International Journal of Mathematical Archive-6(3), 2015, 25-31
Available online through www.ijma.info ISSN 2229 – 5046

UTILIZING LAGUERRE TRANSFORM IN TWO-SPATIAL DIMENSIONS PARTIAL DIFFERENTIAL EQUATIONS AND IMAGE PROCESSING

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(Received On: 19-02-15; Revised & Accepted On: 24-03-15)

ABSTRACT

The Laguerre Transform can be used to solve Partial differential equations with the initial and the boundary conditions to find the approximation solution of the problem. The technique is verified by solving example and obtaining the results give a good accuracy when they compared with the numerical solution. It is well known in the field of image processing using the possibilities of the heat equation in terms of improving the image of a blur or remove distortions, so we will later apply the heat equation on some images after the dissolution of the equation by using the Laguerre Transform.

Key words: *laguerre transform, partial differential equations, image processing.*

1. INTRODUCTION

Historically, PDEs were not consciously created as a subject but emerged in the 18th century as ODEs failed to describe the physical principles being studied, [2]. They are the basis of many models that mathematically describes different phenomena occurring in various fields of engineering and natural sciences. More recently their use has been spread into economics, financial forecasting, image processing and other fields, [6]. Many phenomena in nature may be described mathematically by functions of a small number of independent variables which often result in equations, usually containing a large variety of derivatives with respect to these variables. One particular important case in real life is when the independent variables are the spatial positions and the time, and where a transient solution is required, [5].

The applications of PDEs in computer vision and image processing dated back to the 1960s, [8]. Digital image processing is an important example of applied numerical computing. It has been successfully applied to many problems in computer vision and image processing includes image restoration, image enhancement, image compressing, image coding…etc. Image enhancement is one of the most interesting and visually appealing areas of image processing, which is to bring out details that are obscured or simply to highlight certain features of interest in an image, [3].

2. Utilizing Laguerre Transform in a Technique for Solving Two Spatial Dimensions PDEs

Laguerre Transform is the sequence of numbers *f(n)* that are defined by the following equation:-

$$\hat{f}(n) = L\{F(x)\} = \int_0^\infty e^{-x} L_n(x) F(x)\,dx \qquad n=0, 1, 2,\ldots \tag{1}$$

Where $L_n(x)$ denotes the Laguerre Polynomials of nth degree. The Laguerre Transform of a function *F(x)* exists if *F(x)* is continuous in every finite interval in the range $x \geq 0$, and its inverse transform is represented here by the fourier laguerre expansion of *F(x)*, [7,1,4]:

$$F(x) = \sum_{n=0}^{\infty} \hat{f}(n) L_n(x) \qquad \text{where} \quad 0 < x < \infty \tag{2}$$

In this section, we can use the Laguerre Transform for solving the heat equation in two spatial dimensions:-

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + b^2 \frac{\partial^2 u}{\partial y^2} \tag{3}$$

where a , b are arbitrary constant not equal to zero, and with a given initial and boundary conditions.

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Starting the solution technique by separation of the time variable from the spatial dimensions.

Let: u(x, y, t) =V(x, y)W(t) where $W(t) \neq 0$, $\forall$ t and $V(x, y) \neq 0$ $\forall$ x, y

Then

$$\frac{1}{W(t)}\frac{\partial W(t)}{\partial t} = \frac{a^2}{V(x,y)}\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{b^2}{V(x,y)}\frac{\partial^2 V(x,y)}{\partial y^2} \tag{4}$$

Since each side is a function of different independent variables, then each side must be a constant, say $-\alpha^2$. Now we have two equations as following:

$$\frac{\partial^2 V(x,y)}{\partial x^2} + r\frac{\partial^2 V(x,y)}{\partial y^2} = -\frac{\alpha^2}{a^2}V(x,y) \quad \text{where } r = \frac{b^2}{a^2} \tag{5}$$

