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A STUDY OF AN n-NORM ON l^p SPACE

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ABSTRACT

As, we know that, every findings in Mathematics are valuable for review and reanalysis. Here, we reviewed some n-norms defined on l^p, and introduced a new n-norms on l^p. Which contains some different properties than others.

Keywords: *2-normed spaces, n-normed spaces, Cauchy sequence, completeness.*

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1. INTRODUCTION

The theory of 2-normed spaces was initially developed by Gähler [1] in the mid of 1960's. After that, theory of 2-normed spaces was generalized to n- normed spaces and studied by Misiak [6], A. Malćeski [5], H. Gunawan [2],[3],[4] and so many others.

Let X be a vector space over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ of dimension d ≥ n. A non-negative real valued function $\|.,.,\dots,.\|$ on X^n satisfying the four conditions:

(N_1) $\|x^1,x^2,\dots,x^n\| = 0$ iff $x^1,x^2,\dots,x^n$ are linearly dependent ;
(N_2) $\|x^1,x^2,\dots,x^n\|$ is invarient under permutation of $x^1,x^2,\dots,x^n$;
(N_3) $\|\alpha x^1,x^2,\dots,x^n\| = |\alpha|\|x^1,x^2,\dots,x^n\|$;
(N_4) $\|x^1+y,x^2,\dots,x^n\| \le \|x^1,x^2,\dots,x^n\| + \|y,x^2,\dots,x^n\|$; $\forall\, x^1,x^2,\dots,x^n,y \in X$ and $\forall \alpha \in \mathbb{K}$
is *called an n-norm* on X , and the pair $(X, \|.,.,\dots,.\|)$ is *called an n-normed space.*

Example 1.1: Taking $\boldsymbol{X} = \mathbb{R}^{\boldsymbol{n}}$, let $\boldsymbol{x^i} = \langle x_0^i\ ,x_1^i,x_2^i,x_3^i,\dots,x_{n-1}^i\rangle$; $i = 1,2,\dots,n$ if we define $\|\boldsymbol{x^1},\boldsymbol{x^2},\dots,\boldsymbol{x^n}\| = \left|\boldsymbol{det}\,(x_j^i)\right|$ then $\|\boldsymbol{x^1},\boldsymbol{x^2},\dots,\boldsymbol{x^n}\|$ forms an n-norm on $\mathbb{R}^{\boldsymbol{n}}$.

1.2. Here we shall study the Banach space $\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$, $1 \le p < \infty$; where

$$\boldsymbol{l^p} = \left\{\boldsymbol{x} = (x_i)_{i=0}^{\infty} \left|\sum_{i=0}^{\infty}|x_i|^p < \infty \text{ and } x_i \in \mathbb{K}, \forall\, i = \{0,1,2,3,\dots\}\right.\right\}$$

With norm

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=0}^{\infty}|x_i|^p\right)^{1/p}.$$

Again, $(\boldsymbol{l^p}, \|\,.\,\|_\infty)$ forms a normed space, where $\|\boldsymbol{x}\|_\infty = \sup_{0\le i<\infty}|x_i|$.

Let $\boldsymbol{x^1},\boldsymbol{x^2},\dots,\boldsymbol{x^h}$are **h**-vectors in $\boldsymbol{l^p}$, if we define $\boldsymbol{u} = \left(\boldsymbol{x^1}.\boldsymbol{x^2}\dots\boldsymbol{x^h}\right)$ as term wise multiplication of these **h**-vectors, that is $\boldsymbol{u} = (u_i)_{i=0}^{\infty} = \left(x_i^1.x_i^2\dots x_i^h\right)_{i=0}^{\infty}$; where $\boldsymbol{x^j} = \left(x_i^j\right)_{i=0}^{\infty}$; $j = 1,2,\dots,h$.

Now from simple calculation, we can show that $\|\boldsymbol{u}\|_p \le \|\boldsymbol{x^1}\|_p.\|\boldsymbol{x^2}\|_p\dots\left\|\boldsymbol{x^h}\right\|_p$ as well as $\|\boldsymbol{u}\|_p \le \|\boldsymbol{x^{\pi_1}}\|_p.\|\boldsymbol{x^{\pi_2}}\|_\infty\dots\|\boldsymbol{x^{\pi_h}}\|_\infty$; Where $\boldsymbol{\pi_1},\boldsymbol{\pi_2},\dots,\boldsymbol{\pi_h}$ is a permutation of$1,2,\dots,h$.

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1.3. Let us consider the set {0, 1, 2, 3, 4, 5 . . .} of whole no. as a sequence –
$\mathbb{N} = < 0,1,2,3,4, \dots > = < l >_0^\infty$.

