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# GROWTH OF GENERALIZED ITERATED ENTIRE FUNCTIONS 

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#### Abstract

We consider generalized iteration of two entire functions of ( $p, q$ )-order and study some growth properties of generalized iterated entire functions to improve some earlier results.


Key words: Entire functions, (p, q) order, Generalized iteration.
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## 1. INTRODUCTION AND DEFINITION

It is well known that for any two transcendental entire functions $f(z)$ and $g(z), \lim _{r \rightarrow \infty} \frac{M(r, f \circ g)}{M(r, f)}=\infty$. In a paper [5] Clunie proved that the same is also true when maximum modulus functions are replaced by their characteristic functions. Singh [9] proved some results dealing with the ratios of $\log T(r, f \circ g)$ and $T(r, f)$ under some restrictions on the orders of $f$ and $g$. In a recent paper [2] Banerjee and Mondal generalize the results of A. P. Singh [9] for iterated entire functions imposing some restrictions on (p, q)-orders and lower (p, q)-orders of $f$ and $g$. In the present paper we extend the results of Banerjee and Mondal for generalized iterated entire functions under some restrictions on (p, q)-orders and lower (p, q)-orders of $f$ and $g$. Following Sato [8], we write $\log ^{[0]} x=x, \exp ^{[0]} x=x$ and for positive integer $m, \log ^{[m]} x=\log \left(\log ^{[m-1]} x\right), \exp ^{[m]} x=\exp \left(\exp ^{[m-1]} x\right)$.

Let $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ be an entire function. Then the (p,q)-order and lower (p,q)-order of $f(z)$ are denoted by $\rho_{(p, q)}(f)$ and $\lambda_{(p, q)}(f)$ respectively and defined by [4]

$$
\rho_{(p, q)}(f)=\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p]} T(r, f)}{\log ^{[q]} r} \text { and } \lambda_{(p, q)}(f)=\liminf _{r \rightarrow \infty} \frac{\log ^{[p]} T(r, f)}{\log ^{[q]} r}, p \geq q \geq 1 .
$$

According to Lahiri and Banerjee [7] if $f(z)$ and $g(z)$ be entire functions then the iteration of $f$ with respect to $g$ is defined as follows:
$f_{1}(z)=f(z)$
$f_{2}(z)=f(g(z))=f\left(g_{1}(z)\right)$
$f_{3}(z)=f(g(f(z)))=f\left(g_{2}(z)\right)$
$f_{4}(z)=f(g(f(g(z))))=f\left(g_{3}(z)\right)$
$f_{n}(z)=f\left(g(f(g(\ldots(f(z)\right.$ or $g(z)$ according as $n$ is odd or even $))))$, and so are $g_{n}(z)$.
Clearly all $f_{n}(z)$ and $g_{n}(z)$ are entire functions.

In paper [1] Banerjee and Mondal introduced a more general type of iteration, called generalized iteration as follows:
Let $f$ and $g$ be two non-constant entire functions and $\alpha$ be any real number satisfying $0<\alpha \leq 1$. Then the generalized iteration of $f$ with respect to $g$ is defined as follows:
$f_{1, g}(z)=(1-\alpha) z+\alpha f(z)$
$f_{2, g}(z)=(1-\alpha) g_{1, f}(z)+\alpha f\left(g_{1, f}(z)\right)$
$f_{3, g}(z)=(1-\alpha) g_{2, f}(z)+\alpha f\left(g_{2, f}(z)\right)$
$f_{n, g}(\mathrm{z})=(1-\alpha) g_{n-1, f}(\mathrm{z})+\alpha f\left(g_{n-1, f}(\mathrm{z})\right)$
and so are
$\mathbf{g}_{1, f}(z)=(1-\alpha) z+\alpha g(z)$
$g_{2, f}(z)=(1-\alpha) f_{1, g}(z)+\alpha g\left(f_{1, g}(z)\right)$
$g_{3, f}(z)=(1-\alpha) f_{2, g}(z)+\alpha g\left(f_{2, g}(z)\right)$
$\mathrm{g}_{n, f}(\mathrm{z})=(1-\alpha) f_{n-1, g}(\mathrm{z})+\alpha g\left(f_{n-1, g}(\mathrm{z})\right)$.

