International Journal of Mathematical Archive-6(2), 2015, 68-72
Available online through www.ijma.info ISSN 2229 – 5046

STEADY-STATE OF STOCHASTIC DIFFERENCE EQUATIONS USING GENERATING FUNCTIONS

M. Reni Sagayaraj*[1], P. Manoharan[2], A. George Maria Selvam[3], S. Anand Gnana Selvam[4]
[1,2,3,4]Department of Mathematics,
Sacred Heart College, Tirupattur - 635601, Vellore District.Tamil Nadu, S. India.

(Received On: 06-02-15; Revised & Accepted On: 21-02-15)

ABSTRACT

In the present paper we consider a general analytic iterative method for solving stochastic hereditary integrodifferential equation of the Ito type. The proposed method helps to determine probability of the solution to the original equation. The generality of this method is in the sense that Manu well-known iterative methods are its special cases, the Picard-Lindel of method of successive approximations. This method also uses the iterative; including linearization of the coefficients of the original equation, whenever required is computation.

Key words: *Stochastic differential equation, iterative method, steady-state difference equations.*

INTRODUCTION

The motivation for this work is the desire for ultimately investigating mechanical systems under stochastic excitations depending on parameters. The purpose of this article is thus to consider SDEs [9] whose initial value xt_0 and coefficients f and G depend on parameters. The uncertainty of these parameters can be modeled by random variables which require the assumption of certain probability distributions. But in practice, there may only be scarce information available like a small sample size or estimates on the mean value and the variance. Hence, the classical probabilistic [8] approach might involve tacit assumptions that cannot be verified and the need for alternative uncertainty models may arise for a general discussion see for example .Among those alternative models are random sets which can be interpreted as imprecise observations of random variables, that is, instead of a single value one assigns a set which is supposed to include the actual value to each of the elements of the underlying probability space.

1.1 ITERATIVE METHOD OF SOLVING THE STOCHASTIC STEADY-STATE DIFFERENCE EQUATIONS FOR $\{\tau_n\}$

In the present work , an attempt has been made to obtained, that we iteratively use the global balance equations given by Equations

$$0 = -(\lambda+\mu)\tau_n + \mu\tau_{n+1} + \lambda\tau_{n-1} \qquad (n \geq 1) \tag{1.1}$$
$$0 = -\lambda\tau_0 + \mu\tau_1$$

Equivalently the local balance equation $\lambda\tau_n = \mu\tau_{n+1}$ to obtain a sequence of state probabilities, $\tau_1, \tau_2, \tau_3, ...$ each in terms of τ_0. When we believe that we have enough information about the form of these state probabilities, we make a conjecture on their general form for all mathematical induction or the like. But this is precisely the sort of thing that we define for the general birth-death process in where we verified that Equation is the appropriate formula for the steady-state probability $\{\tau_n\}$ for any birth-death process with birth rates $\{\lambda_n, n = 0,1,2...\}$ and death rates $\{\mu_n, n = 1,2,3...\}$ Since the M/M/1 system is indeed a birth-death process with constant birth and death rates, we can directly apply with $\{\lambda_n = \lambda\}$ and $\{\mu_n = \mu\}$ for all n. It follows that

$$\tau_n = \tau_0 \prod_{i=1}^{n} \left(\frac{\lambda}{\mu}\right) \quad (n \geq 1)$$
$$= \tau_0 \left(\frac{\lambda}{\mu}\right)^n$$

Corresponding Author: M. Reni Sagayaraj[1]*

To get τ_0 now, we utilize the fact that probabilities must sum to 1 and it follows that

$$1=\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n \tau_0$$

Recall our earlier definition in Section 1.5 of the traffic intensity or utilization rate ρ as the ratio $\frac{\lambda}{\mu}$ for single-server queues. Rewriting thus gives $\tau_0 = \frac{1}{\sum_{n=0}^{\infty}\rho^n}$

Now $\sum_{n=0}^{\infty}\rho^n$ is the geometric series $1+\tau+\tau^2+\tau^3+.....$ and converges if and only if $\rho<1$. Thus for the existence of a steady-state solution, $\rho=\frac{\lambda}{\mu}$ must be less than 1, or equivalently, λ less than μ. This makes intuitive sense, for if $\lambda<\mu$ the mean arrival rate is greater than the mean service rate, and the server will get further and further behind. That is to say, the system size will keep building up without limit. It is not as intuitive, however, why no steady-state solution exists when $\lambda=\mu$. One possible way to explain infinite build up when $\lambda=\mu$ is that as the queue grows; it is more and more difficult for the server to decrease the queue because the average service rate is no higher than the average arrival rate.