Here we shall denote the sequence $\mathbb{N} = < l >_0^\infty$ in the form of *two consecutive terms notation* as –
$\mathbb{N} = < 0,1,2,3, \dots, 2l, 2l+1, \dots > = < 2l, 2l+1 >_{l=0}^\infty$

We shall express
$\mathbb{N} = < 2l, 2l+1 >_{l=0}^\infty$ as –
$\mathbb{N} = < 2.0 = 0 , 2.0+1 = 1 , 2.1 = 2 , 2.1+1 = 3 , 2.2 = 4 , 2.2+1 = 5 , \dots >$

We shall denote $\overline{\overline{\mathbb{N}}} = < \overline{\overline{m}}_{2k} , \overline{\overline{m}}_{2k+1} >_{k=0}^\infty$ as a rearrangement of the sequence $\mathbb{N} = < 2l, 2l+1 >_{l=0}^\infty$.

Similarly, for any $x \in l^p$, $x = (x_i)_{i=0}^\infty$, we denote it as- $x = (x_{2l} , x_{2l+1})_{l=0}^\infty$ and express as-

$$x = \langle x_{2.0} , x_{2.0+1} , x_{2.1} , x_{2.1+1} , \dots , x_{2l} , x_{2l+1} , \dots \rangle = \langle x_0, x_1, x_2, x_3, \dots , x_{2l}, x_{2l+1}, \dots \rangle.$$

1.4. Parallel rearranged sequences: Let $\boldsymbol{x^1} , \boldsymbol{x^2} \in \boldsymbol{l^p}$; where
$\boldsymbol{x^1} = \langle x_{2l}^1, x_{2l+1}^1 \rangle_{l=0}^\infty$ and $\boldsymbol{x^2} = \langle x_{2l}^2, x_{2l+1}^2 \rangle_{l=0}^\infty$.

Now, related to $\boldsymbol{x^1}, \boldsymbol{x^2} \in \boldsymbol{l^p}$ and corresponding to $\overline{\overline{\mathbb{N}}} = < \overline{\overline{m}}_{2k} , \overline{\overline{m}}_{2k+1} >_{k=0}^\infty$ we define –

$$\overline{\overline{\boldsymbol{x}}}^1 = \langle x_{\overline{\overline{m}}_0}^1 , x_{\overline{\overline{m}}_1}^1, x_{\overline{\overline{m}}_2}^1, x_{\overline{\overline{m}}_3}^1, \dots , x_{\overline{\overline{m}}_{2k}}^1, x_{\overline{\overline{m}}_{2k+1}}^1, \dots \rangle = \langle x_{\overline{\overline{m}}_{2k}}^1, x_{\overline{\overline{m}}_{2k+1}}^1 \rangle_{k=0}^\infty$$

and

$$\overline{\overline{\boldsymbol{x}}}^2 = \langle x_{\overline{\overline{m}}_0}^2 , x_{\overline{\overline{m}}_1}^2, x_{\overline{\overline{m}}_2}^2, x_{\overline{\overline{m}}_3}^2, \dots , x_{\overline{\overline{m}}_{2k}}^2, x_{\overline{\overline{m}}_{2k+1}}^2, \dots \rangle = \langle x_{\overline{\overline{m}}_{2k}}^2, x_{\overline{\overline{m}}_{2k+1}}^2 \rangle_{k=0}^\infty .$$

Then we say ; $\overline{\overline{x}}^1, \overline{\overline{x}}^2$ are parallel rearrangements of the sequences x^1 , x^2 respectively.

Next, let us define a series, corresponding to parallel rearranged sequences $\overline{\overline{x}}^1, \overline{\overline{x}}^2$ as –

$$\sum_{k=0}^{\infty} \left| det \begin{pmatrix} x_{\overline{\overline{m}}_{2k}}^1 & x_{\overline{\overline{m}}_{2k+1}}^1 \\ x_{\overline{\overline{m}}_{2k}}^2 & x_{\overline{\overline{m}}_{2k+1}}^2 \end{pmatrix} \right|^p = \left(\left| det \begin{pmatrix} x_{\overline{\overline{m}}_0}^1 & x_{\overline{\overline{m}}_1}^1 \\ x_{\overline{\overline{m}}_0}^2 & x_{\overline{\overline{m}}_1}^2 \end{pmatrix}_{k=0} \right|^p + \left| det \begin{pmatrix} x_{\overline{\overline{m}}_2}^1 & x_{\overline{\overline{m}}_3}^1 \\ x_{\overline{\overline{m}}_2}^2 & x_{\overline{\overline{m}}_3}^2 \end{pmatrix}_{k=1} \right|^p + \dots \right).$$

