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GENERATING RELATIONS INVOLVING GENERALIZED ZERNIKE OR DISK POLYNOMIALS

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ABSTRACT

Generating relations play a large role in the study of special functions. The present paper deals with the derivation of some novel generating relations of generalized Zernike or disc polynomials by the application of group-theoretic method introduced by Louis Weisner .In fact , by suitably interpreting the indices (m) and (n) of the polynomials under consideration we define six linear partial differential operators and on showing that they generate a Lie algebra , we obtain a new generating relation (2.4) as the main result of our investigation. Furthermore , some generating relations of Laguerre 2D polynomials $L_{m,n}(z,z^*)$ Jacobi polynomials $P_n^{(\alpha,\beta)}(u)$ and ${}_2F_1(a,b;c;x)$ the hyper geometric function are obtained as the special cases of our main result.

Keywords and phrases: *Generalized Zernike or dick Polynomials, Generating relations.*

AMS Subject Classification: *33C45, 33C50, 33C80.*

1. INTRODUCTION

The generalized Zernike or disc polynomials $P_n^{(\alpha,\beta)}(z,z^*),(m,n=0,1,2,...)$ defined by Wunsche [13,p.137(2.1)] are orthogonal 2D polynomials in the unit disc $0 \le \sqrt{zz^*} < 1$ with weights $(1-zz^*)^{\alpha}$ in complex coordinates $z \equiv x+iy$, $z^* \equiv x-iy$, where $\alpha > -1$ is a real free parameter which are defined as the following (agrees with Dunkle and Xu [5] and up to standardization with Koornwinder [8])

$$\begin{aligned} P_{m,n}^{\alpha}(z,z^*) &= \frac{n!\alpha!}{(n+\alpha)!} z^{m-n} P_n^{(\alpha,m-n)}(2zz^*-1) \\ &= \frac{m!\alpha!}{(m+\alpha)!} z^{*n-m} P_m^{(\alpha,n-m)}(2zz^*-1) \\ &= z^m z^{*n} {}_2F_1\left(-m,-n;\alpha+1;1-\frac{1}{zz^*}\right), \ (m,n=0,1,2,...) \end{aligned} \tag{1.1}$$

where $P_n^{(\alpha,\beta)}(u)$ denotes the Jacobi polynomials and ${}_2F_1(a,b;c;x)$ the hyper geometric function(e.g., [1,2,6,9,10]). And it is related to Laguerre 2D polynomials $L_{m,n}(z,z^*)$ by the following relation [13, p.140 (2.14)]

$$L_{m,n}(z,z^*) = \lim_{|\alpha|\to\infty} \left(\sqrt{\alpha}\right)^{m+n} P_{m,n}^{\alpha}\left(\frac{z}{\sqrt{\alpha}},\frac{z^*}{\sqrt{\alpha}}\right) \tag{1.2}$$

These polynomials defined by (1.1) satisfy the following simultaneous partial differential equations

$$\begin{aligned} & s^2 zz^* \frac{\partial^2 w}{\partial s^2} + (1-zz^*)\left[sz\frac{\partial^2}{\partial z \partial s} - sz^* \frac{\partial^2}{\partial s \partial z^*} - (1-zz^*)\frac{\partial^2}{\partial z \partial z^*}\right] w + (2zz^* + \alpha zz^* - 1) s \frac{\partial w}{\partial s} \\ & + (1+\alpha) z \left[z^* + \frac{(1-zz^*)}{s} + (1-zz^*)\frac{\partial}{\partial z}\right] w - m(m+\alpha) w = 0 \end{aligned} \tag{1.3}$$

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and

$$\begin{aligned} & t^2 zz^*\frac{\partial^2 w}{\partial t^2}+(1-zz^*)\left[tz^*\frac{\partial^2}{\partial z^*\partial t}-tz\frac{\partial^2}{\partial t\partial z}-(1-zz^*)\frac{\partial^2}{\partial z\partial z^*}\right]w+(2zz^*+\alpha zz^*-1)t\frac{\partial w}{\partial t} \\ & +(1+\alpha)z^*\left[z+\frac{(1-zz^*)}{t}+(1-zz^*)\frac{\partial}{\partial z^*}\right]w-n(n+\alpha)w=0 \end{aligned} \tag{1.4}$$

More recent representations of the Zernike polynomials with application to the optical aberrations and with graphical representations are given in [7, 14, 15] . This is carefully summarized in Weisstein's Encyclopedia [12].