Making use of the well-known expression for the sum of the terms of a geometric progression,

$$\sum_{n=0}^{\infty}\rho^n=\frac{1}{1-\rho} \qquad (\rho<1),$$

We have

$$\tau_0 = 1-\rho \qquad (\rho=\lambda/\mu<1) \tag{1.2}$$

Which confirms the general result for τ_0 we derived previously for all G/G/1 queues. Thus the full steady-state solution for the M/M/1 system is the geometric probability function.

$$\tau_n=(1-\rho)\rho^n \qquad (\rho=\lambda/\mu<1) \tag{1.3}$$

2. STEADY-STATE STOCHASTIC DIFFERENCE EQUATIONS FOR BY GENERATING FUNCTIONS

The probability generating function, $\tau(z)=\sum_{n=0}^{\infty}\tau_n z^n$ (z complex with $|z|\le 1$), can be utilized to find the $\{\tau_n\}$. The procedure involves finding a closed expression for P(z) from Equation (1.1) and then finding the power series expansion to “pick off ” the $\{\tau_n\}$ which are the coefficients. For some models, it is relatively easy to find a closed expression for $\tau(z)$, but quite difficult to find its series expansion to obtain the $\{\tau_n\}$. However, even if the series expansion cannot be found, $\tau(z)$ still provides useful information. For example, $d\tau(z)/dz$ evaluated at z=1 gives the expected number in the system, $L=\sum n\tau_n$. For the M/M/1 model considered here, we can completely solve for the $\{\tau_n\}$ using $\tau(z)$.

$$\tau_{n+1}=(\rho+1)\tau_n-\rho\tau_{n-1}, \qquad (n\ge 1) \tag{2.1}$$

When both sides of the line of (2.1) are multiplied by z^n we find

$$\tau_{n+1}z^n=(\rho+1)\tau_n z^n-\rho\tau_{n-1}z^n,$$
$$z^{-1}\tau_{n+1}z^{n+1}=(\rho+1)\tau_n z^n-\rho z\tau_{n-1}z^{n-1}.$$

When both sides of the above equation are summed from n=1 to ∞ it is found that

$$z^{-1}\sum_{n=1}^{\infty}\tau_{n+1}z^{n+1} = (\rho+1)\sum_{n=1}^{\infty}\tau_n z^n - \rho z\sum_{n=1}^{\infty}\tau_{n-1}z^{n-1}.$$

$$z^{-1}\left[\tau(z)-\tau_1 z-\tau_0\right] = (\rho+1)\left[\tau(z)-\tau_0\right]-\left[\rho z\tau(z)\right] \tag{2.2}$$

We know from (2.1) that $\tau_1 = \rho\tau_0$ and hence

$$z^{-1}\left[\tau(z)-(\rho z+1)\tau_0\right] = (\rho+1)\left[\tau(z)-\tau_0\right]-\left[\rho z\tau(z)\right]$$

Solving for $\tau(z)$ we finally have

$$\tau(z) = \frac{\tau_0}{1-z\rho} \tag{2.3}$$

To find τ_0 now, we use the boundary condition that probabilities sum to 1 in the following way. Consider $\tau(1)$ which can be seen to be

$$\tau(1) = \sum_{n=0}^{\infty}\tau_n 1^n = \sum_{n=0}^{\infty}\tau_n = 1$$

Thus from (2.3), we have

$$\tau(1) = 1 = \frac{\tau_0}{1-\rho} \tag{2.4}$$

So that $\tau_0 = 1-\rho$. Because the $\{\tau_n\}$ are probabilities, $\tau(z) > 0$ for $z > 0$ hence $\tau(1) > 0$ from (2.4) we see that $\tau(1) = \tau_0/(1-\rho) > 0$ therefore c must be <1, since τ_0 is a probability and is >0. Thus

$$\tau(z) = \frac{1-\rho}{1-\rho z} \qquad (\rho < 1,\ \ |z| \le 1) \tag{2.5}$$

It is a easy to expand (2.5) as a power series by simple long division or to recognize it as the sum of a geometric series, since $|\rho z| < 1$. So doing yields

$$\frac{1}{1-\rho} = 1+\rho z+(\rho z)^2+(\rho z)^3+.......$$

and thus the probability generating function is

$$\tau(z) = \sum_{n=0}^{\infty}(1-\rho)\rho^n z^n \tag{2.6}$$

The coefficient of z^n is therefore given by

$$\tau_n = (1-\rho)\rho^n \qquad\qquad (\rho = \lambda/\mu < 1)$$

Which is what was previously obtained in Equation (1.3).