Now by Minkowski Inequality, it is clear that;

$$\left(\sum_{k=0}^{\infty} \left| det \begin{pmatrix} x_{\overline{\overline{m}}_{2k}}^1 & x_{\overline{\overline{m}}_{2k+1}}^1 \\ x_{\overline{\overline{m}}_{2k}}^2 & x_{\overline{\overline{m}}_{2k+1}}^2 \end{pmatrix} \right|^p \right)^{1/p} = \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 - x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{1/p}$$

$$\leq \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 \right|^p \right)^{1/p} + \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{1/p}. \quad (1)$$

(Provided $\left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 \right|^p \right)^{1/p}, \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{1/p} exist.$)

But taking $u = \langle x_{\overline{\overline{m}}_{2k}}^1 \rangle_{k=0}^\infty$ and $v = \langle x_{\overline{\overline{m}}_{2k+1}}^2 \rangle_{k=0}^\infty$, we see that u and v are rearrangements of some subsequences of x^1 and x^2 respectively and therefore, $\|u\|_p \leq \|x^1\|_p$ and $\|v\|_p \leq \|x^2\|_p$.

Next taking $w = u.v = \left(x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 \right)_{k=0}^\infty$,

We have -

$$\|w\|_p = \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 \right|^p \right)^{1/p} \leq \|u\|_p . \|v\|_p \leq \|x^1\|_p . \|x^2\|_p . \quad (2)$$

Similarly, $\left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{1/p} \leq \|x^1\|_p . \|x^2\|_p$. (3)

Now using (2) and (3) in (1), we get,

$$\left(\sum_{k=0}^{\infty} \left| det \begin{pmatrix} x_{\overline{\overline{m}}_{2k}}^1 & x_{\overline{\overline{m}}_{2k+1}}^1 \\ x_{\overline{\overline{m}}_{2k}}^2 & x_{\overline{\overline{m}}_{2k+1}}^2 \end{pmatrix} \right|^p \right)^{\frac{1}{p}} = \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 - x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k}}^1 . x_{\overline{\overline{m}}_{2k+1}}^2 \right|^p \right)^{\frac{1}{p}} + \left(\sum_{k=0}^{\infty} \left| x_{\overline{\overline{m}}_{2k+1}}^1 . x_{\overline{\overline{m}}_{2k}}^2 \right|^p \right)^{\frac{1}{p}}$$

$$\leq \|\boldsymbol{x^1}\|_p . \|\boldsymbol{x^2}\|_p + \|\boldsymbol{x^1}\|_p . \|\boldsymbol{x^2}\|_p = \boldsymbol{2}\|\boldsymbol{x^1}\|_p . \|\boldsymbol{x^2}\|_p .$$

Thus, for any arbitrary parallel rearranged sequences $\bar{\bar{x}}^1, \bar{\bar{x}}^2$ of x^1 and x^2 respectively, we have –

$$\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\bar{\bar{m}}_{2k}} & x^1_{\bar{\bar{m}}_{2k+1}} \\ x^2_{\bar{\bar{m}}_{2k}} & x^2_{\bar{\bar{m}}_{2k+1}} \end{pmatrix}\right|^p\right)^{1/p} \leq 2\|x^1\|_{\mathrm{p}}.\|x^2\|_{\mathrm{p}} \tag{4}$$

We will denote,

$$|\bar{\bar{x}}^1, \bar{\bar{x}}^2| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\bar{\bar{m}}_{2k}} & x^1_{\bar{\bar{m}}_{2k+1}} \\ x^2_{\bar{\bar{m}}_{2k}} & x^2_{\bar{\bar{m}}_{2k+1}} \end{pmatrix}\right|^p\right)^{\frac{1}{p}} \leq 2\|x^1\|_{\mathrm{p}}.\|x^2\|_{\mathrm{p}} \tag{5}$$

Similarly, we can also show that $|\bar{\bar{x}}^1, \bar{\bar{x}}^2| \leq 2\|x^{\pi_1}\|_{\mathrm{p}}.\|x^{\pi_2}\|_{\infty}$; where π_1, π_2 is a permutation of 1, 2.