The generalized Zernike or disc polynomials may find multiple applications in cases when one has to do with given functions over a circular disc and when one wants to make expansions into an orthonormalized set of functions over this disc , in particular , in geometrical and wave optics for systems with circular apertures [3,4,7,15], for example , in Kirchoff's diffraction integrals where one has to insert the field and its first derivative within the aperture.

The aim at presenting the article is to apply L.Weisner's group theoretic method [11] with suitable interpretations of the indices (m) and (n) in the study of generalized Zernike or disc polynomials. The principal interest in the given results lies in the fact that a number of special cases listed in section 3 would yield many new results of the theory of special functions.

2. GROUP-THEORETIC METHOD

Replacing m by $s\frac{\partial}{\partial s}$ and n by $t\frac{\partial}{\partial t}$ in (1.3) and (1.4) respectively, we get

$$\begin{aligned} & s^2 zz^*\frac{\partial^2 w}{\partial s^2}+(1-zz^*)\left[sz\frac{\partial^2}{\partial z\partial s}-sz^*\frac{\partial^2}{\partial s\partial z^*}-(1-zz^*)\frac{\partial^2}{\partial z\partial z^*}\right]w+(2zz^*+\alpha zz^*-1)s\frac{\partial w}{\partial s} \\ & +(1+\alpha)z\left[z^*+\frac{(1-zz^*)}{s}+(1-zz^*)\frac{\partial}{\partial z}\right]w-s\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}+\alpha\right)w=0 \end{aligned} \tag{2.1}$$

$$\begin{aligned} & t^2 zz^*\frac{\partial^2 w}{\partial t^2}+(1-zz^*)\left[tz^*\frac{\partial^2}{\partial z^*\partial t}-tz\frac{\partial^2}{\partial t\partial z}-(1-zz^*)\frac{\partial^2}{\partial z\partial z^*}\right]w+(2zz^*+\alpha zz^*-1)t\frac{\partial w}{\partial t} \\ & +(1+\alpha)z^*\left[z+\frac{(1-zz^*)}{t}+(1-zz^*)\frac{\partial}{\partial z^*}\right]w-t\frac{\partial}{\partial t}\left(t\frac{\partial}{\partial t}+\alpha\right)w=0 \end{aligned} \tag{2.2}$$

We see that $u(z,z^*,s,t)=P^{\alpha}_{m,n}(z,z^*)s^m t^n$ is a solution of (2.1) and (2.2), since $P^{\alpha}_{m,n}(z,z^*)$ is a solution of (1.3) and (1.4).

We first consider the following first order linear differential operators

$$A_1=s\frac{\partial}{\partial s}+\frac{(\alpha+1)}{2}$$

$$A_2=t\frac{\partial}{\partial t}+\frac{(\alpha+1)}{2}$$

$$A_3=z^*\frac{\partial}{\partial s}+\frac{(1-zz^*)}{s}\frac{\partial}{\partial z}$$

$$A_4=s^2 z\frac{\partial}{\partial s}-(1-zz^*)s\frac{\partial}{\partial z^*}+(1+\alpha)sz$$

$$A_5=z\frac{\partial}{\partial t}+\frac{(1-zz^*)}{t}\frac{\partial}{\partial z^*}$$

$$A_6=t^2 z^*\frac{\partial}{\partial t}-(1-zz^*)t\frac{\partial}{\partial z}+(1+\alpha)tz^*$$

such that

$$A_1[f(z,z^*,s,t)] = \left(m + \frac{(\alpha+1)}{2}\right) P_{m,n}^{\alpha}(z,z^*) s^m t^n$$

$$A_2[f(z,z^*,s,t)] = \left(n + \frac{(\alpha+1)}{2}\right) P_{m,n}^{\alpha}(z,z^*) s^m t^n$$

$$A_3[f(z,z^*,s,t)] = m P_{m-1,n}^{\alpha}(z,z^*) s^{m-1} t^n$$

$$A_4[f(z,z^*,s,t)] = (m+1+\alpha) P_{m+1,n}^{\alpha}(z,z^*) s^{m+1} t^n$$

$$A_5[f(z,z^*,s,t)] = n P_{m,n-1}^{\alpha}(z,z^*) s^m t^{n-1}$$

$$A_6[f(z,z^*,s,t)] = (n+1+\alpha) P_{m,n+1}^{\alpha}(z,z^*) s^m t^{n+1}$$

where the operators $\{A_i : i = 1,2,3,4,5,6\}$ satisfy the following commutation relations

$[A_1, A_2] = 0$ $\quad [A_2, A_3] = -A_3$ $\quad [A_3, A_4] = 2A_1$ $\quad [A_4, A_5] = 0$ $\quad [A_5, A_6] = 2A_2$
$[A_1, A_3] = -A_3$ $\quad [A_2, A_4] = A_4$ $\quad [A_3, A_5] = 0$ $\quad [A_4, A_6] = 0$
$[A_1, A_4] = A_4$ $\quad [A_2, A_5] = -A_5$ $\quad [A_3, A_6] = 0$
$[A_1, A_5] = -A_5$ $\quad [A_2, A_6] = A_6$
$[A_1, A_6] = A_6$

where $[A, B] = AB - BA$.

