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GROWTH OF GENERALIZED ITERATED ENTIRE FUNCTIONS

Dibyendu Banerjee*[1] and Nilkanta Mondal[2]

[1]Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India.

[2]Ballavpur R.G.S. Vidyapith (H.S.), Ballavpur, Raniganj, Burdwan, West Bengal, India.

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ABSTRACT

We consider generalized iteration of two entire functions of (p, q)-order and study some growth properties of generalized iterated entire functions to improve some earlier results.

Key words: *Entire functions, (p, q) order, Generalized iteration.*

AMS Subject Classification: *30D15.*

1. INTRODUCTION AND DEFINITION

It is well known that for any two transcendental entire functions $f(z)$ and $g(z)$, $\lim\limits_{r\to\infty}\dfrac{M(r, f\circ g)}{M(r, f)} = \infty$. In a paper [5] Clunie proved that the same is also true when maximum modulus functions are replaced by their characteristic functions. Singh [9] proved some results dealing with the ratios of $\log T(r, f\circ g)$ and $T(r, f)$ under some restrictions on the orders of f and g. In a recent paper [2] Banerjee and Mondal generalize the results of A. P. Singh [9] for iterated entire functions imposing some restrictions on (p, q)-orders and lower (p, q)-orders of f and g. In the present paper we extend the results of Banerjee and Mondal for generalized iterated entire functions under some restrictions on (p, q)-orders and lower (p, q)-orders of f and g. Following Sato [8], we write $\log^{[0]} x = x, \exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x), \exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then the (p, q)-order and lower (p, q)-order of $f(z)$ are denoted by $\rho_{(p,q)}(f)$ and $\lambda_{(p,q)}(f)$ respectively and defined by [4]

$$\rho_{(p,q)}(f) = \limsup_{r\to\infty}\frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_{(p,q)}(f) = \liminf_{r\to\infty}\frac{\log^{[p]} T(r,f)}{\log^{[q]} r},\ p \ge q \ge 1.$$

According to Lahiri and Banerjee [7] if $f(z)$ and $g(z)$ be entire functions then the iteration of f with respect to g is defined as follows:

$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f(g_1(z))$$
$$f_3(z) = f(g(f(z))) = f(g_2(z))$$
$$f_4(z) = f(g(f(g(z)))) = f(g_3(z))$$
$$\ldots\ldots\quad \ldots\ldots\quad \ldots\ldots$$
$$f_n(z) = f(g(f(g(\ldots(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even})))), \text{ and so are } g_n(z).$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

Corresponding Author: Dibyendu Banerjee[1]*

In paper [1] Banerjee and Mondal introduced a more general type of iteration, called generalized iteration as follows:

Let f and g be two non-constant entire functions and α be any real number satisfying $0 < \alpha \leq 1$. Then the generalized iteration of f with respect to g is defined as follows:

$$
\begin{aligned}
& f_{1,g}(z) = (1-\alpha)z + \alpha f(z) \\
& f_{2,g}(z) = (1-\alpha) g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\
& f_{3,g}(z) = (1-\alpha) g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\
& \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
& f_{n,g}(z) = (1-\alpha) g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z))
\end{aligned}
$$

and so are

$$
\begin{aligned}
& g_{1,f}(z) = (1-\alpha)z + \alpha g(z) \\
& g_{2,f}(z) = (1-\alpha) f_{1,g}(z) + \alpha g(f_{1,g}(z)) \\
& g_{3,f}(z) = (1-\alpha) f_{2,g}(z) + \alpha g(f_{2,g}(z)) \\
& \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \\
& g_{n,f}(z) = (1-\alpha) f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)).
\end{aligned}
$$

Note-1: For $\alpha = 1$, generalized iteration reduces to relative iteration.

Definition 1[1]: We say a real valued function $\phi(r)$ is said to have the property P if

(i) $\phi(r)$ is non-negative and continuous for $r \geq r_0$, say;

(ii) $\phi(r)$ is strictly increasing and $\phi(r) \to \infty$ as $r \to \infty$; and

(iii) $\phi(r) < e^{\delta[\phi(\frac{r}{2})]^{\lambda}}$

hold for all $\lambda, \delta > 0$ and for all sufficiently large values of r.