Note-1: For $\alpha=1$, generalized iteration reduces to relative iteration.
Definition 1[1]: We say a real valued function $\phi(r)$ is said to have the property P if
(i) $\phi(r)$ is non-negative and continuous for $r \geq r_{0}$, say;
(ii) $\phi(r)$ is strictly increasing and $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$; and
(iii) $\phi(r)<e^{\delta\left[\phi\left(\frac{r}{2}\right)\right]^{2}}$
hold for all $\lambda, \delta>0$ and for all sufficiently large values of $r$.
The purpose of this paper is to compare the characteristic function of generalized iterated entire functions with that of the generating functions. Throughout we assume $f$ and $g$ are non constant entire functions having finite (p, q)-orders.

## 2. LEMMAS

Following two lemmas will be needed during the proof of our theorems.
Lemma 1[6]: If $f(z)$ be regular in $|z| \leq R$, then for $0 \leq r<R$

$$
T(r, f) \leq \log ^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f)
$$

In particular if $f$ be entire, then for all large values of $r$

$$
T(r, f) \leq \log M(r, f) \leq 3 T(2 r, f)
$$

Lemma 2[3]: If $f$ is meromorphic and $g$ is entire then for all large values of $r$

$$
T(r, f \circ g) \leq(1+o(1)) \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)
$$

Since $g$ is entire so using Lemma 1, we have

$$
T(r, f \circ g) \leq(1+o(1)) T(M(r, g), f)
$$

## 3. MAIN RESULTS

Theorem 1: Let $f(z)$ and $g(z)$ be two entire functions with $\rho_{(p, q)}(g)<\lambda_{(p, q)}(f)$. Then for even $n$

$$
\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(2^{n-2} r, f\right)} \leq \rho_{(p, q)}(f)
$$

Proof: We have

$$
\begin{aligned}
T\left(r, f_{n, g}\right) & =T\left(r,(1-\alpha) g_{n-1, f}+\alpha f\left(g_{n-1, f}\right)\right) \\
& \leq T\left(r,(1-\alpha) g_{n-1, f}\right)+T\left(r, \alpha f\left(g_{n-1, f}\right)\right)+\log 2 \\
& \leq T\left(r, g_{n-1, f}\right)+T\left(r, f\left(g_{n-1, f}\right)\right)+O(1) \\
& \leq T\left(r, g_{n-1, f}\right)+(1+o(1)) T\left(M\left(r, g_{n-1, f}\right), f\right)+O(1), \text { by Lemma } 2
\end{aligned}
$$

or, $\log ^{[p]} T\left(r, f_{n, g}\right) \leq \log ^{[p]} T\left(r, g_{n-1, f}\right)+\log ^{[p]} T\left(M\left(r, g_{n-1, f}\right), f\right)+O(1)$

$$
\begin{align*}
& <\log ^{[p]} T\left(r, g_{n-1, f}\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q]} M\left(r, g_{n-1, f}\right)+O(1), \\
& \quad \text { for all large values of } r \text { and } \varepsilon>0 \\
& \leq \log ^{[q]} T\left(r, g_{n-1, f}\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q-1]}\left\{3 T\left(2 r, g_{n-1, f}\right)\right\}+O(1), \text { by Lemma1 } \\
& \leq \log ^{[q-1]} T\left(2 r, g_{n-1, f}\right)+3\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q-1]} T\left(2 r, g_{n-1, f}\right)+O(1) \\
& =\left[3\left(\rho_{(p, q)}(f)+\varepsilon\right)+1\right] \log ^{[q-1]} T\left(2 r, g_{n-1, f}\right)+O(1) \tag{1}
\end{align*}
$$

or, $\log ^{[p+(p+1-q)]} T\left(r, f_{n, g}\right)<\log ^{[p]} T\left(2 r, g_{n-1, f}\right)+O(1)$

