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# SEMI SYMMETRIC NON-METRIC $s$-CONNECTION ON AN UNIFIED CONTACT RIEMANNIAN MANIFOLD 

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#### Abstract

In this paper, we define a semi-symmetric non-metric $S$-connection $\tilde{B}$ on an unified contact Riemannian manifold $M_{n}$ [6] and define the curvature tensor of $M_{n}$ with respect to semi-symmetric non-metric $S$-connection. We obtain a relation connecting the curvature tensors of $M_{n}$ with respect to semi-symmetric non-metric $S$-connection and Riemannian connection. Further we obtain the expression for curvature tensors associated with semi-symmetric nonmetric $S$-connection. Further it has been shown that if an unified contact Riemannian manifold admits a semisymmetric non-metric $S$-connection whose curvature tensor is locally isometric to the unit sphere $S^{n}(1)$, then the conformal and con-harmonic curvature tensor with respect to Riemannian connection are identical iff $n+\frac{a^{2}}{c}(n+2)=0$. Also it has been shown that if an unified contact Riemannian manifold admits a semi-symmetric non-metric $S$-connection whose curvature tensor is locally isometric to the unit sphere $S^{n}(1)$, then the con-circular curvature tensor coincides with the Riemannian connection if $n+\frac{a^{2}}{c}(n+2)=0$.


Some other useful results on projective curvature tensor $W$ and con-circular curvature tensor $C$ with respect to semisymmetric non-metric $S$-connection have been obtained.

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## 1. INTRODUCTION

We consider a differentiable manifold $M_{n}$ of differentiability class $C^{\infty}$. Let there exist in $M_{n}$ a vector valued $C^{\infty}$. linear function $\Phi$, a $C^{\infty}$ - vector field $\eta$ and a $C^{\infty}$-one form $\xi$ such that
$\Phi^{2}(X)=a^{2} X+c \xi(X) \eta$
$\bar{\eta}=0$
$\xi(\eta)=-\frac{a^{2}}{c}$
$\xi(\bar{X})=0$
Where $\Phi(X)=\bar{X}, a$ is a nonzero complex number and $c$ is an integer.

Let us agree to say that $\Phi$ gives to $M_{n}$ a differentiable structure define by algebraic equation (1.1). We shall call $(\Phi, \eta, a, c, \xi)$ as an unified contact structure. It may be noted that an unified contact structure $(\Phi, \eta, a, c, \xi)$ gives an almost contact structure [4], almost Para-contact structure [5] or hyperbolic contact structure [1] according as $(a= \pm i, c=1),(a= \pm 1, c=-1)$ or $(a= \pm 1, c=1)$ respectively. The manifold $M_{n}$ equipped with an unified contact structure will be called an unified contact structure manifold.

Let us define the metric $G$ in $M_{n}$ by

$$
\begin{equation*}
G(\bar{X}, \bar{Y})=-a^{2} G(X, Y)-c \xi(X) \xi(Y) \tag{1.5a}
\end{equation*}
$$

where

$$
\begin{equation*}
G(X, \eta) \xlongequal{\text { def }} \xi(X) \tag{1.5b}
\end{equation*}
$$

Then the unified contact structure $M_{n}$ will be called an unified contact Riemannian manifold.
Remark 1.1: An unified contact Riemannian manifold gives an almost contact metric manifold, an almost hyperbolic paracontact metric manifold or an almost hyperbolic contact metric manifold according as
$(a= \pm i, c=1),(a= \pm 1, c=-1)$ or $(a= \pm 1, c=1)$
Definition 1.1: A $C^{\infty}$-manifold $M_{n}$, satisfying
$D_{X} \eta=\Phi(X)$
will be denoted by $M_{n}{ }^{*}$
In $M_{n}{ }^{*}$, we can easily shown that
$\left(D_{X} \xi\right)(Y)=` \Phi(X, Y)=-\left(D_{Y} \xi\right)(X)$
where

$$
\begin{equation*}
\lceil(X, Y) \xlongequal{d \kappa} G(\bar{X}, Y)=-G(X, \bar{Y})=` \Phi(Y, X) \tag{1.8}
\end{equation*}
$$

