# A COMPARATIVE STUDY OF THE LEFT AND RIGHT EIGENVECTORS <br> OF THE SQUARE MATRICES A AND $A^{T}$ CORRESPONDING TO EACH OF THE EIGENVALUES 

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#### Abstract

In this paper, our aim is to determine numerically the left and right eigenvectors of $A$ and $A^{T}$ corresponding to each of the eigenvalues, where $A$ is a square matrix. Finally, the result shows that the transpose of a right eigenvector of $A$ is a left eigenvector of $A^{T}$ corresponding to the same eigenvalue.


Keywords: Eigenvalues and eigenvectors of a square matrix, Left and right eigenvectors of a square matrix, Square matrix, Transpose of a matrix.

## 1. INTRODUCTION

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application. For example, this problem is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). Other areas such as physics, sociology, biology, economics and statistics have focused considerable attention on "eigenvalues" and "eigenvectors"-their applications and their computations. We know that many physical systems arising in engineering applications can be represented as discrete models involving matrices. Some key parameters describing physical systems (e.g., the resonance frequency) are closely related to eigenvalues of the matrix representing the system. That is why the eigenvalue analysis is ubiquitous in all branches of modern engineering. For example, the natural frequency of the bridge is the eigenvalue of smallest magnitude of a system that models the bridge. The engineers exploit this knowledge to ensure the stability of their constructions. Eigenvalue analysis is also used in the design of car stereo systems, where it helps to reduce the vibration of the car due to the music. In electrical engineering, the application of eigenvalues and eigenvectors is useful for decoupling three-phase systems through symmetrical component transformation.

In section 2 of this paper, we have discussed some basic concepts regarding eigenvalues and eigenvectors required to understand the concepts that are discussed. In section 3, we have presented one example for determining the left and right eigenvectors of $A$ and $A^{T}$ corresponding to each of the eigenvalues. Finally, in section 4, we summarized some concluding remarks that are used in practice.

## 2. PRELIMINARIES

In this section, we recall some basic concepts which would be used in the sequel.
Definition 2.1: A matrix is said to be a square matrix if the number of rows is same as the number of columns.
Definition 2.2: If the rows (or columns) of a matrix are changed to corresponding columns (or rows), the resulting matrix is defined to be the transpose of the original matrix. $A^{T}$ denotes the transpose of the matrix $A$. If $A$ is of size $m \times n$, then $A^{T}$ is of size $n \times m$. Further, if $A=\left(a_{i j}\right)$ and $A^{T}=\left(b_{i j}\right)$, then $b_{i j}=a_{j i}$.

Definition 2.3: Let $A$ be a square matrix of order $n$. The sum of the elements of $A$ lying along the principal diagonal is called the trace of $A$. We shall write the trace of $A$ as $\operatorname{tr} A$. Thus, if $A=\left(a_{i j}\right)_{n \times n}$, then
$\operatorname{tr} A=\sum_{i=1}^{n} a_{i i}=a_{11}+a_{22}+\ldots \ldots \ldots+a_{n n}$.

### 2.4. LEFT AND RIGHT EIGENVECTORS

A nonzero column vector $X$ is an eigenvector (or right eigenvector or right characteristic vector) of a square matrix $A$ if there exists a scalar $\lambda$ such that

$$
\begin{equation*}
A X=\lambda X \tag{1}
\end{equation*}
$$

Then $\lambda$ is an eigenvalue (or characteristic value) of $A$. Eigenvalues may be zero; an eigenvector may not be the zero vector.

The characteristic equation of an $n \times n$ matrix $A$ is the $n$ th-degree polynomial equation

$$
\begin{equation*}
\operatorname{det}(A-\lambda I)=0 \tag{2}
\end{equation*}
$$

Solving the characteristic equation for $\lambda$ gives the eigenvalues of $A$, which may be real, complex, or multiples of each other. Once an eigenvalue is determined, it may be substituted into equation (1), and then that equation may be solved for the corresponding eigenvectors. The polynomial $\operatorname{det}(A-\lambda I)$ is called the characteristic polynomial of $A$.

A left eigenvector of a matrix $A$ is a nonzero row vector $X$ having the property that

$$
X A=\lambda X
$$

or, equivalently, that

$$
\begin{equation*}
X(A-\lambda I)=0 \tag{3}
\end{equation*}
$$

for some scalar $\lambda$. Again $\lambda$ is an eigenvalue for $A$. Once $\lambda$ is determined, it is substituted into equation (3), and then that equation is solved for $X$, i.e., for left eigenvector.

### 2.5. TWO IMPORTANT PROPERTIES OF EIGENVALUES

Property-I: A matrix and its transpose have the same eigenvalues.
Proof: We have,

$$
\begin{aligned}
& (A-\lambda I)^{T}=A^{T}-\lambda I^{T}=A^{T}-\lambda I \\
\therefore & \left|(A-\lambda I)^{T}\right|=\left|A^{T}-\lambda I\right| \\
\Rightarrow & |A-\lambda I|=\left|A^{T}-\lambda I\right| \quad\left[\because\left|\mathrm{B}^{\mathrm{T}}\right|=|B|\right] \\
\therefore & |A-\lambda I|=0 \text { if and only if }\left|\mathrm{A}^{\mathrm{T}}-\lambda I\right|=0
\end{aligned}
$$

i.e., $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^{T}$.

