy topological, geometrical and analytical structure in our future work.

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