$$
<\left[3\left(\rho_{(p, q)}(g)+\varepsilon\right)+1\right] \log ^{[q-1]} T\left(2^{2} r, f_{n-2, g}\right)+O(1), \text { using }(1)
$$

or, $\log ^{[p+2(p+1-q]]} T\left(r, f_{n, g}\right)<\log ^{[p]} T\left(2^{2} r, f_{n-2, g}\right)+O(1)$.
Proceeding similarly after some steps we get

$$
\begin{align*}
& \log ^{[p+(n-2)(p+1-q]} T\left(r, f_{n, g}\right)<\log ^{[p]} T\left(2^{n-2} r, f_{2, g}\right)+O(1) \\
& =\log ^{[p]} T\left(2^{n-2} r,(1-\alpha) g_{1, f}+\alpha f\left(g_{1, f}\right)\right)+O(1) \\
& \leq \log ^{[p]} T\left(2^{n-2} r, g_{1, f}\right)+\log ^{[p]} T\left(2^{n-2} r, f\left(g_{1, f}\right)\right)+O(1) \\
& \leq \log ^{[p]} T\left(2^{n-2} r, g_{1, f}\right)+\log ^{[p]} T\left(M\left(2^{n-2} r, g_{1, f}\right), f\right)+O(1) \\
& <\log ^{[p]} T\left(2^{n-2} r, g_{1, f}\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q]} M\left(2^{n-2} r, g_{1, f}\right)+O(1) \\
& =\log ^{[p]} T\left(2^{n-2} r,(1-\alpha) z+\alpha g(z)\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \\
& \times \log ^{[q]} M\left(2^{n-2} r,(1-\alpha) z+\alpha g(z)\right)+O(1) \\
& \leq \log ^{[p]} T\left(2^{n-2} r, z\right)+\log ^{[p]} T\left(2^{n-2} r, g\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \\
& \times \log ^{[q]} M\left(2^{n-2} r, z\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q]} M\left(2^{n-2} r, g\right)+O(1) \\
& <\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right) \\
& +\left(\rho_{(p, q)}(f)+\varepsilon\right) \exp ^{[p-q]}\left\{\log ^{[q-1]}\left(2^{n-2} r\right)\right\}^{\rho_{(p, q)}(g)+\varepsilon}+O(1)  \tag{2}\\
& <\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\left(\rho_{(p, q)}(f)+2 \varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)\right. \\
& +\left(\rho_{(p, q)}(f)+\varepsilon\right) \exp ^{[p-q]}\left\{\log ^{[q-1]}\left(2^{n-2} r\right)\right\}^{\lambda_{(p, q)}(f)-\varepsilon}+O(1)
\end{align*}
$$

by choosing $\varepsilon>0$ so small that $\rho_{(p, q)}(g)+\varepsilon<\lambda_{(p, q)}(f)-\varepsilon$.

On the other hand
$T(r, f)>\exp ^{[p-1]}\left(\log ^{[q-1]} r\right)^{\lambda_{p(p)}(f) r}$, for all $r \geq r_{0}$.
or, $\log ^{[q-1]} T(r, f)>\exp ^{[p-q]}\left(\log ^{[q-1]} r\right)^{\lambda_{p, q, q}(f) \varepsilon}$, for all $r \geq r_{0}$.

Therefore, from above
$\frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(2^{n-2} r, f\right)}$
$<\frac{\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\left(\rho_{(p, q)}(f)+2 \varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \exp ^{[p-q]}\left\{\log ^{[q-1]}\left(2^{n-2} r\right)\right\}^{\lambda_{p, q}(p)-\varepsilon}+O(1)\right.}{\exp ^{[p-q]}\left(\log ^{[q-1]}\left(2^{n-2} r\right)\right)^{\lambda_{p, q, q}(f)-\varepsilon}}$,
for all $r \geq r_{0}$.
Hence, $\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(2^{n-2} r, f\right)} \leq \rho_{(p, q)}(f)+\varepsilon$.
The theorem now follows since $\varepsilon(>0)$ is arbitrary.
Note-2: From the hypothesis it is clear that $f$ must be transcendental.