Example 1.5: Let us take

$$\boldsymbol{x^1} = \langle 1, 0, 0, 0, \ldots, 0, 0, \ldots \rangle = \langle \delta^0_{2l}, \delta^0_{2l+1} \rangle_{l=0}^{\infty}$$

And

$$\boldsymbol{x^2} = \langle 0, 0, 4, 0, \ldots, 0, 0, \ldots \rangle = \langle 4.\delta^2_{2l}, 4.\delta^2_{2l+1} \rangle_{l=0}^{\infty} = 4\langle \delta^2_{2l}, \delta^2_{2l+1} \rangle_{l=0}^{\infty} .$$

Taking $\bar{\bar{\mathbb{N}}} = \mathbb{N}$, we have $\boldsymbol{\bar{\bar{x}}^1} = \boldsymbol{x^1}$ and $\boldsymbol{\bar{\bar{x}}^2} = \boldsymbol{x^2}$, then

$$|\boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2}| = |\boldsymbol{x^1}, \boldsymbol{x^2}| = \left(\sum_{l=0}^{\infty}\left|det\begin{pmatrix} x^1_{2l} & x^1_{2l+1} \\ x^2_{2l} & x^2_{2l+1} \end{pmatrix}\right|^p\right)^{1/p} = 0$$

Again taking $\bar{\bar{\mathbb{N}}}' = < 0, 2, 1, 3, 4, 5, 6, 7, \ldots > = < \bar{\bar{m}}'_{2k}, \bar{\bar{m}}'_{2k+1} >_{k=0}^{\infty}$, then corresponding to $\bar{\bar{\mathbb{N}}}'$, parallel rearranged sequences are given by –

$$\bar{\bar{x}}'^1 = \langle 1, 0, 0, 0, \ldots, 0, 0, \ldots \rangle \text{ and } \bar{\bar{x}}'^2 = \langle 0, 4, 0, 0, \ldots, 0, 0, \ldots \rangle$$

Then we have –

$$\left|\bar{\bar{x}}'^1, \bar{\bar{x}}'^2\right| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\bar{\bar{m}}'_{2k}} & x^1_{\bar{\bar{m}}'_{2k+1}} \\ x^2_{\bar{\bar{m}}'_{2k}} & x^2_{\bar{\bar{m}}'_{2k+1}} \end{pmatrix}\right|^p\right)^{1/p} = 4.$$

In 1997, A. Malćeski [5] studied, l^{∞} as n-normed spaces and proved the following lemma:

Lemma 1.6: Any h vectors $x^j = \left(x^j_i\right)_{i=1}^{\infty} \in l^{\infty}$, j = 1, 2, ... , h ; h ∈ N , are linearly dependent iff :

$$\begin{vmatrix} x^1_{i_1} & x^1_{i_2} & \ldots & x^1_{i_h} \\ x^2_{i_1} & x^2_{i_2} & \ldots & x^2_{i_h} \\ \ldots & \ldots & \ldots & \ldots \\ x^h_{i_1} & x^h_{i_2} & \ldots & x^h_{i_h} \end{vmatrix} = 0$$

For every natural numbers $i_1, i_2, \ldots, i_h \in \mathbb{N}$. (6)

2. $\boldsymbol{l^p}$, as 2-normed space

Let us define a real valued function $\overline{\overline{\|.,.\|}}_{\boldsymbol{p}}$ on $\boldsymbol{l^p}$ x $\boldsymbol{l^p}$ as –

$$\overline{\overline{\|\boldsymbol{x^1}, \boldsymbol{x^2}\|}}_{\boldsymbol{p}} = \sup\{|\boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2}| : \boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2} \text{ are parallel rearranged sequences of } \boldsymbol{x^1}, \boldsymbol{x^2} \text{ respectively.}\}$$

$$= \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\bar{\bar{m}}_{2k}} & x^1_{\bar{\bar{m}}_{2k+1}} \\ x^2_{\bar{\bar{m}}_{2k}} & x^2_{\bar{\bar{m}}_{2k+1}} \end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2} \text{ are parallel rearranged sequences}\right\}. \tag{7}$$

Theorem 2.2: The function $\overline{\overline{\|\,.,.\,\|}}_{\boldsymbol{p}}$ defined by **(7)** forms a 2-norm on $\boldsymbol{l^p}$.

Proof: First of all from (4) and (5), we see that for every arbitrary parallel rearranged sequence $\bar{\bar{x}}^1, \bar{\bar{x}}^2$

$$0 \leq |\bar{\bar{x}}^1, \bar{\bar{x}}^2| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\bar{\bar{m}}_{2k}} & x^1_{\bar{\bar{m}}_{2k+1}} \\ x^2_{\bar{\bar{m}}_{2k}} & x^2_{\bar{\bar{m}}_{2k+1}} \end{pmatrix}\right|^p\right)^{1/p} \leq 2\|x^1\|_{\mathrm{p}}.\|x^2\|_{\mathrm{p}}$$

Therefore,

$$0 \leq \overline{\overline{\|x^1, x^2\|}}_p \leq 2\|x^1\|_{\mathrm{p}}.\|x^2\|_{\mathrm{p}}, \text{ for every } x^1, x^2 \in l^p. \tag{8}$$

Thus $\overline{\overline{\| .,. \|}}_p$ is well-defined.