The purpose of this paper is to compare the characteristic function of generalized iterated entire functions with that of the generating functions. Throughout we assume f and g are non constant entire functions having finite (p, q)-orders.

2. LEMMAS

Following two lemmas will be needed during the proof of our theorems.

Lemma 1[6]: If $f(z)$ be regular in $|z| \leq R$, then for $0 \leq r < R$

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

In particular if f be entire, then for all large values of r

$$T(r, f) \leq \log M(r, f) \leq 3T(2r, f).$$

Lemma 2[3]: If f is meromorphic and g is entire then for all large values of r

$$T(r, f \circ g) \leq (1 + o(1)) \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Since g is entire so using Lemma 1, we have

$$T(r, f \circ g) \leq (1 + o(1)) T(M(r, g), f).$$

3. MAIN RESULTS

Theorem 1: Let $f(z)$ and $g(z)$ be two entire functions with $\rho_{(p,q)}(g) < \lambda_{(p,q)}(f)$. Then for even n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(2^{n-2} r, f)} \le \rho_{(p,q)}(f).$$

Proof: We have

$$\begin{aligned} T(r, f_{n,g}) &= T(r,(1-\alpha)g_{n-1,f} + \alpha f(g_{n-1,f}))\\ &\le T(r,(1-\alpha)g_{n-1,f}) + T(r,\alpha f(g_{n-1,f})) + \log 2\\ &\le T(r, g_{n-1,f}) + T(r, f(g_{n-1,f})) + O(1)\\ &\le T(r, g_{n-1,f}) + (1+o(1))T(M(r, g_{n-1,f}), f) + O(1), \quad \text{by Lemma 2} \end{aligned}$$

$$\begin{aligned} \text{or, } \log^{[p]} T(r,f_{n,g}) &\le \log^{[p]} T(r,g_{n-1,f}) + \log^{[p]} T(M(r,g_{n-1,f}),f) + O(1)\\ &< \log^{[p]} T(r,g_{n-1,f}) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q]} M(r,g_{n-1,f}) + O(1),\\ &\qquad \text{for all large values of } r \text{ and } \varepsilon > 0\\ &\le \log^{[q]} T(r,g_{n-1,f}) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q-1]}\{3T(2r,g_{n-1,f})\} + O(1), \text{by Lemma 1}\\ &\le \log^{[q-1]} T(2r,g_{n-1,f}) + 3(\rho_{(p,q)}(f)+\varepsilon)\log^{[q-1]} T(2r,g_{n-1,f}) + O(1)\\ &= [3(\rho_{(p,q)}(f)+\varepsilon)+1]\log^{[q-1]} T(2r, g_{n-1,f}) + O(1) \end{aligned} \tag{1}$$

$$\begin{aligned} \text{or, } \log^{[p+(p+1-q)]} T(r,f_{n,g}) &< \log^{[p]} T(2r,g_{n-1,f}) + O(1)\\ &< [3(\rho_{(p,q)}(g)+\varepsilon)+1]\log^{[q-1]} T(2^2 r,f_{n-2,g}) + O(1), \text{ using (1)} \end{aligned}$$

$$\text{or, } \log^{[p+2(p+1-q)]} T(r, f_{n,g}) < \log^{[p]} T(2^2 r,f_{n-2,g}) + O(1).$$

Proceeding similarly after some steps we get

$$\begin{aligned} \log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g}) &< \log^{[p]} T(2^{n-2} r,f_{2,g}) + O(1)\\ &= \log^{[p]} T(2^{n-2} r,(1-\alpha)g_{1,f} + \alpha f(g_{1,f})) + O(1)\\ &\le \log^{[p]} T(2^{n-2} r,g_{1,f}) + \log^{[p]} T(2^{n-2} r,f(g_{1,f})) + O(1)\\ &\le \log^{[p]} T(2^{n-2} r,g_{1,f}) + \log^{[p]} T(M(2^{n-2} r,g_{1,f}),f) + O(1)\\ &< \log^{[p]} T(2^{n-2} r,g_{1,f}) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q]} M(2^{n-2} r,g_{1,f}) + O(1)\\ &= \log^{[p]} T(2^{n-2} r,(1-\alpha)z + \alpha g(z)) + (\rho_{(p,q)}(f)+\varepsilon)\\ &\qquad \times \log^{[q]} M(2^{n-2} r,(1-\alpha)z + \alpha g(z)) + O(1)\\ &\le \log^{[p]} T(2^{n-2} r,z) + \log^{[p]} T(2^{n-2} r,g) + (\rho_{(p,q)}(f)+\varepsilon)\\ &\qquad \times \log^{[q]} M(2^{n-2} r,z) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q]} M(2^{n-2} r,g) + O(1)\\ &< \log^{[p+1]}(2^{n-2} r) + (\rho_{(p,q)}(g)+\varepsilon)\log^{[q]}(2^{n-2} r) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q]}(2^{n-2} r)\\ &\qquad + (\rho_{(p,q)}(f)+\varepsilon)\exp^{[p-q]}\{\log^{[q-1]}(2^{n-2} r)\}^{\rho_{(p,q)}(g)+\varepsilon} + O(1) \end{aligned} \tag{2}$$