And

$$\frac{\partial W(t)}{\partial t} = -\, W(t)\, \alpha^2 \tag{6}$$

Solving equation (5) starts with taking its Laguerre Transform w. r. t. (x), when $n=0$ we obtain the following equation:

$$\frac{d^2\hat{v}(0,y)}{dy^2} + \left(\frac{1}{r} + \frac{\alpha^2}{b^2}\right)\hat{v}(0,y) = 0 \tag{7}$$

The solution of equation (7) is:

$$\hat{v}(0,y) = A0 \sin(\omega\, y) + B0 \cos(\omega\, y) \tag{8}$$

where $$\omega = \sqrt{\frac{1}{r} + \frac{\alpha^2}{b^2}} \tag{9}$$

Taking the Laguerre Transform of the boundary conditions and substitution in equation (8) yields that $B0=0$, and $\hat{v}(0,1) = A0 \sin(\omega)=0$, the above equation has the following solutions: $\omega_i = i\pi$ where $i=1,2,\ldots,\infty$.

And the total solution of equation (7) is:

$$\hat{v}(0,y) = \sum_{i=1}^{\infty} A0_i \sin(\omega_i\, y) \tag{10}$$

Substitution of the value of ω_i in equation (9), one obtain $\alpha_i^2 = (i\pi)^2 b^2 - a^2$

Now taking the Laguerre Transform of equation (5) w. r. t. (x) when $n=1$

$$\frac{d^2\hat{v}(1,y)}{dy^2} + \left(\frac{1}{r} + \frac{\alpha^2}{b^2}\right) + \hat{v}(1,y) = -2\hat{v}(0,y) \tag{11}$$

Equation (11) has two solutions, namely the homogeneous solution and the particular solution.

$$\hat{v}(1,y) = \hat{v}_h(1,y) + \hat{v}_p(1,y) \tag{12}$$

Using the variation of parameters technique to find $\hat{v}_p(1,y)$ as:

$$\hat{v}_p(1,y) = \frac{\sin(\omega y)A0_i}{-\omega}\int_0^{\infty} 2\cos(\omega y)\sin(\omega_i\, y)dy - \frac{\cos(\omega y)A0_i}{-\omega}\int_0^{\infty} 2\sin(\omega y)\sin(\omega_i\, y)dy$$

Using trigonometric identities, perform the integrals and simplify, as follow:

$$\hat{v}_p(1,y) = \frac{2A0_i \sin(\omega_i y)}{\omega_i^2 - \omega^2} \tag{13}$$

and

$$\hat{v}_h(1,y) = A1 \sin(\omega\, y) + B1 \cos(\omega\, y) \tag{14}$$

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Substitution of equations (14) and (13) in equation (12), and applied the boundary conditions in equation (12) yields that $B1= 0$ and

$$\hat{v}(1,1) = A1\sin(\grave{u}) + \frac{2A0_i \, Sin(\omega_i)}{\omega_i^2 - \omega^2} = 0 \tag{15}$$

Equation (15) has the solutions $\omega_{i_1} = i_1\pi$ where $i_1=1, 2,\ldots, \infty$ and $i_1 \neq i$. So we get the total solution as:

$$\hat{v}(1,y) = \sum_{i_1=1}^{\infty} (A1_{i_1} \sin(\omega_{i_1} y) + \sum_{\substack{i=1,\\ i\neq i_1}}^{\infty} \frac{2A0_i \sin(\omega_i y)}{\omega_i^2 - \omega_{i_1}^2}) \tag{16}$$

Now taking the Laguerre Transform of equation (5) w. r. t. (x) when $n=2$

$$\frac{d^2\hat{v}(2,y)}{dy^2} + \left(\frac{1}{r} + \frac{\alpha^2}{b^2}\right)\hat{v}(2,y) = -3\hat{v}(0,y) - 2\hat{v}(1,y) \tag{17}$$