The above commutation relations show that the set of operators $\{A_i : i = 1,2,3,4,5,6\}$ generate a Lie-algebra λ and the sets of operators $\{\mathrm{A}_1, \mathrm{A}_3, \mathrm{A}_4\}$ and $\{\mathrm{A}_2, \mathrm{A}_5, \mathrm{A}_6\}$ form a sub algebras of λ. It is clear that the differential operators

$$L_1 = s^2 zz^* \frac{\partial^2 w}{\partial s^2} + (1 - zz^*)\left[sz \frac{\partial^2}{\partial z \partial s} - sz^* \frac{\partial^2}{\partial s \partial z^*} - (1 - zz^*) \frac{\partial^2}{\partial z \partial z^*} \right] w$$
$$+ (2zz^* + \alpha zz^* - 1) s \frac{\partial w}{\partial s} + (1+\alpha) z \left[z^* + \frac{(1 - zz^*)}{s} + (1 - zz^*) \frac{\partial}{\partial z} \right] w - s \frac{\partial}{\partial s}\left(s \frac{\partial}{\partial s} + \alpha \right) w = 0$$

and

$$L_2 = t^2 zz^* \frac{\partial^2 w}{\partial t^2} + (1 - zz^*)\left[tz^* \frac{\partial^2}{\partial z^* \partial t} - tz \frac{\partial^2}{\partial t \partial z} - (1 - zz^*) \frac{\partial^2}{\partial z \partial z^*} \right] w$$
$$+ (2zz^* + \alpha zz^* - 1) t \frac{\partial w}{\partial t} + (1+\alpha) z^* \left[z + \frac{(1 - zz^*)}{t} + (1 - zz^*) \frac{\partial}{\partial z^*} \right] w - t \frac{\partial}{\partial t}\left(t \frac{\partial}{\partial t} + \alpha \right) w = 0$$

which can be expressed as:

$L_1 = A_3 A_4 - m(m+\alpha)$ and $L_2 = A_5 A_6 - n(n+\alpha)$

Commutes with $\{A_i : i = 1,2,3,4,5,6\}$ that is

$$\begin{cases} [L_1, A_i] = 0, i = 1,2,3,4,5,6 \\ [L_2, A_i] = 0, i = 1,2,3,4,5,6 \end{cases} \tag{2.3}$$

The extended form of the groups generated by $\{A_i : i = 1,2,3,4,5,6\}$ are as follows:

$$e^{a_1 A_1} f(z, z^*, s, t) = \exp\left[\left(\frac{\alpha+1}{2}\right)a_1\right] . f\left(z, z^*, se^{a_1}, t\right)$$

$$e^{a_2 A_2} f(z, z^*, s, t) = \exp\left[\left(\frac{\alpha+1}{2}\right)a_2\right] . f\left(z, z^*, s, te^{a_2}\right)$$

$$e^{a_3 A_3} f(z, z^*, s, t) = f\left(\frac{1}{z^*}\left(1 - \frac{s(1-zz^*)}{s + a_3 z^*}\right), z^*, s + a_3 z^*, t\right)$$

$$e^{a_4 A_4} f(z, z^*, s, t) = (1 - a_4 sz)^{-(1+\alpha)} f\left(z, \frac{1}{z}\left(1 - \frac{1-zz^*}{1 - a_4 sz}\right), \frac{s}{1 - a_4 sz}, t\right)$$

$$e^{a_5 A_5} f(z, z^*, s, t) = f\left(z, \frac{1}{z}\left(1 - \frac{t(1-zz^*)}{t + a_5 z}\right), s, t + a_5 z\right)$$

$$e^{a_6 A_6} f(z, z^*, s, t) = \left(1 - a_6 tz^*\right)^{-(1+\alpha)} f\left(\frac{1}{z^*}\left(1 - \frac{1-zz^*}{1 - a_6 tz^*}\right), z^*, s, \frac{t}{1 - a_6 tz^*}\right)$$

where $f(z, z^*, s, t)$ is an arbitrary function.