We make a number of concluding observations about the algebraic form of the generating function given in (2.5). First, we note that this expression is the quotient of two (simple) polynomials (i.e., it is a rational function), the numerator being a constant, while the denominator is the linear form $1-\rho z$ It thus follows that the denominator has the single zero of $1/\rho(>1)$ which is clearly just the reciprocal of the traffic intensity. These points are important, because (2.5) is the first of many generating functions we shall be seeing as we develop more in the text.

3.1 SOLVING STEADY-STATE DIFFERENCE EQUATIONS FOR $\left\{\tau_n\right\}$ BY THE USE OF OPERATORS

We begin here by considering a linear operator D defined on the sequence $\left\{a_0 + a_1 + a_2 + ...\right\}$ such that $Da_n = a_{n+1}$ (for all n)

Then the general linear difference equation with constant coefficients

$$C_n a_n + C_{n+1} a_{n+1} + ... + C_{n+k} a_{n+k} = \sum_{i=n}^{n+k} C_i a_i = 0 \tag{3.1}$$

May be written as $\left(\sum_{i=n}^{n+k} C_i D^{i-n}\right) a_n = 0$

Since $D^m a_n = a_{n+m}$ (for all n and m) .For example, in the event that (3.1) is of the form

$$C_2 a_{n+2} + C_1 a_{n+1} + C_0 a_n = 0 \tag{3.2}$$

Then

$$\left(C_2 D^2 + C_1 D + C_0\right) a_n = 0 \tag{3.3}$$

And if the quadratic in D has the real roots r_1 and r_2 then it is also true that

$$\left(D - C_1\right)\left(D - r_2\right) a_n = 0$$

Since r_1 and r_2 are roots of Equation (3.3), $d_1 r_1^n$ and $d_2 r_2^n$ are solutions to (3.1), where d_1 and d_2 are arbitrary constants. This can be verified by the substitution of $d_1 r_1^n$ and $d_2 r_2^n$ into (3.2), where $a_n = d_1 r_1^n$ or $d_2 r_2^n$. For example, letting $a_n = d_1 r_1^n$ we have, upon substitution in (3.2)

$$c_2 d_1 r_1^{n+2} + c_1 d_1 r_1^{n+1} + c_0 d_1 r_1^n = 0$$
$$d_1 r_1^n (c_2 r_1^2 + c_1 r_1 + c_0) = 0$$

Which is of the form of (3.3). Similarly, one can show that $a_n = d_2 r_2^n$ is a solution and hence that their sum $a_n = d_1 r_1^n + d_2 r_2^n$ is also a solution. It can be shown in a manner similar to that used for ordinary linear differential equations that this sum is the most general solution.

CONCLUSION

A system with two statistically identical components, each with an exponential failure law with parameter μ only one component is required to be operative for the system. The other component is kept it as spare one. When the operative component is failed, then it is replaced by the spare. Assume that the spare component does not fail and that the failure detection and switching equipment is perfect. This method also uses the Lagrange's method to interpolate the missing vlue of repair rates of the system wherever required in computation. Results thus obtained are found to be efficient for studying the steady-state behaviour of stochastic difference equations using generating functions.

REFERENCES

1. M.T.Abulima'atti and L.S.Qamber, reliability and Availability analysis of some system with common cause failure using SPICE circuit simulation program. Active and Passive electronic components. 22(1999) 31-49.
2. Cox. D (1995.The analysis of non-markovian stochastic processes by inclusion of supplymentary variable. Cambridge Univ. Press.51(3) (1954) 433-441.
3. P.Gupta A.K.Lal, R.K.Sharma and J.Singh (2005). Numerical analysis of reliability and availability of the seriel processesin butter-oil processing plant, International journal of quality & reliability management. 22(3), 303-316.

4. C.E.Ebeling, An Introduction to reliability and maintainability Engineering, Tata McGraw-Hill Education, 2004.
5. M.N .Gopalan and U.D. Kumar , On the Transient behaviour of a repairable system with warm standby, Microelectron reliability, 36(4) (19960 525-532.
6. Y.Narahari and N.Wisvanathan, Transient Analysis of manufacturing system performance, IEEE Trans, Robot, Automat, 10(1994), 230-224.
7. A.L.Reibman and K.S.Trivedi, Transient analysis of cumulative measure of markov chain behaviour, Stochastic models. 5(4),(1989)683-710.
8. V.Sridharan and P.Mohanavadivu, Reliability and availability analysis for two non-identical unit parallel system with common cause failure and human errors, Microelectron. Reliability. 37(5) (1997) 747-752.
9. B.Tombuyses and J.Devooght, solving Markovian system of ODE for availability and reliability calculation, reliability Engineering and system safety. 48(1995) 47-55.

Source of support: Nil, Conflict of interest: None Declared