The solution of equation (17) is:

$$\hat{v}(2,y) = \hat{v}_h(2,y) + \hat{v}_p(2,y) \tag{18}$$

and $\hat{v}_h(2,y) = A2 \sin(\omega y) + B2 \cos(\omega y)$ (19)

and using a variation of parameters method to find $\hat{v}_p(2,y)$, to get the following equation:

$$\hat{v}_p(2,y) = \frac{3A0_i \sin(\omega_i y)}{\omega_i^2 - \omega^2} + \frac{2A1_{i_1} \sin(\omega_{i_1} y)}{\omega_{i_1}^2 - \omega^2} + \frac{4A0_i \sin(\omega_i y)}{(\omega_i^2 - \omega_{i_1}^2)(\omega_i^2 - \omega^2)} \tag{20}$$

Substitution of equation (19) and equation (20) in equation (18) and using the boundary conditions yields that:

The solution of equation (24) is $\omega_{i2} = i_2\pi$, $i_2=1, 2,\ldots,\infty$. and $i_2 \neq i_1$, $i_2 \neq i$, then we get:

$$\hat{v}(2,y) = \sum_{i2=1}^{\infty}[A2_{i2} \sin(\omega_{i2} y) + \sum_{\substack{i=1,\\ i\neq i_2}}^{\infty} \frac{3A0_i \sin(\omega_i y)}{\omega_i^2 - \omega_{i_2}^2} + \sum_{\substack{i=1,\\ i\neq i_2}}^{\infty} [(\frac{3A1_i \sin(\omega_i y)}{\omega_i^2 - \omega_{i_2}^2})$$

$$+ \sum_{\substack{i=1,\\ i\neq i_2,\\ i\neq i_1}}^{\infty} (\frac{4A0_i \sin(\omega_i y)}{(\omega_i^2 - \omega_{i_1}^2)(\omega_i^2 - \omega_{i_2}^2)})]] \tag{21}$$

Follow the same procedures to get $\hat{v}(n,y)$ for $n=3, 4,\ldots$. The solution of equation (5) will be:

$$V_1(x,y) = \sum_{n=0}^{\infty} \hat{v}(n,y) L_n(x)$$

Similar solution can be found by taking LT of equation (5) w. r. t. (y), and where $\psi = \sqrt{r + \frac{\alpha^2}{a^2}}$

where $n=0$

$$\hat{v}(x,0) = \sum_{j=1}^{\infty} C0_j \sin(\psi_j x) \tag{22}$$

n=1

$$\hat{v}(x,1) = \sum_{j_1=1}^{\infty} (C1_{j_1} \sin(\psi_{j_1} x) + \sum_{\substack{j=1,\\ j\neq j_1}}^{\infty} \frac{2C0_j \sin(\psi_j x)}{\psi_j^2 - \psi_{j_1}^2}) \tag{23}$$

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n=2

$$\hat{v}(x,2)=\sum_{j_2=1}^{\infty}[C2_{j_2}\sin(\psi_{j_2}x)+\sum_{\substack{j=1,\\ j\neq j_2}}^{\infty}\frac{3C0_j\sin(\psi_j x)}{\psi_j^2-\psi_{j_2}^2}+\sum_{\substack{j=1,\\ j\neq j_2}}^{\infty}[(\frac{3C1_j Sin(\psi_j x)}{\psi_j^2-\psi_{j_2}^2})$$

$$+\sum_{\substack{j=1,\\ j\neq j_2,\\ j\neq j_1}}^{\infty}(\frac{4C0_j\sin(\psi_j x)}{(\psi_j^2-\psi_{j_1}^2)(\psi_j^2-\psi_{j_2}^2)})]] \tag{24}$$

Follow the same procedures to get $\hat{v}(x,n)$ *where n=3, 4,…*, the second solution of equation (5) is:

$$V_2(x,y)=\sum_{n=0}^{\infty}\hat{v}(x,n)L_n(y) \tag{25}$$

And the total solution of equation (5) will be:

$$V(x,y)=\sum_{n=0}^{\infty}[\hat{v}(n,y)L_n(x)+\hat{v}(x,n)L_n(y)] \tag{26}$$

Equation (6) has the following simple solution:

$W(t)=k\,e^{-\alpha^2 t}$ (without using LT) and k is an arbitrary constant,
where $\alpha^2=((i\pi)^2b^2-a^2)+((i\pi)^2a^2-b^2)$ (27)

The total solution will be

$$u(x,y,t)=\sum_{n=0}^{\infty}[\hat{v}(n,x)L_n(x)+\hat{v}(x,n)L_n(y)]ke^{-\alpha^2 t} \tag{28}$$

To solve the following equation

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} \tag{29}$$

With the boundary conditions $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$,and the initial condition $u(x,y,0)=(3x-3x^3)(2y^2-2y^3)$

The approximate solution, equation (29), for different values of *t*, are shown in Figure (1) and Figure (2). The absolute difference values between the approximate solution by the proposed method and that by a finite difference method are listed in Tables (1) and (2) for $N = 2$. And where $thf= 0.693/(\pi^2-1)$, it is the time required for the fundamental component of a function to decay to a half of its original value.

n_y \ n_x	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.0019	0.0041	0.0055	0.0062	0.0062	0.0057	0.0048	0.0038	0.003	0.0023	0.002	0.0019	0.0018
2	0.0033	0.0073	0.0099	0.0111	0.011	0.01	0.0084	0.0067	0.0052	0.0041	0.0036	0.0035	0.0035
3	0.0042	0.0096	0.0129	0.0143	0.014	0.0124	0.0101	0.0076	0.0056	0.0043	0.004	0.0043	0.0046
4	0.0047	0.011	0.0147	0.0161	0.0154	0.0132	0.0102	0.0072	0.0048	0.0035	0.0034	0.0043	0.0053
5	0.0049	0.012	0.0156	0.0168	0.0157	0.0129	0.0093	0.0058	0.0032	0.002	0.0024	0.0039	0.0057
6	0.0047	0.012	0.0156	0.0166	0.0151	0.0119	0.0079	0.0041	0.0014	0.0004	0.0011	0.0033	0.0058
7	0.0044	0.0113	0.0151	0.0159	0.0141	0.0106	0.0064	0.0025	0.0002	0.0011	0	0.0026	0.0057
8	0.004	0.0106	0.0141	0.0147	0.0129	0.0094	0.0052	0.0014	0.0012	0.0019	0.0007	0.0022	0.0055
9	0.0035	0.0097	0.0129	0.0135	0.0117	0.0085	0.0046	0.001	0.0014	0.002	0.0007	0.002	0.0051
10	0.0031	0.0086	0.0116	0.0121	0.0107	0.0079	0.0046	0.0015	0.0006	0.0012	0.0002	0.0021	0.0047
11	0.0027	0.0073	0.01	0.0107	0.0097	0.0076	0.005	0.0026	0.0009	0.0003	0.0009	0.0024	0.0042
12	0.0022	0.0058	0.0080	0.0088	0.0084	0.0072	0.0055	0.0039	0.0026	0.002	0.0021	0.0028	0.0025
13	0.0015	0.0038	0.0053	0.0061	0.0063	0.0059	0.0053	0.0045	0.0038	0.0032	0.0029	0.0027	0.0025

Table (1): *The absolute difference between the numerical solution using a finite difference method and the approximate solution obtained by the proposed method when N=2, at t=0.5 thf. (note nx=1+12x and ny=1+12y)*