$$\begin{aligned} &< \log^{[p+1]}(2^{n-2} r) + (\rho_{(p,q)}(g) + (\rho_{(p,q)}(f)+2\varepsilon)\log^{[q]}(2^{n-2} r)\\ &\qquad + (\rho_{(p,q)}(f)+\varepsilon)\exp^{[p-q]}\{\log^{[q-1]}(2^{n-2} r)\}^{\lambda_{(p,q)}(f)-\varepsilon} + O(1) \end{aligned}$$

by choosing $\varepsilon > 0$ so small that $\rho_{(p,q)}(g)+\varepsilon < \lambda_{(p,q)}(f) - \varepsilon$.

On the other hand

$$T(r,f) > \exp^{[p-1]}(\log^{[q-1]} r)^{\lambda_{(p,q)}(f)-\varepsilon}, \text{ for all } r \ge r_0.$$

$$\text{or, } \log^{[q-1]} T(r,f) > \exp^{[p-q]}(\log^{[q-1]} r)^{\lambda_{(p,q)}(f)-\varepsilon}, \text{ for all } r \ge r_0.$$

Therefore, from above

$$\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(2^{n-2}r, f)}$$

$$< \frac{\log^{[p+1]}(2^{n-2}r) + (\rho_{(p,q)}(g) + (\rho_{(p,q)}(f) + 2\varepsilon)\log^{[q]}(2^{n-2}r) + (\rho_{(p,q)}(f)+\varepsilon)\exp^{[p-q]}\{\log^{[q-1]}(2^{n-2}r)\}^{\lambda_{(p,q)}(f)-\varepsilon} + O(1)}{\exp^{[p-q]}(\log^{[q-1]}(2^{n-2}r))^{\lambda_{(p,q)}(f)-\varepsilon}},$$

for all $r \ge r_0$.

$$\text{Hence, } \limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(2^{n-2}r, f)} \le \rho_{(p,q)}(f) + \varepsilon.$$

The theorem now follows since $\varepsilon(>0)$ is arbitrary.

Note-2: From the hypothesis it is clear that f must be transcendental.

Theorem 2: Let f and g be two entire functions with $\rho_{(p,q)}(f) < \lambda_{(p,q)}(g)$. Then for odd n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(2^{n-2}r, g)} \le \rho_{(p,q)}(g).$$

The proof of the theorem is on the same line as that of Theorem 1.

Theorem 3: Let $f(z)$ and $g(z)$ be two transcendental entire functions such that $\lambda_{(p,q)}(g) > 0.$ Then for even n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-1)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(2^{n-2}r, g)} \le \frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(g)}.$$

Proof: From (2) we have for all $r \ge r_0$

$$\begin{aligned}\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g}) &< \log^{[p+1]}(2^{n-2}r) + (\rho_{(p,q)}(g)+\varepsilon)\log^{[q]}(2^{n-2}r) + (\rho_{(p,q)}(f)+\varepsilon)\log^{[q]}(2^{n-2}r)\\ &\quad + (\rho_{(p,q)}(f)+\varepsilon)\exp^{[p-q]}\{\log^{[q-1]}(2^{n-2}r)\}^{\rho_{(p,q)}(g)+\varepsilon} + O(1)\\ &= \log^{[p+1]}(2^{n-2}r) + (\rho_{(p,q)}(g) + \rho_{(p,q)}(f) + 2\varepsilon)\log^{[q]}(2^{n-2}r) + (\rho_{(p,q)}(f)+\varepsilon)\\ &\quad \times \exp^{[p-q]}\{\log^{[q-1]}(2^{n-2}r)\}^{\rho_{(p,q)}(g)+\varepsilon} + O(1)\end{aligned}$$