Definition 1.2: An affine connection $\tilde{B}$ is said to be metric if
$\tilde{B}_{X} G=0$
The affine metric connection $\tilde{B}$ satisfying
$\left(\tilde{B}_{X} \Phi\right)(Y)=\xi(Y) X-G(X, Y) \eta$
is called metric $S$ - connection.
A metric $S$-connection $\tilde{B}$ is called semi-symmetric non-metric $S$-connection if
$\tilde{B}_{X} Y=D_{X} Y-\xi(Y) X-G(X, Y) \eta$
Where $D$ is the Riemannian connection.
Also

$$
\left(\tilde{B}_{X} G\right)(Y, Z)=2 \xi(Y) G(X, Z)+2 \xi(Z) G(X, Y)
$$

which implies

$$
\begin{equation*}
S(X, Y)=\xi(Y) \bar{X}-\xi(X) \bar{Y} \tag{1.12}
\end{equation*}
$$

where $S$ is the torsion tensor of connection $\tilde{B}$.
The curvature tensor with respect to the semi-symmetric non-metric $S$-connection is defined as
$\tilde{R}(X, Y, Z) \xlongequal{\text { def }} \tilde{B}^{\prime} \tilde{B}_{Y} Z-\tilde{B}_{Y} \tilde{B}_{X} Z-\tilde{B}_{[X, Y]^{Z}}$

Using (1.11) in (1.13), we get
$\tilde{R}(X, Y, Z)=K(X, Y, Z)-\beta(X, Z) Y+\beta(Y, Z) X-G(Y, Z)\left(D_{X} \eta-\xi(X) \eta\right)+G(X, Z)\left(D_{Y} \eta-\xi(Y) \eta\right)$
where

$$
\beta(X, Y)=\left(D_{X} \xi\right)(Y)+\xi(X) \xi(Y)+G(X, Y) \xi(\eta)(1.15)
$$

and

$$
\begin{equation*}
\left.K(X, Y, Z) \xlongequal{\text { def }} D_{X} D_{Y} Z-D_{Y} D_{X} Z-D_{[X, Y}\right]^{Z} \tag{1.16}
\end{equation*}
$$

where $\tilde{R}$ and $K$ be the curvature tensors with respect to the connection $\tilde{B}$ and $D$ respectively.
Using (1.6) in (1.14), we get
$\tilde{R}(X, Y, Z)=K(X, Y, Z)-\beta(X, Z) Y+\beta(Y, Z) X-G(Y, Z)(\bar{X}-\xi(X) \eta)+G(X, Z)(\bar{Y}-\xi(Y) \eta)$
If $\tilde{R}(X, Y, Z)=0$ then above equation becomes
$K(X, Y, Z)-\beta(X, Z) Y+\beta(Y, Z) X-G(Y, Z)(\bar{X}-\xi(X) \eta)+G(X, Z)(\bar{Y}-\xi(Y) \eta)=0$
Contracting above equation with respect to $X$, we get
$\operatorname{Ric}(Y, Z)-\beta(Y, Z)+n \beta(Y, Z)-\frac{a^{2}}{c} G(Y, Z)+G(\bar{Y}, Z)-\xi(Y) \xi(Z)=0$
Using (1.15) in (1.19), we get
$c R i c(Y, Z)+c(n-1)\left[\hookrightarrow(Y, Z)+\xi(Y) \xi(Z)-\frac{a^{2}}{c} G(Y, Z)\right]+G(\bar{Y}, \bar{Z})+c G(\bar{Y}, Z)=0$
Contracting above equation with respect to $Z$, we get
$r Y+n\left(-\frac{a^{2}}{c} Y+\bar{Y}\right)+(n-2) \xi(Y) \eta=0$
Contracting above equation with respect to $Y$, we get
$R=\frac{a^{2}}{c}(n+2)(n-1)$
Where Ric and $R$ are Ricci tensor and scalar curvature of the manifold respectively.
The Projective curvature tensor $W$, Con-circular curvature tensor $C$, Conformal curvature tensor $Q$, Con-harmonic curvature tensor $L$ in a Riemannian manifold are given by [2], [3].