Theorem 2: Let $f$ and $g$ be two entire functions with $\rho_{(p, q)}(f)<\lambda_{(p, q)}(g)$. Then for odd $n$

$$
\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(2^{n-2} r, g\right)} \leq \rho_{(p, q)}(g) .
$$

The proof of the theorem is on the same line as that of Theorem 1.
Theorem 3: Let $f(z)$ and $g(z)$ be two transcendental entire functions such that $\lambda_{(p, q)}(g)>0$. Then for even $n$

$$
\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p+(n-1)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(2^{n-2} r, g\right)} \leq \frac{\rho_{(p, q)}(g)}{\lambda_{(p, q)}(g)} .
$$

Proof: From (2) we have for all $r \geq r_{0}$

$$
\begin{aligned}
& \log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)< \log ^{[p+1]}\left(2^{n-2} r\right)+ \\
&\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right) \\
&+\left(\rho_{(p, q)}(f)+\varepsilon\right) \exp ^{[p-q]}\left\{\log ^{[q-1]}\left(2^{n-2} r\right)\right\}^{\rho_{(p, q)}(g)+\varepsilon}+O(1) \\
&=\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\rho_{(p, q)}(f)+2 \varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(f)+\varepsilon\right) \\
& \times \exp ^{[p-q]}\left\{\log ^{[q-1]}\left(2^{n-2} r\right)\right\}^{\rho_{(p, q)}(g)+\varepsilon}+O(1)
\end{aligned}
$$

or, $\log ^{[p+(n-1)(p+1-q)]} T\left(r, f_{n, g}\right)<\log ^{[2 p+2-q]}\left(2^{n-2} r\right)+\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+O(1)$.
On the other hand,
$\log ^{[p]} T(r, g)>\left(\lambda_{(p, q)}(g)-\varepsilon\right) \log ^{[q]} r$, for all $r \geq r_{0}$.

Therefore,

$$
\frac{\log ^{[p+(n-1)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(2^{n-2} r, g\right)}<\frac{\log ^{[2 p+2-q]}\left(2^{n-2} r\right)+\log ^{[p+1]}\left(2^{n-2} r\right)+\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)+O(1)}{\left(\lambda_{(p, q)}(g)-\varepsilon\right) \log ^{[q]}\left(2^{n-2} r\right)} .
$$

Thus $\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p+(n-1)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(2^{n-2} r, g\right)} \leq \frac{\rho_{(p, q)}(g)}{\lambda_{(p, q)}(g)}$, since $\varepsilon>0$ is arbitrary small.

Theorem 4: Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\lambda_{(p, q)}(f)>0$. Then for odd $n$

$$
\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-1)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(2^{n-2} r, f\right)} \leq \frac{\rho_{(p, q)}(f)}{\lambda_{(p, q)}(f)}
$$

The proof is omitted.
Note-3: If $\rho_{(p, q)}(g)>\rho_{(p, q)}(f)$ holds in Theorem 1 we shall show that the limit superior will tend to infinity.
Now we prove the following four theorems where we assume that the maximum modulus functions of $f, g$ and all of their generalized functions satisfy property P .

Theorem 5: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders with $\rho_{(p, q)}(g)>\rho_{(p, q)}(f)$. Then for even $n$

$$
\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(\frac{r}{4^{n-1}}, f\right)}=\infty
$$

Proof: We have,

$$
\begin{aligned}
T\left(r, f_{n, g}\right) & =T\left(r,(1-\alpha) g_{n-1, f}+\alpha f\left(g_{n-1, f}\right)\right) \\
& \geq T\left(r, \alpha f\left(g_{n-1, f}\right)\right)-T\left(r,(1-\alpha) g_{n-1, f}\right)+O(1) \\
& \geq T\left(r, f\left(g_{n-1, f}\right)\right)-T\left(r, g_{n-1, f}\right)+O(1) \\
& >\frac{1}{3} \exp ^{[p-1]}\left\{\log ^{[q-1]} M\left(\frac{r}{4}, g_{n-1, f}\right)\right\}^{\lambda_{(p, q)}(f)-\varepsilon}-T\left(r, g_{n-1, f}\right)+O(1) . \quad \text { \{see[9], page100\} }
\end{aligned}
$$