$$\text{or, } \log^{[p+(n-1)(p+1-q)]} T(r,f_{n,g}) < \log^{[2p+2-q]}(2^{n-2}r) + \log^{[p+1]}(2^{n-2}r) + (\rho_{(p,q)}(g)+\varepsilon)\log^{[q]}(2^{n-2}r) + O(1).$$

On the other hand,

$$\log^{[p]} T(r,g) > (\lambda_{(p,q)}(g) - \varepsilon)\log^{[q]} r, \text{ for all } r \ge r_0.$$

Therefore,

$$\frac{\log^{[p+(n-1)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(2^{n-2}r, g)} < \frac{\log^{[2p+2-q]}(2^{n-2}r) + \log^{[p+1]}(2^{n-2}r) + (\rho_{(p,q)}(g)+\varepsilon)\log^{[q]}(2^{n-2}r) + O(1)}{(\lambda_{(p,q)}(g)-\varepsilon)\log^{[q]}(2^{n-2}r)}.$$

$$\text{Thus } \limsup_{r\to\infty} \frac{\log^{[p+(n-1)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(2^{n-2}r, g)} \le \frac{\rho_{(p,q)}(g)}{\lambda_{(p,q)}(g)}, \text{ since } \varepsilon > 0 \text{ is arbitrary small.}$$

Theorem 4: Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\lambda_{(p,q)}(f) > 0$. Then for odd n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-1)(p+1-q)]} T(r, f_{n,g})}{\log^{[p]} T(2^{n-2} r, f)} \le \frac{\rho_{(p,q)}(f)}{\lambda_{(p,q)}(f)}.$$

The proof is omitted.

Note-3: If $\rho_{(p,q)}(g) > \rho_{(p,q)}(f)$ holds in Theorem 1 we shall show that the limit superior will tend to infinity.

Now we prove the following four theorems where we assume that the maximum modulus functions of f, g and all of their generalized functions satisfy property P.

Theorem 5: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders with $\rho_{(p,q)}(g) > \rho_{(p,q)}(f)$. Then for even n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r, f_{n,g})}{\log^{[q-1]} T(\frac{r}{4^{n-1}}, f)} = \infty.$$

Proof: We have,

$$\begin{aligned} T(r,f_{n,g}) &= T(r,(1-\alpha)g_{n-1,f} + \alpha f(g_{n-1,f})) \\ &\ge T(r,\alpha f(g_{n-1,f})) - T(r,(1-\alpha)g_{n-1,f}) + O(1) \\ &\ge T(r,f(g_{n-1,f})) - T(r,g_{n-1,f}) + O(1) \\ &> \frac{1}{3}\exp^{[p-1]}\{\log^{[q-1]} M(\frac{r}{4},g_{n-1,f})\}^{\lambda_{(p,q)}(f)-\varepsilon} - T(r,g_{n-1,f}) + O(1). \qquad \{\text{see[9], page100}\} \end{aligned}$$

Therefore,

$$\begin{aligned} \log^{[p]} T(r,f_{n,g}) &> \log\{\log^{[q-1]} M(\frac{r}{4},g_{n-1,f})\}^{\lambda_{(p,q)}(f)-\varepsilon} - \log^{[p]} T(r,g_{n-1,f}) + O(1) \\ &= (\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4},g_{n-1,f}) - \log^{[p]} T(r,g_{n-1,f}) + O(1) \\ &\ge (\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4},g_{n-1,f}) - \frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4},g_{n-1,f}) + O(1). \\ &= \frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M\left(\frac{r}{4},g_{n-1,f}\right) + O(1). \end{aligned} \tag{3}$$