$$
\begin{align*}
W(X, Y, Z)= & K(X, Y, Z)-\frac{1}{(n-1)}[\operatorname{Ric}(Y, Z) X-\operatorname{Ric}(X, Z) Y]  \tag{1.23}\\
C(X, Y, Z)= & K(X, Y, Z)-\frac{R}{n(n-1)}[G(Y, Z) X-G(X, Z) Y]  \tag{1.24}\\
Q(X, Y, Z)= & K(X, Y, Z)-\frac{1}{(n-2)}[\operatorname{Ric}(Y, Z) X-\operatorname{Ric}(X, Z) Y \\
& +G(Y, Z) r(X)-G(X, Z) r(Y)]+\frac{R}{(n-1)(n-2)}[G(Y, Z) X-G(X, Z) Y]  \tag{1.25}\\
L(X, Y, Z)= & K(X, Y, Z)-\frac{1}{(n-2)}[\operatorname{Ric}(Y, Z) X-\operatorname{Ric}(X, Z) Y+G(Y, Z) r(X)-G(X, Z) r(Y)] \tag{1.26}
\end{align*}
$$

where

$$
\begin{align*}
& W(X, Y, Z, T) \xlongequal{d \kappa} G(W(X, Y, Z), T)  \tag{1.27}\\
& C(X, Y, Z, T) \xlongequal{\underline{d k}} G(C(X, Y, Z), T) \tag{1.28}
\end{align*}
$$

$$
\begin{align*}
& Q(X, Y, Z, T) \xlongequal{d 飞} G(Q(X, Y, Z), T)  \tag{1.29}\\
& L(X, Y, Z, T) \underline{\underline{d \kappa}} G(L(X, Y, Z), T) \tag{1.30}
\end{align*}
$$

## 2. CURVATURE TENSORS

Theorem 2.1: If an unified contact Riemannian manifold admits a semi-symmetric non-metric $S$-connection whose curvature tensor is locally isometric to the unit sphere $S^{(n)}(1)$, then the Conformal and Con-harmonic curvature tensors with respect to the Riemannian connection are identical iff $n+\frac{a^{2}}{c}(n+2)=0$

Proof: If the curvature tensor with respect to the semi-symmetric non metric $S$ - connection is locally isometric to the unit sphere $S^{(n)}(1)$, then

$$
\begin{equation*}
\tilde{R}(X, Y, Z)=G(Y, Z) X-G(X, Z) Y \tag{2.1}
\end{equation*}
$$

Using (2.1) in (1.17), we get

$$
\begin{align*}
G(Y, Z) X-G(X, Z) Y= & K(X, Y, Z)-\beta(X, Z) Y+\beta(Y, Z) X \\
& -G(Y, Z)(\bar{X}-\xi(X) \eta)+G(X, Z)(\bar{Y}-\xi(Y) \eta) \tag{2.2}
\end{align*}
$$

Contracting above with respect to $X$ and using (1.3) and (1.8), we get

$$
\begin{equation*}
c R i c(Y, Z)=c(n-1)\left[G(Y, Z)-\Phi \Phi(Y, Z)-\xi(Y) \xi(Z)+\frac{a^{2}}{c} G(Y, Z)\right]-G(\bar{Y}, \bar{Z})-c G(\bar{Y}, Z) \tag{2.3}
\end{equation*}
$$

Contracting above equation with respect to $Z$, we get

$$
\begin{equation*}
c r Y=c n(Y-\bar{Y})-(n-2) c \xi(Y) \eta+\left(a^{2} n-c\right) Y \tag{2.4}
\end{equation*}
$$

Contracting above equation with respect to $Y$, we get
$R=(n-1)\left[n+\frac{a^{2}}{c}(n+2)\right]$
Where Ric and $R$ are Ricci tensor and scalar curvature of the manifold respectively.
From (2.5), (1.25) and (1.26), we obtain the necessary part of the theorem. Converse part is obvious from (1.25) and (1.26).

Now, using (1.18) and (1.20) in (1.23), we get

$$
\begin{align*}
W(X, Y, Z)= & \beta(X, Z) Y-\beta(Y, Z) X+G(Y, Z) \bar{X}-G(Y, Z) \xi(X) \eta \\
& -G(X, Z) \bar{Y}+G(X, Z) \xi(Y) \eta+\xi(Y) \xi(Z) X-\xi(X) \xi(Z) Y \\
& +\frac{n}{(n-1)}[G(\bar{Y}, Z) X-G(\bar{X}, Z) Y]+\frac{1}{c(n-1)}[G(\bar{Y}, \bar{Z}) X-G(\bar{X}, \bar{Z}) Y] \\
& +\frac{a^{2}}{c}[G(X, Z) Y-G(Y, Z) X] \tag{2.6}
\end{align*}
$$