Therefore,

$$
\begin{align*}
\log ^{[p]} T\left(r, f_{n, g}\right) & >\log \left\{\log ^{[q-1]} M\left(\frac{r}{4}, g_{n-1, f}\right)\right\}^{\lambda_{(p, q)}(f)-\varepsilon}-\log ^{[p]} T\left(r, g_{n-1, f}\right)+O(1) \\
& =\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4}, g_{n-1, f}\right)-\log ^{[p]} T\left(r, g_{n-1, f}\right)+O(1) \\
& \geq\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4}, g_{n-1, f}\right)-\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4}, g_{n-1, f}\right)+O(1) \\
& =\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4}, g_{n-1, f}\right)+O(1) \tag{3}
\end{align*}
$$

So, we have for all $r \geq r_{0}$,

$$
\begin{aligned}
\log ^{[p+(p+1-q)]} T\left(r, f_{n, g}\right) & >\log ^{[p]}\left[\log M\left(\frac{r}{4}, g_{n-1, f}\right)\right]+O(1) \\
& \geq \log ^{[p]} T\left(\frac{r}{4}, g_{n-1, f}\right)+O(1) \\
& >\frac{1}{2}\left(\lambda_{(p, q)}(g)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4^{2}}, f_{n-2, g}\right)+O(1), \text { using (3) }
\end{aligned}
$$

or, $\log ^{[p+2(p+1-q)]} T\left(r, f_{n, g}\right)>\log ^{[p]} T\left(\frac{r}{4^{2}}, f_{n-2, g}\right)+O(1)$

$$
>\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4^{3}}, g_{n-3, f}\right)+O(1), \text { using (3) }
$$

Proceeding similarly after some steps we get
$\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)>\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4^{n-1}}, g_{1, f}\right)+O(1)$

$$
\begin{align*}
& =\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right) \log ^{[q]} M\left(\frac{r}{4^{n-1}},(1-\alpha) z+\alpha g(z)\right)+O(1) \\
& \geq \frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right)\left[\log ^{[q]} M\left(\frac{r}{4^{n-1}}, g\right)-\log ^{[q]} M\left(\frac{r}{4^{n-1}}, z\right)\right]+O(1) \\
& >\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right)\left[\exp ^{[p-q]}\left\{\log ^{[q-1]}\left(\frac{r}{4^{n-1}}\right)\right\}^{\rho_{(p q)}(g)-\varepsilon}-\log ^{[q]}\left(\frac{r}{4^{n-1}}\right)\right]+O(1) \tag{4}
\end{align*}
$$

for a sequence of values of $r \rightarrow \infty$.
On the other hand for all $r \geq r_{0}$ we have,
$T(r, f)<\exp ^{[p-1]}\left(\log ^{[q-1]} r\right)^{\rho_{p, p,(f)+\varepsilon}}$
or, $\log ^{[q-1]} T(r, f)<\exp ^{[p-q]}\left(\log ^{[q-1]} r\right)^{\rho_{p q}(\theta)+\varepsilon}$.
So, from (4) and (5) we have for a sequence of values of $r \rightarrow \infty$,

$$
\frac{\log ^{[p+(n-2)(p+1-q]]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(\frac{r}{4^{n-1}}, f\right)}>\frac{\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right)\left[\exp ^{[p-q]}\left\{\log ^{[q-1]}\left(\frac{r}{4^{n-1}}\right)\right\}^{\rho_{(p, q)}(q)-\varepsilon}-\log ^{[q]}\left(\frac{r}{4^{n-1}}\right)\right]}{\exp ^{[p-q]}\left(\log ^{[q-1]} \frac{r}{4^{n-1}}\right)^{\rho_{(p, q)}(f)+\varepsilon}}+o(1)
$$