So, we have for all $r \ge r_0$,

$$\begin{aligned} \log^{[p+(p+1-q)]} T(r,f_{n,g}) &> \log^{[p]}[\log M(\frac{r}{4},g_{n-1,f})] + O(1) \\ &\ge \log^{[p]} T(\frac{r}{4},g_{n-1,f}) + O(1) \\ &> \frac{1}{2}(\lambda_{(p,q)}(g)-\varepsilon)\log^{[q]} M(\frac{r}{4^2},f_{n-2,g}) + O(1), \text{ using (3)} \end{aligned}$$

$$\begin{aligned} \text{or, } \log^{[p+2(p+1-q)]} T(r,f_{n,g}) &> \log^{[p]} T(\frac{r}{4^2},f_{n-2,g}) + O(1) \\ &> \frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4^3},g_{n-3,f}) + O(1), \text{ using (3)} \end{aligned}$$

Proceeding similarly after some steps we get

$$\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g}) > \frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4^{n-1}},g_{1,f}) + O(1)$$

$$=\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)\log^{[q]} M(\frac{r}{4^{n-1}},(1-\alpha)z+\alpha g(z))+O(1)$$

$$\geq\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)[\log^{[q]} M(\frac{r}{4^{n-1}},g)-\log^{[q]} M(\frac{r}{4^{n-1}},z)]+O(1)$$

$$>\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\rho_{(p,q)}(g)-\varepsilon}-\log^{[q]}(\frac{r}{4^{n-1}})]+O(1) \quad (4)$$

for a sequence of values of $r\to\infty$.

On the other hand for all $r\geq r_0$ we have,

$T(r,f)<\exp^{[p-1]}(\log^{[q-1]} r)^{\rho_{(p,q)}(f)+\varepsilon}$

or, $\log^{[q-1]} T(r,f)<\exp^{[p-q]}(\log^{[q-1]} r)^{\rho_{(p,q)}(f)+\varepsilon}$. (5)

So, from (4) and (5) we have for a sequence of values of $r\to\infty$,

$$\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(\frac{r}{4^{n-1}},f)}>\frac{\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\rho_{(p,q)}(g)-\varepsilon}-\log^{[q]}(\frac{r}{4^{n-1}})]}{\exp^{[p-q]}(\log^{[q-1]}\frac{r}{4^{n-1}})^{\rho_{(p,q)}(f)+\varepsilon}}+o(1)$$

and so $\limsup\limits_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(\frac{r}{4^{n-1}},f)}=\infty,$

since we can choose $\varepsilon(>0)$ such that $\rho_{(p,q)}(g)-\varepsilon>\rho_{(p,q)}(f)+\varepsilon$.

This proves the theorem.

An immediate consequence of Theorem 5 for odd n is the following theorem.

Theorem 6: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders with $\rho_{(p,q)}(f)>\rho_{(p,q)}(g)$. Then for odd n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[q-1]} T(\frac{r}{4^{n-1}},g)}=\infty.$$

Theorem 7: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders. Then for even n

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(\frac{r}{4^{n-1}},g)}=\infty.$$

Proof: From (4), we have for a sequence of values of $r\to\infty$

$$\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})>\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\rho_{(p,q)}(g)-\varepsilon}-\log^{[q]}(\frac{r}{4^{n-1}})+O(1).$$

Also, $\log^{[p]} T(r,g)<(\rho_{(p,q)}(g)+\varepsilon)\log^{[q]} r$, for all $r\geq r_0$.

Thus for a sequence of values of $r\to\infty$

$$\frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(\frac{r}{4^{n-1}},g)}\geq\frac{\frac{1}{2}(\lambda_{(p,q)}(f)-\varepsilon)[\exp^{[p-q]}\{\log^{[q-1]}(\frac{r}{4^{n-1}})\}^{\rho_{(p,q)}(g)-\varepsilon}-\log^{[q]}(\frac{r}{4^{n-1}})+O(1)}{(\rho_{(p,q)}(g)+\varepsilon)\log^{[q]}(\frac{r}{4^{n-1}})}$$

which tends to infinity as $r\to\infty$, through this sequence since $\rho_{(p,q)}(g)>0$.

Theorem 8: Let $f(z)$ and $g(z)$ be two entire functions of positive lower (p, q)-orders. Then for odd n

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p+1-q)]} T(r,f_{n,g})}{\log^{[p]} T(\frac{r}{4^{n-1}}, f)} = \infty.$$

The proof is omitted.

Note-4: If we put $\alpha = 1$ in the above Theorems we get the results of Banerjee and Mondal [2].

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