Now to prove, the function $\overline{\overline{\| .,. \|}}_p$ defined by (7) forms a 2-norm on l^p , we have to show that the function $\overline{\overline{\| .,. \|}}_p$ satisfies the four properties of the 2-norm.

(N$_1$) $\overline{\overline{\|x^1 , x^2\|}}_p = 0$ iff x^1 , x^2 are linearly dependent: Let

$$\overline{\overline{\|x^1 , x^2\|}}_p = 0 \Leftrightarrow \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\} = 0$$

$$\Leftrightarrow \left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right| = 0;\ \forall\ \overline{\overline{x}}^1, \overline{\overline{x}}^2\ and\ \forall\ \ \overline{\overline{m}}_{2k}, \overline{\overline{m}}_{2k+1} \in \overline{\overline{\mathbb{N}}}$$

[But $\overline{\overline{\mathbb{N}}} = \langle \overline{\overline{m}}_0, \overline{\overline{m}}_1, \overline{\overline{m}}_2, \dots\rangle$ is arbitrary rearrangement of $\mathbb{N} = <0,1,2,3, \dots>$]

$$\Leftrightarrow \begin{vmatrix} x^1_l & x^1_{l'} \\ x^2_l & x^2_{l'}\end{vmatrix} = 0 ; \forall\, l, l' \in \mathbb{N} \quad \text{[By Lemma 1.6]}$$

$\Leftrightarrow$ x^1 , x^2 are linearly dependent.

(N$_2$) $\overline{\overline{\|x^1 , x^2\|}}_p$ is invariant under permutation of x^1 , x^2: We know that if any two rows (or two columns) of determinant are interchanged the value of new determinant is multiplied by -1. Therefore

$$\overline{\overline{\|x^1 , x^2\|}}_p = \sup\{|\overline{\overline{x}}^1, \overline{\overline{x}}^2| : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences of } x^1, x^2 \text{ respectively.}\}$$

$$= \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\}$$

$$= \sup\left\{\left(\sum_{k=0}^{\infty}\left|-det\begin{pmatrix} x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}} \\ x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\}$$

$$= \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}} \\ x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^2, \overline{\overline{x}}^1 \text{ are parallel rearranged sequences}\right\}$$

$$= \overline{\overline{\|x^2 , x^1\|}}_p.$$

(N$_3$) $\overline{\overline{\|\alpha x^1 , x^2\|}}_p = |\alpha|\overline{\overline{\|x^1 , x^2\|}}_p\ \forall \alpha \in \mathbb{K}$ and $\forall x^1, x^2 \in l^p$:

Consider

$\boldsymbol{x^1} = \langle x^1_0 , x^1_1, x^1_2, x^1_3, \dots, x^1_{2l}, x^1_{2l+1}, \dots\rangle = \langle x^1_{2l}, x^1_{2l+1}\rangle_{l=0}^{\infty}$

Then, $\boldsymbol{\alpha x^1} = \boldsymbol{\alpha}\langle x^1_{2l}, x^1_{2l+1}\rangle_{l=0}^{\infty}$, therefore for every $\overline{\overline{\mathbb{N}}} =< \overline{\overline{m}}_{2k} , \overline{\overline{m}}_{2k+1} >_{k=0}^{k=\infty}$, it is obvious that –

$\overline{\overline{\boldsymbol{\alpha x}}}^1 = \langle \alpha x^1_{\overline{\overline{m}}_0} , \alpha x^1_{\overline{\overline{m}}_1}, \alpha x^1_{\overline{\overline{m}}_2}, \alpha x^1_{\overline{\overline{m}}_3}, \dots, \alpha x^1_{\overline{\overline{m}}_{2k}}, \alpha x^1_{\overline{\overline{m}}_{2k+1}}, \dots\rangle = \boldsymbol{\alpha}\langle x^1_{\overline{\overline{m}}_{2k}}, x^1_{\overline{\overline{m}}_{2k+1}}\rangle_{k=0}^{\infty} = \boldsymbol{\alpha}\overline{\overline{\boldsymbol{x}}}^{\boldsymbol{1}}$.

Again, we know that if all the elements of one row (or one column) of a determinant are multiplied by a scalar α then the value of new determinant is α times the value of the original determinant.