Then we have

$$\begin{aligned} &e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} f(z, z^*, s, t) \\ &\qquad = (\beta)^{-(1+\alpha)} (\beta')^{-(1+\alpha)} \exp\left[\left(\frac{\alpha+1}{2}\right)a_1\right] \exp\left[\left(\frac{\alpha+1}{2}\right)a_2\right] \\ &\qquad \ldots f\left(\frac{1}{\gamma}\left(1 - \frac{\frac{s}{\beta}(1-\gamma z)}{\frac{s}{\beta} + a_3\gamma}\right), \frac{1}{\gamma'}\left(1 - \frac{\frac{t}{\beta'}(1-\gamma' z^*)}{\frac{t}{\beta'} + a_5\gamma'}\right), \left(\frac{s}{\beta} + a_3\gamma\right)e^{a_1}, \left(\frac{t}{\beta'} + a_5\gamma'\right)e^{a_2}\right) \end{aligned} \tag{2.4}$$

where $\beta = 1 - a_4 sz$, $\gamma = \frac{1}{z}\left(1 - \frac{1-zz^*}{1 - a_4 sz}\right)$, $\beta' = 1 - a_6 tz^*$, $\gamma' = \frac{1}{z^*}\left(1 - \frac{1-zz^*}{1 - a_6 tz^*}\right)$

3. GENERATING FUNCTIONS

From the above discussion, we see that $u(z, z^*, s, t) = P_{m,n}^{\alpha}(z, z^*) s^m t^n$ is a solution of the following systems

$$\begin{cases} L_1 u = 0 \\ (A_3 A_4 - m(m+\alpha))u = 0 \end{cases} \qquad \begin{cases} L_2 u = 0 \\ (A_5 A_6 - n(n+\alpha))u = 0 \end{cases}$$

From (2.3), we easily see that

$$SL_1\left(P_{m,n}^{\alpha}(z, z^*) s^m t^n\right) = L_1 S\left(P_{m,n}^{\alpha}(z, z^*) s^m t^n\right)$$

and

$$SL_2\left(P_{m,n}^{\alpha}(z, z^*) s^m t^n\right) = L_2 S\left(P_{m,n}^{\alpha}(z, z^*) s^m t^n\right),$$

where $S = e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}$

Therefore, the transformation $S\left(P_{m,n}^{\alpha}(z, z^*) s^m t^n\right)$ is also annulled by L_1 and L_2.

By setting $\{a_i = 0 : i = 1,2,3,4; a_3 = c, a_4 = b\}$ and writing $f(z, z^*, s, t) = P_{m,n}^{\alpha}(z, z^*) s^m t^n$ in (2.4), we get

$$e^{bA_4}e^{cA_3}\left(P_{m,n}^{\alpha}(z,z*)s^m t^n\right)=(1-bsz)^{-(1+\alpha)}.P_{m,n}^{\alpha}\left(\frac{1}{\gamma}\left(1-\frac{\frac{s}{\beta}(1-\gamma z)}{\frac{s}{\beta}+c\gamma}\right),\gamma\right)\left(\frac{s}{\beta}+c\gamma\right)^m t^n \tag{3.1}$$

But

$$e^{bA_4}e^{cA_3}\left(P_{m,n}^{\alpha}(z,z*)s^m t^n\right)=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{b^k}{k!}\frac{c^l}{l!}\prod_{k=1}^{\infty}(m+\alpha+k)\prod_{l=1}^{\infty}(m-(l-1))P_{m+k-l,n}^{\alpha}(z,z*)s^{m+k-l}t^n \tag{3.2}$$

Combining the above two relations (3.1) and (3.2), we get

$$(1-bsz)^{-(1+\alpha)}\left(\frac{s}{\beta}+c\gamma\right)^m.P_{m,n}^{\alpha}\left(\frac{1}{\gamma}\left(1-\frac{\frac{s}{\beta}(1-\gamma z)}{\frac{s}{\beta}+c\gamma}\right),\gamma\right)$$

$$=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{b^k}{k!}\frac{c^l}{l!}\prod_{k=1}^{\infty}(m+\alpha+k)\prod_{l=1}^{\infty}(m-(l-1))P_{m+k-l,n}^{\alpha}(z,z*)s^{m+k-l} \tag{3.3}$$

where $\beta=1-bsz$, $\gamma=\frac{1}{z}\left(1-\frac{1-zz*}{1-bsz}\right)$, $|b|<\infty$ and $|c|<\infty$.