Now operating $G$ on both the sides of above equation and using (1.5b) and (1.27), we get

$$
\begin{align*}
W(X, Y, Z, T) & =\beta(X, Z) G(Y, T)-\beta(Y, Z) G(X, T)+G(Y, Z) G(\bar{X}, T) \\
& -G(Y, Z) \xi(X) \xi(T)-G(X, Z) G(\bar{Y}, T)+G(X, Z) \xi(Y) \xi(T)+\xi(Y) \xi(Z) G(X, T) \\
& -\xi(X) \xi(Z) G(Y, T)+\frac{n}{(n-1)}[G(\bar{Y}, Z) G(X, T)-G(\bar{X}, Z) G(Y, T)] \\
& +\frac{1}{c(n-1)}[G(\bar{Y}, \bar{Z}) G(X, T)-G(\bar{X}, \bar{Z}) G(Y, T)]+\frac{a^{2}}{c}[G(X, Z) G(Y, T)-G(Y, Z) G(X, T)] \tag{2.7}
\end{align*}
$$

Theorem 2.2: On a $C^{\infty}$-manifold $M_{n}$, we have

$$
\begin{align*}
W(X, Y, Z, \eta)= & \beta(X, Z) \xi(Y)-\beta(Y, Z) \xi(X)+\frac{n}{(n-1)}[\Phi(Y, Z) \xi(X)-` \Phi(X, Z) \xi(Y)] \\
& +\frac{1}{c(n-1)}[G(\bar{Y}, \bar{Z}) \xi(X)-G(\bar{X}, \bar{Z}) \xi(Y)]  \tag{2.8a}\\
W(\eta, Y, Z, \eta)= & \beta(\eta, Z) \xi(Y)+\frac{a^{2}}{c} \beta(Y, Z)+\frac{a^{2}}{c}\left(\frac{n}{n-1}\right) G(\bar{Y}, Z)-\frac{a^{2}}{c^{2}}\left(\frac{1}{n-1}\right) G(\bar{Y}, \bar{Z})  \tag{2.8b}\\
W(\bar{X}, \bar{Y}, Z, \eta)= & 0  \tag{2.8c}\\
W(X, Y, \eta, \eta)= & \beta(X, \eta) \xi(Y)-\beta(Y, \eta) \xi(X)  \tag{2.8d}\\
W(\bar{X}, \bar{Y}, \eta, \eta)= & 0  \tag{2.8e}\\
W(\eta, Y, Z, T)= & \beta(\eta, Z) G(Y, T)-\beta(Y, Z) \xi(T)-\xi(Z) G(\bar{Y}, T)+2 \xi(Z) \xi(Y) \xi(T) \\
& +\frac{2 a^{2}}{c} G(Y, T) \xi(Z)+\frac{n}{(n-1)} G(\bar{Y}, Z) \xi(T)+\frac{1}{c(n-1)} G(\bar{Y}, \bar{Z}) \xi(T)  \tag{2.8f}\\
W(\eta, Y, Z, \eta)= & \beta(\eta, Z) \xi(Y)+\frac{a^{2}}{c} \beta(Y, Z)-\frac{a^{2}}{c(n-1)}[n \mathbb{G}(\bar{Y}, Z)+G(\bar{Y}, \bar{Z})] \tag{2.8g}
\end{align*}
$$

Proof: Replacing $T$ by $\eta$ in (2.7) and using (1.3), (1.4), (1.5b) and (1.8) we get (2.8a).
Replacing $X$ by $\eta$ in (2.8a) and using (1.2), (1.3), (1.4), (1.5b) and (1.8) we get (2.8b).
Replacing $X$ by $\bar{X}$ and $Y$ by $\bar{Y}$ by in (2.8a) and using (1.4), we get (2.8c).
Replacing $Z$ by $\eta$ in (2.8a) and using (1.2), (1.4) and (1.8), we get (2.8d).
Replacing $X$ by $\bar{X}$ and $Y$ by $\bar{Y}$ in (2.8d), we get (2.8e)
Replacing $X$ by $\eta$ in (2.7) and using (1.2), (1.5b), (1.6), we get (2.8f).
Replacing $T$ by $\eta$ in (2.8f) and using (1.3), (1.4) and (1.5b), we get (2.8g).
Theorem 2.3: If an unified contact Riemannian manifold admits a semi-symmetric non metric $S$-connection whose curvature tensor is locally isometric to the unit sphere $S^{(n)}(1)$, then the Con-circular curvature tensor coincides with respect to the Riemannian connection if $n+\frac{a^{2}}{c}(n+2)=0$