Now by definition,

$$\overline{\overline{\|\alpha x^1 , x^2\|}}_p = \sup\{|\overline{\overline{\alpha x}}^1, \overline{\overline{x}}^2| : \overline{\overline{\alpha x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences of } \alpha x^1 , x^2 \text{ respectively.}\}$$

$$= \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} \alpha x^1_{\overline{\overline{m}}_{2k}} & \alpha x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \alpha\overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\}$$

$$= \sup\left\{|\alpha|\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\}$$

$$= |\alpha| \sup\left\{\left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{2k}} & x^1_{\overline{\overline{m}}_{2k+1}} \\ x^2_{\overline{\overline{m}}_{2k}} & x^2_{\overline{\overline{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{\frac{1}{p}} : \overline{\overline{x}}^1, \overline{\overline{x}}^2 \text{ are parallel rearranged sequences}\right\}$$

$$= |\alpha|\overline{\overline{\|x^1 , x^2\|}}_p$$

(N$_4$) $\overline{\overline{\|x+y,z\|_p}} \le \overline{\overline{\|x,z\|_p}} + \overline{\overline{\|y,z\|_p}}\ \forall x,y,z \in l^p$: Let $x = (x_{2l}, x_{2l+1})_{l=0}^{\infty}$, $y = (y_{2l}, y_{2l+1})_{l=0}^{\infty}$, and $z = (z_{2l}, z_{2l+1})_{l=0}^{\infty}$ then for any $\bar{\bar{\mathbb{N}}} = < \bar{\bar{m}}_{2k}, \bar{\bar{m}}_{2k+1} >_{k=0}^{\infty}$, we have

$$\begin{aligned}\overline{\overline{x+y}} &= \left(x_{\bar{\bar{m}}_{2k}} + y_{\bar{\bar{m}}_{2k}}, x_{\bar{\bar{m}}_{2k+1}} + y_{\bar{\bar{m}}_{2k+1}}\right)_{k=0}^{\infty}\\ &= \left(x_{\bar{\bar{m}}_{2k}}, x_{\bar{\bar{m}}_{2k+1}}\right)_{k=0}^{\infty} + \left(y_{\bar{\bar{m}}_{2k}}, y_{\bar{\bar{m}}_{2k+1}}\right)_{k=0}^{\infty}\\ &= \bar{\bar{x}} + \bar{\bar{y}}\end{aligned}$$

Again by the property of determinant, we have –

$$det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21} & a_{22}\end{pmatrix} = det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} + det\begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22}\end{pmatrix} \text{ using these results, we have}$$

$$\begin{aligned}\left|\overline{\overline{x+y}}, \bar{\bar{z}}\right| &= \left(\sum_{k=0}^{\infty} \left|det\begin{pmatrix} x_{\bar{\bar{m}}_{2k}} + y_{\bar{\bar{m}}_{2k}} & x_{\bar{\bar{m}}_{2k+1}} + y_{\bar{\bar{m}}_{2k+1}} \\ z_{\bar{\bar{m}}_{2k}} & z_{\bar{\bar{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{1/p}\\ &\le \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x_{\bar{\bar{m}}_{2k}} & x_{\bar{\bar{m}}_{2k+1}} \\ z_{\bar{\bar{m}}_{2k}} & z_{\bar{\bar{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{1/p} + \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} y_{\bar{\bar{m}}_{2k}} & y_{\bar{\bar{m}}_{2k+1}} \\ z_{\bar{\bar{m}}_{2k}} & z_{\bar{\bar{m}}_{2k+1}}\end{pmatrix}\right|^p\right)^{1/p}\end{aligned}$$

(By Minkowski inequality).

i. e. $\left|\overline{\overline{x+y}}, \bar{\bar{z}}\right| \le |\bar{\bar{x}}, \bar{\bar{z}}| + |\bar{\bar{y}}, \bar{\bar{z}}|$ for every $\bar{\bar{\mathbb{N}}} = < \bar{\bar{m}}_{2k}, \bar{\bar{m}}_{2k+1} >_{k=0}^{\infty}$

Therefore, taking supremum over $\bar{\bar{\mathbb{N}}}$ on both sides; we get –

$$\overline{\overline{\|x+y,z\|_p}} \le \overline{\overline{\|x,z\|_p}} + \overline{\overline{\|y,z\|_p}}.$$

Hence, the function $\overline{\overline{\|.,.\|_p}}$ defined by (7) forms a 2-norm on l^p.

3. ***l^p*, as *n*-normed space:**

Let $\bar{\bar{\mathbb{N}}} = < \boldsymbol{\bar{\bar{m}}_{nk}}, \boldsymbol{\bar{\bar{m}}_{nk+1}}, \dots, \boldsymbol{\bar{\bar{m}}_{nk+(n-1)}} >_{\boldsymbol{k=0}}^{\boldsymbol{\infty}}$ is a rearrangement of $\mathbb{N}$, written in the form of *n-consecutive terms notation*. Let $\boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2}, \dots, \boldsymbol{\bar{\bar{x}}^n}$ are parallel rearrangements of $\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}$ respectively. As we studied above, in same manner, we define:

$$|\boldsymbol{\bar{\bar{x}}^1}, \boldsymbol{\bar{\bar{x}}^2}, \dots, \boldsymbol{\bar{\bar{x}}^n}| = \left(\sum_{\boldsymbol{k=0}}^{\infty}\left|\boldsymbol{det}\begin{pmatrix} \boldsymbol{x^1_{\bar{\bar{m}}_{nk}}} & \boldsymbol{x^1_{\bar{\bar{m}}_{nk+1}}} & \dots & \boldsymbol{x^1_{\bar{\bar{m}}_{nk+(n-1)}}} \\ \boldsymbol{x^2_{\bar{\bar{m}}_{nk}}} & \boldsymbol{x^2_{\bar{\bar{m}}_{nk+1}}} & \dots & \boldsymbol{x^2_{\bar{\bar{m}}_{nk+(n-1)}}} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{x^n_{\bar{\bar{m}}_{nk}}} & \boldsymbol{x^n_{\bar{\bar{m}}_{nk+1}}} & \dots & \boldsymbol{x^n_{\bar{\bar{m}}_{nk+(n-1)}}}\end{pmatrix}\right|^{\boldsymbol{p}}\right)^{\boldsymbol{\frac{1}{p}}} \tag{9}$$

3.1: Next, let us define a function $\overline{\overline{\|.,.,\dots,.\|_p}}$ on l^p x l^p x ... xl^p (n-times) as –

$$\overline{\overline{\|x^1, x^2, \dots, x^n\|_p}} = \sup\left\{|\bar{\bar{x}}^1, \bar{\bar{x}}^2, \dots, \bar{\bar{x}}^n| : \bar{\bar{x}}^1, \bar{\bar{x}}^2, \dots, \bar{\bar{x}}^n \text{ are parallel rearrangements of } x^{i'} \text{s resp.}\right\}. \tag{10}$$

Now, in same manner, as discussed above, we show that $\overline{\overline{\|.,.,\dots,.\|_p}}$ forms an *n-norm* on l^p. Next, as we know that expansion of a determinant of order 'n' consists of sum of $\lfloor n$ terms, among which each term is again a product of n terms; therefore from (9) and using Minkowski inequality, we obtain following results:

$$\begin{aligned}&\overline{\overline{\|x^1, x^2, \dots, x^n\|_p}} \le \lfloor n\, \|x^1\|_p . \|x^2\|_p \dots \|x^n\|_p;\\ &\overline{\overline{\|x^1, x^2, \dots, x^n\|_p}} \le \lfloor n \|x^{\pi_1}\|_p . \|x^{\pi_2}\|_\infty \dots \|x^{\pi_n}\|_\infty;\\ &\overline{\overline{\|x^1, x^2, \dots, x^n\|_p}} \le \lfloor n \|x^{\pi_1}\|_p . \|x^{\pi_2}\|_{p/\infty} \dots \|x^{\pi_n}\|_{p/\infty}\end{aligned} \tag{11}$$

Where; $\boldsymbol{\pi_1}, \boldsymbol{\pi_2}, \dots, \boldsymbol{\pi_n}$ is a permutation of $1,2,\dots,n$; and $\|\boldsymbol{x^{\pi_t}}\|_{\mathbf{p}/\infty}$ means either p-norm or supremum norm of $\boldsymbol{x^{\pi_t}}$ is taken.

3.2: Convergence in n-normed space: Let $(X, \|.,.,\dots,.\|)$ is an *n-normed space.* Then a sequence $(x^l)_{l=0}^{\infty}$ in X is called a Cauchy sequence in X iff $\left\|x^l - x^{l'}, a^1, a^2, \dots, a^{n-1}\right\| \to 0$ as $l, l' \to \infty$ and $\forall a^1, a^2, \dots, a^{n-1} \in X$.

And the sequence $(x^l)_{l=0}^{\infty}$ in X is said to be convergent at $x \in X$ iff –
$\|x^l - x, a^1, a^2, \dots, a^{n-1}\| \to 0$ as $l \to \infty$ and $\forall a^1, a^2, \dots, a^{n-1} \in X$.

The space$(X, \|.,.,\dots,.\|)$ is called an *n-Banach space (or complete n-normed space)* iff every Cauchy sequence converge X.

Theorem 3.3:

(I) If $(x^l)_{l=0}^{\infty}$ is a Cauchy sequence in $(l^p, \|.\|_p)$ then $(x^l)_{l=0}^{\infty}$ is a Cauchy sequence in *n-normed space* $(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$ also.

(II) If $x^l \longrightarrow x \ as\ l \longrightarrow \infty$ in $(l^p, \|.\|_p)$, then $x^l \longrightarrow x \ as\ l \longrightarrow \infty$ in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$.