If we put $b=0, s=1$ in equation (3.3), we get

$$(1+cz*)^m.P_{m,n}^{\alpha}\left(\frac{1}{z*}\left(1-\frac{1-zz*}{1+cz*}\right),z*\right)=\sum_{l=1}^{\infty}\frac{c^l}{l!}\prod_{l=1}^{\infty}(m-(l-1))P_{m-l,n}^{\alpha}(z,z*),\ \text{where } |c|<\infty . \tag{3.4}$$

If we put $c=0, s=1$ in equation (3.3), we get

$$(1-bz)^{-(1+\alpha+m)}.P_{m,n}^{\alpha}\left(z,\frac{1}{z}\left(1-\frac{1-zz*}{1-bz}\right)\right)=\sum_{k=1}^{\infty}\frac{b^k}{k!}\prod_{k=1}^{\infty}(m+\alpha+k)P_{m+k,n}^{\alpha}(z,z*)\ \text{ where } |b|<\infty . \tag{3.5}$$

Again putting $\{a_i=0: i=1,2,3,4; a_5=c', a_6=b'\}$ and writing $f(z,z*,s,t)=P_{m,n}^{\alpha}(z,z*)s^m t^n$ in (2.4), we get

$$e^{b'A_6}e^{c'A_5}\left(P_{m,n}^{\alpha}(z,z*)s^m t^n\right)=(1-b'tz*)^{-(1+\alpha)}.P_{m,n}^{\alpha}\left(\gamma',\frac{1}{\gamma'}\left(1-\frac{\frac{t}{\beta'}(1-\gamma' z*)}{\frac{t}{\beta'}+c'\gamma'}\right)\right)s^m\left(\frac{t}{\beta'}+c'\gamma'\right)^n \tag{3.6}$$

But

$$e^{b'A_6}e^{c'A_5}\left(P_{m,n}^{\alpha}(z,z*)s^m t^n\right)=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{b^k}{k!}\frac{c^l}{l!}\prod_{k=1}^{\infty}(n+\alpha+k)\prod_{l=1}^{\infty}(n-(l-1))P_{m,n+k-l}^{\alpha}(z,z*)s^m t^{n+k-l} \tag{3.7}$$

Combining the above two relations (3.6) and (3.7), we get

$$(1-b'tz*)^{-(1+\alpha)}\left(\frac{t}{\beta'}+c'\gamma'\right)^n P_{m,n}^{\alpha}\left(\gamma',\frac{1}{\gamma'}\left(1-\frac{\frac{t}{\beta'}(1-\gamma' z*)}{\frac{t}{\beta'}+c'\gamma'}\right)\right)$$

$$=\sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\frac{b'^k}{k!}\frac{c'^l}{l!}\prod_{k=1}^{\infty}(n+\alpha+k)\prod_{l=1}^{\infty}(n-(l-1))P_{m,n+k-l}^{\alpha}(z,z*)t^{n+k-l} \tag{3.8}$$

where $\beta'=1-b'tz*$, $\gamma'=\frac{1}{z*}\left(1-\frac{1-zz*}{1-b'tz*}\right)$, $|b'|<\infty$ and $|c'|<\infty$.

If we put $b=0, t=1$ in equation (3.8), we get

$$(1+c'z)^n . P^{\alpha}_{m,n}\left(z, \frac{1}{z}\left(1-\frac{1-zz^*}{1+c'z}\right)\right) = \sum_{l=1}^{\infty} \frac{c^l}{l!} \prod_{l=1}^{\infty} (n-(l-1)) P^{\alpha}_{m,n-l}(z,z^*) \text{ where } |c'| < \infty . \quad (3.9)$$

If we put $c=0, t=1$ in equation (3.8), we get

$$(1-b'z^*)^{-(1+\alpha+n)} . P^{\alpha}_{m,n}\left(\frac{1}{z^*}\left(1-\frac{1-zz^*}{1-b'z^*}\right), z^*\right) = \sum_{k=1}^{\infty} \frac{b^k}{k!} \prod_{k=1}^{\infty} (n+\alpha+k) P^{\alpha}_{m,n+k}(z,z^*) \text{where } |b'| < \infty . \quad (3.10)$$

4. CONCLUSION

We have seen that Weisner's group theoretic method is a power full tool in getting generating functions. It is also interesting to define a new function which forms generalization for the generalized Zernike or disc polynomials under consideration and then by using Lie theoretic technique, we can obtain generating functions. We will deal with this aspect in the subsequent communication.

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