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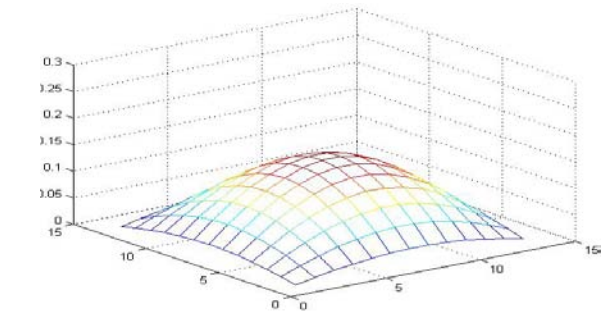
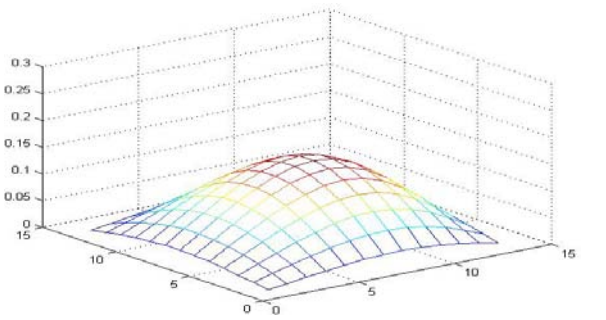
n_x \ n_y	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.0007	0.0017	0.0024	0.0029	0.0031	0.0031	0.0031	0.0029	0.0027	0.0025	0.0023	0.002	0.0015
2	0.0012	0.0031	0.0044	0.0052	0.0056	0.0057	0.0056	0.0053	0.005	0.0046	0.0042	0.0037	0.0029
3	0.0016	0.0041	0.0058	0.0068	0.0073	0.0073	0.0071	0.0067	0.0063	0.0059	0.0055	0.0049	0.0039
4	0.0018	0.0047	0.0066	0.0078	0.0082	0.0082	0.0079	0.0075	0.007	0.0066	0.0063	0.0057	0.0047
5	0.0018	0.005	0.0071	0.0083	0.0087	0.0086	0.0082	0.0077	0.0073	0.0070	0.0067	0.0062	0.0052
6	0.0018	0.0051	0.0073	0.0084	0.0088	0.0086	0.0082	0.0077	0.0073	0.0070	0.0068	0.0064	0.0055
7	0.0018	0.0051	0.0073	0.0084	0.0087	0.0085	0.0081	0.0076	0.0072	0.0069	0.0067	0.0064	0.0056
8	0.0018	0.005	0.0071	0.0082	0.0085	0.0083	0.0079	0.0074	0.007	0.0067	0.0066	0.0063	0.0055
9	0.0017	0.0048	0.0068	0.0079	0.0082	0.0081	0.0077	0.0072	0.0068	0.0065	0.0063	0.006	0.0052
10	0.0016	0.0045	0.0063	0.0074	0.0078	0.0077	0.0074	0.007	0.0066	0.0063	0.006	0.0055	0.0046
11	0.0015	0.0040	0.0057	0.0067	0.0071	0.0071	0.0069	0.0065	0.0062	0.0058	0.0054	0.0048	0.0039
12	0.0013	0.0033	0.0047	0.0056	0.006	0.0061	0.006	0.0057	0.0054	0.005	0.0045	0.0039	0.003
13	0.001	0.0022	0.0032	0.0038	0.0042	0.0044	0.0043	0.0042	0.0039	0.0035	0.0031	0.0026	0.0018

Table (2): *The absolute difference between the numerical solution using a finite difference method and the approximate solution obtained by the proposed method when N=2, at t=1.0 thf. (note nx=1+12x and ny=1+12y).*

In the following Figures show the solutions of the equation (29).

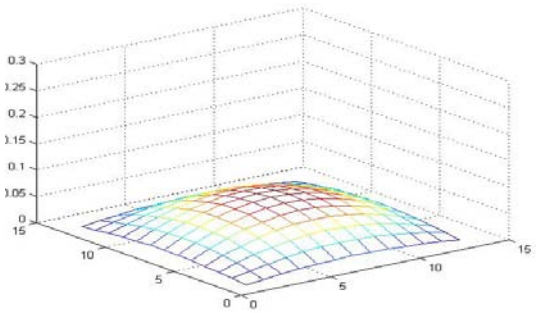


(a)



(b)

Fig. (1): (a) Show the approximate solution by the proposed method.(b)The solution by finite difference method.