and so $\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(\frac{r}{4^{n-1}}, f\right)}=\infty$,
since we can choose $\varepsilon(>0)$ such that $\rho_{(p, q)}(g)-\varepsilon>\rho_{(p, q)}(f)+\varepsilon$.
This proves the theorem.
An immediate consequence of Theorem 5 for odd $n$ is the following theorem.
Theorem 6: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders with $\rho_{(p, q)}(f)>\rho_{(p, q)}(g)$. Then for odd $n$

$$
\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[q-1]} T\left(\frac{r}{4^{n-1}}, g\right)}=\infty .
$$

Theorem 7: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders. Then for even $n$

$$
\limsup _{r \rightarrow \infty} \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(\frac{r}{4^{n-1}}, g\right)}=\infty
$$

Proof: From (4), we have for a sequence of values of $r \rightarrow \infty$
$\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)>\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right)\left[\exp ^{[p-q]}\left\{\log ^{[q-1]}\left(\frac{r}{4^{n-1}}\right)\right\}^{\rho_{(p, q)}(g)-\varepsilon}-\log ^{[q]}\left(\frac{r}{4^{n-1}}\right)+O(1)\right.$.
Also, $\log ^{[p]} T(r, g)<\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]} r$, for all $r \geq r_{0}$.
Thus for a sequence of values of $r \rightarrow \infty$
$\frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(\frac{r}{4^{n-1}}, g\right)} \geq \frac{\frac{1}{2}\left(\lambda_{(p, q)}(f)-\varepsilon\right)\left[\exp ^{[p-q]}\left\{\log ^{[q-1]}\left(\frac{r}{4^{n-1}}\right)\right\}^{\rho_{\rho(q, q)}(g)-\varepsilon}-\log ^{[q]}\left(\frac{r}{4^{n-1}}\right)+O(1)\right.}{\left(\rho_{(p, q)}(g)+\varepsilon\right) \log ^{[q]}\left(\frac{r}{4^{n-1}}\right)}$
which tends to infinity as $r \rightarrow \infty$, through this sequence since $\rho_{(p, q)}(g)>0$.

Theorem 8: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p,q)-orders. Then for odd $n$

$$
\underset{r \rightarrow \infty}{\limsup } \frac{\log ^{[p+(n-2)(p+1-q)]} T\left(r, f_{n, g}\right)}{\log ^{[p]} T\left(\frac{r}{4^{n-1}}, f\right)}=\infty .
$$

The proof is omitted.
Note-4: If we put $\alpha=1$ in the above Theorems we get the results of Banerjee and Mondal [2].

## REFERENCES

1. Banerjee D. and Mondal N., Maximum modulus and maximum term of generalized iterated entire functions, Bulletin of the Allahabad Mathematical Society, Vol. 27, No. 1 (2012), p.p.117-131.
2. Banerjee D. and Mondal N., On the growth of iterated entire functions, International Journal of Analysis and Applications, Vol. 4, No 1 (2014), p.p. 21-25.
3. Bergweiler W., On the Nevanlinna Characteristic of a composite function, Complex Variables, Vol. 10 (1988), 225-236.
4. Bergweiler W., Gerhard Jank and Lutz Volkmann, Wachstumsverhalten zusam-mengesetzter Funktionen, Results Math., Vol. 7 (1984), p.p. 35-53.
5. Clunie J., The composition of entire and meromorphic functions, Mathematical Essays dedicated to A. J. Macintyre, Ohio University Press, 1970, p.p. 75-92.
6. Hayman W. K., Meromorphic Functions, Oxford University Press, 1964.
7. Lahiri B. K. and Banerjee D., Relative fix points of entire functions, J. Indian Acad. Math., Vol. 19, No. 1 (1997), p.p.87-97.
8. Sato D., On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., Vol. 69 (1963), p.p. 411-414.
9. Singh A. P., Growth of composite entire functions, Kodai Math. J., Vol. 8 (1985), p.p.99-102.

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