Proof: From (11), we have $0 \le \overline{\overline{\|a^1, a^2, \dots, a^n\|}}_p \le \lfloor n \, \|a^1\|_p . \|a^2\|_p \dots \|a^n\|_p$, for every $a^1 \ , a^2, \dots, a^n \in l^p$; Therefore, convergence and Cauchy criterion of a sequence in$(l^p, \|.\|_p)$ to *n-normed space* $(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$ preserved.

In [4], H. Gunawan defined a natural *n*-norm on $l^p$and found that convergence and Cauchy criterion of a sequence in$(l^p, \|.\|_p)$ to Gunawan's natural n-normed space preserved and vice-versa. Here a question arises that, "is the converse of theorem 3.3 True? ".

Theorem 3.4: Converse of Theorem3.3 need not be true, That is –

(I) If $(x^r)_{r=0}^{\infty}$ is a Cauchy sequence in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$ then, $(x^r)_{r=0}^{\infty}$ need not be a Cauchy sequence in $(l^p, \|.\|_p)$;

(II) If $(x^r)_{r=0}^{\infty}$ is a convergent sequence, such that $x^r \longrightarrow x$ in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$ then, $(x^r)_{r=0}^{\infty}$ need not be convergent sequence in $(l^p, \|.\|_p)$.

We shall prove above theorem by giving a counter example.

Counter example3.5: let us take a sequence $(x^r)_{r=0}^{\infty}$ inl^p, where –
$x^0 = \langle 0,0, \dots \rangle$ and $x^1 = \langle 0,0, \dots \rangle$ for $r \ge 2$; x^r is defined as –

$x^r = \langle x_{nl}^r, x_{nl+1}^r, \dots, x_{nl+(n-1)}^r \rangle$; Where if $r = 2m$, then $x_i^r = \begin{cases} \frac{1}{m^{1/p}} & for\ i \le m-1 \\ 0 & for\ i \ge m \end{cases}$

Next for, $r = 2m+1$, then $x_i^r = \begin{cases} \frac{-1}{m^{1/p}} & for\ i \le m-1 \\ 0 & for\ i \ge m \end{cases}$ (12)

Let $a^2, \dots, a^n$are arbitrary elements ofl^p, suppose $\varepsilon > 0$ is given, then $\exists N \in \mathbb{N}$, such that

$$\frac{2}{N^{1/p}} < \frac{\varepsilon}{\lfloor n \ (1 + \|a^2\|_p \dots \|a^n\|_p)} \tag{13}$$

Now,$\forall r, s \ \ge 2N$ we have$|x_i^r - x_i^s| \le \frac{2}{N^{1/p}}$, $\forall i \in \mathbb{N}$; therefore$\|x^r - x^s\|_\infty \le \frac{2}{N^{1/p}}$. (14)

Hence from (11), (13) and (14), $\forall r, s \ \ge 2N$ we have:

$$\overline{\overline{\|x^r - x^s, a^2, \dots, a^n\|}}_p \le \lfloor n \, \|x^r - x^s\|_\infty . \|a^2\|_p \dots \|a^n\|_p < \varepsilon \tag{15}$$

That is, $(x^r)_{r=0}^{\infty}$is Cauchy sequence in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$. In similar manner, we can show that $x^r \longrightarrow 0$ in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$.

While, taking $\varepsilon = 1$; then for each $N \in \mathbb{N}$; $\exists\ 2N, 2N+1 > N$ such that$\|x^{2N} - x^{2N+1}\|_p = 2 > \varepsilon$. Which exhibits that,$(x^r)_{r=0}^{\infty}$ is not Cauchy sequence in $(l^p, \|.\|_p)$ hence, not convergent also.

Theorem 3.6: The n-normed spaces, $(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$, is an incomplete space.

Proof: let us take a sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ in$\boldsymbol{l^p}$ such that,

$\boldsymbol{x^r} = \langle x_{nl}^r, x_{nl+1}^r, \dots, x_{nl+(n-1)}^r \rangle$; Where, $x_i^r = \begin{cases} 1 \text{ for } i = 0,1 \\ \frac{1}{i^{1/p}} \text{ for } 1 \le i \le r \\ 0 \text{ for } i \ge r+1 \end{cases}$ (16)

Here, we obtain $x^r - x^s = \langle 0, \dots, 0, \frac{1}{(r+1)^{1/p}}, \dots, \frac{1}{(s)^{1/p}}, 0,0, \dots \rangle$, for $r \le s$. Now, as we studied in above counter example, we can easily find that, above sequence is Cauchy sequence in$(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p)$, but not convergent in$l^p$.
(The convergent point $= \langle 1,1, \frac{1}{2^{1/p}}, \frac{1}{3^{1/p}}, \dots, \frac{1}{i^{1/p}}, \dots \rangle \notin l^p$.)

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