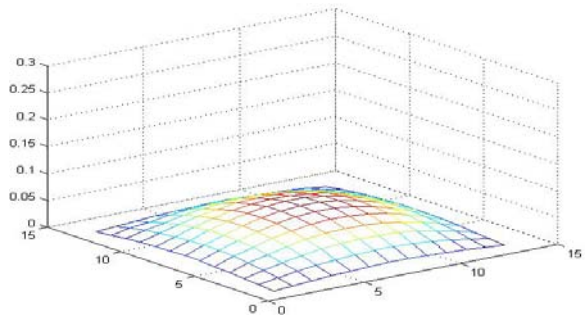


Fig. (2): (a)Show the approximate solution by the proposed method.(b)The solution by finite difference method.

3. THE APPLICATION OF LAGUERRE TRANSFORM IN IMAGE PROCESSING

In this section, we will use the previous technique, to solve the heat equation using Laguerre Transform, in image processing. The basics of the technique is to take the image information (pixels) as an input initial condition of the PDE and where the final solution parameters *k* are to be evaluated from this initial condition. Thus the processing is a type of global operation. Most researchers in the field of digital image processing introduce the two spatial dimension heat equation and solve it using finite difference method in order to achieve smoothing. The program was applied for different images, the results are shown below with the results of a finite difference method for comparison. To clarify the smoothing effect, the frequency histogram for each image is attached to it.