Proof: Using (2.5) in (1.24), we get
$C(X, Y, Z)=K(X, Y, Z)-\frac{\left[n+\frac{a^{2}}{c}(n+2)\right]}{n}[G(Y, Z) X-G(X, Z) Y]$
which is required proves of the theorem.
Now, using (1.18) and (1.22) in (1.24), we get

$$
\begin{align*}
C(X, Y, Z)= & \beta(X, Z) Y-\beta(Y, Z) X+G(Y, Z)(\bar{X})-G(Y, Z) \xi(X) \eta \\
& -G(X, Z) \bar{Y}+G(X, Z) \xi(Y) \eta-\frac{a^{2}}{c}\left(\frac{n+2}{n}\right)[G(Y, Z) X-G(X, Z) Y] \tag{2.10}
\end{align*}
$$

Operating $G$ on both the sides of above equation and using (1.5b), (1.8) and (1.28), we get

$$
\begin{align*}
\subset(X, Y, Z, T)= & \beta(X, Z) G(Y, T)-\beta(Y, Z) G(X, T)+G(Y, Z) \subseteq(X, T) \\
& -G(Y, Z) \xi(X) \xi(T)-G(X, Z) \subseteq \Phi(Y, T)+G(X, Z) \xi(Y) \xi(T) \\
& -\frac{a^{2}}{c}\left(\frac{n+2}{n}\right)[G(Y, Z) G(X, T)-G(X, Z) G(Y, T)] \tag{2.11}
\end{align*}
$$

Theorem 2.4: On $C^{\infty}$-manifold we have

$$
\begin{align*}
& C(X, Y, Z, \eta)=\beta(X, Z) \xi(Y)-\beta(Y, Z) \xi(X)+\frac{a^{2}}{c}\left(\frac{2}{n}\right)[G(X, Z) \xi(Y)-G(Y, Z) \xi(X)](2.12 a) \\
& C(\eta, Y, Z, T)=\beta(\eta, Z) G(Y, T)-\beta(Y, Z) \xi(T)+\frac{a^{2}}{c} G(Y, Z) \xi(T) \\
& -\xi(Z) G(\bar{Y}, T)+\xi(Y) \xi(Z) \xi(T)-\frac{a^{2}}{c}\left(\frac{n+2}{n}\right)[G(Y, Z) \xi(T)-G(Y, T) \xi(Z)]  \tag{2.12b}\\
& C(\eta, Y, Z, \eta)=\beta(\eta, Z) \xi(Y)+\frac{a^{2}}{c} \beta(Y, Z)+\frac{2}{n} \frac{a^{4}}{c^{2}} G(Y, Z)+\frac{2}{n} \frac{a^{2}}{c} \xi(Y) \xi(X)  \tag{2.12c}\\
& \subset(X, Y, \eta, \eta)=\beta(X, \eta) \xi(Y)-\beta(Y, \eta) \xi(X)  \tag{2.12d}\\
& \subset(\bar{X}, \bar{Y}, Z, \eta)=0  \tag{2.12e}\\
& { }^{`} C(\eta, Y, \bar{Z}, \bar{T})=\beta(\eta, \bar{Z}) G(Y, \bar{T}) \tag{2.12f}
\end{align*}
$$

Proof: Replacing $T$ by $\eta$ in (2.8) and using (1.3), (1.4), (1.5b) and (1.8), we get (2.12a).
Replacing $X$ by $\eta$ in (2.8) and using (1.2), (1.3), (1.4), (1.5b) and (1.8), we get (2.12b)

Replacing $T$ by $\eta$ in (2.12b) and using (1.3) and (1.5b), we get (2.12c).

Replacing $Z$ by $\eta$ in (2.12a) and using (1.5b), we get (2.12d).

Replacing $X$ by $\bar{X}$ and $Y$ by $\bar{Y}$ in (2.12a) and using (1.3), we get (2.12e).
Replacing $Z$ by $\bar{Z}$ and $T$ by $\bar{T}$ in (2.12b) and using (1.3), we get (2.12f).

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