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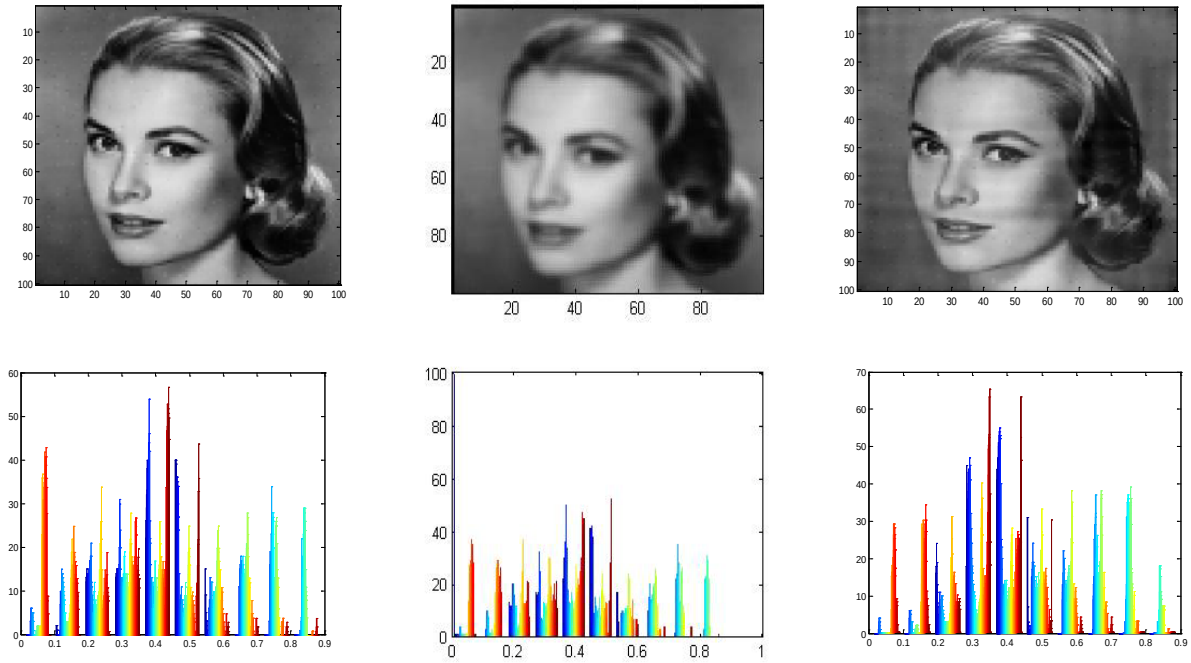
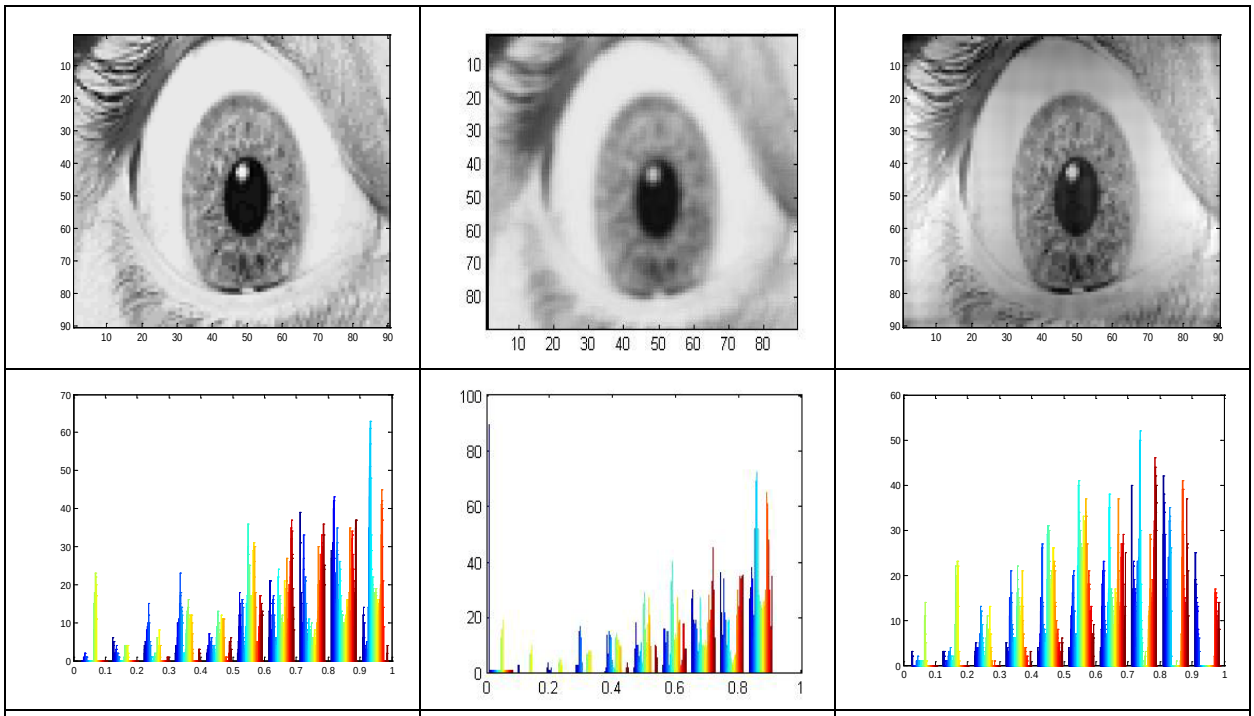


(a) (b) (c)

Figure (3): (a) The original image (input). (b)Images obtained by finite difference method. (c) Image obtained using the proposed method when N=2 and M=35 at thf/500.



(a) (b) (c)

Figure (4): (a) The original image (input). (b) Images obtained by finite difference method. (c) Image obtained using the proposed method when N=2 and M=35 at thf/500.

4. CONCLUSION

In this paper, LT is utilized to build an algorithm for solving different types of two spatial dimensions PDEs.

In the first part the suggested method is applied on the heat equation with two spatial dimensions. Matlab program is built to simulate the method, and the method and the program are tested on a given example. The results are with a good agreement with that of the FD solution. In the second part the same method, with necessary modifications, is used in image processing. The heat equation with two spatial dimensions is selected as a smoothing function for the image,

the solution by the suggested method results in the smoothed image by a global image processing. The method is compared with the most conventional smoothing algorithm, namely the FDM, which is local type processing. The results of the suggested method is "clearly" (less fogy) image than that result from the conventional method, but the long time of computations is the disadvantage of the suggested method.

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Source of support: Nil, Conflict of interest: None Declared

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