Property-II: The sum of the eigenvalues of a matrix is equal to its trace, which is the sum of the elements on its main diagonal.

## 3. DETERMINATION OF LEFT AND RIGHT EIGENVECTORS OF A AND $A^{T}$ CORRESPONDING TO EACH OF THE EIGENVALUES

Example 1: For our convenience, let us consider the matrix $A=\left[\begin{array}{rr}3 & 5 \\ -2 & -4\end{array}\right]$ to find the left and right eigenvectors of $A$ and $A^{T}$ corresponding to each of the eigenvalues

Solution: Right eigenvectors of the matrix $A$ : To find right eigenvectors of $A$ we need to determine the eigenvalues of $A$ first. Now, to find eigenvalues of $A$, we have

$$
\begin{aligned}
& A-\lambda I=\left[\begin{array}{cc}
3 & 5 \\
-2 & -4
\end{array}\right]-\lambda\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{cc}
3-\lambda & 5 \\
-2 & -4-\lambda
\end{array}\right] \\
& \therefore|A-\lambda I|=(3-\lambda)(-4-\lambda)-5(-2) \\
& \quad=\lambda^{2}+\lambda-2
\end{aligned}
$$

The characteristic equation of $A$ is $\lambda^{2}+\lambda-2=0$; when solved for $\lambda$, it gives the two eigenvalues $\lambda=1$ and $\lambda=-2$. As a check, we utilize property II., the trace of $A$ is $3+(-4)=-1$, which is also the sum of the eigenvalues.

The right eigenvectors of $A$ corresponding to $\lambda=1$ are obtained by solving equation (1) for $X=\left[x_{1}, x_{2}\right]^{T}$ with this value of $\lambda$. After substituting and rearranging, we have

$$
\begin{aligned}
& \left(\left[\begin{array}{rr}
3 & 5 \\
-2 & -4
\end{array}\right]-1\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \\
\Rightarrow & {\left[\begin{array}{rr}
2 & 5 \\
-2 & -5
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] }
\end{aligned}
$$

Which is equivalent to the set of linear equations

$$
\begin{array}{r}
2 x_{1}+5 x_{2}=0 \\
-2 x_{1}-5 x_{2}=0
\end{array}
$$

The solution to this system is $x_{1}=-\frac{5}{2} x_{2}$ with $x_{2}$ arbitrary, so the right eigenvectors of $A$ corresponding to $\lambda=1$ are

$$
X=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{c}
-\frac{5}{2} x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{c}
-5 / 2 \\
1
\end{array}\right] \text { with } x_{2} \text { arbitrary. }
$$

When $\lambda=-2$, equation (1) may be written

$$
\begin{aligned}
& \left\{\left[\begin{array}{rr}
3 & 5 \\
-2 & -4
\end{array}\right]-(-2)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right\}\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \\
\Rightarrow & {\left[\begin{array}{rr}
5 & 5 \\
-2 & -2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] } \\
\Rightarrow & {\left[\begin{array}{r}
5 x_{1}+5 x_{2} \\
-2 x_{1}-2 x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] }
\end{aligned}
$$

Which is equivalent to the set of linear equations

$$
\begin{array}{r}
5 x_{1}+5 x_{2}=0 \\
-2 x_{1}-2 x_{2}=0
\end{array}
$$

The solution to this system is $x_{1}=-x_{2}$ with $x_{2}$ arbitrary, so the right eigenvectors of $A$ corresponding to $\lambda=-2$ are

$$
X=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{r}
-x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{r}
-1 \\
1
\end{array}\right] \text { with } x_{2} \text { arbitrary. }
$$

Left eigenvectors of the matrix $A$ : The eigenvalues of $A$ were found in above to be $\lambda=1$ and $\lambda=-2$.
Set $X=\left[x_{1}, x_{2}\right]$.
With $\lambda=1$, equation (3) becomes

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]\left(\left[\begin{array}{rr}
3 & 5 \\
-2 & -4
\end{array}\right]-1\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)=[0,0] } \\
\Rightarrow & {\left[x_{1}, x_{2}\right]\left[\begin{array}{rr}
2 & 5 \\
-2 & -5
\end{array}\right]=[0,0] } \\
\Rightarrow & {\left[2 x_{1}-2 x_{2}, 5 x_{1}-5 x_{2}\right]=[0,0] }
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{aligned}
& 2 x_{1}-2 x_{2}=0 \\
& 5 x_{1}-5 x_{2}=0
\end{aligned}
$$

The solution to this system is $X_{1}=x_{2}$, with $X_{2}$ arbitrary.
The left eigenvectors of $A$ corresponding to $\lambda=1$ are thus $\left[x_{1}, x_{2}\right]=\left[x_{2}, x_{2}\right]=x_{2}[1,1]$ with $x_{2}$ arbitrary .

For $\lambda=-2$, equation (3) reduces to

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]\left\{\left[\begin{array}{rr}
3 & 5 \\
-2 & -4
\end{array}\right]-(-2)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right\}=[0,0] } \\
\Rightarrow & {\left[x_{1}, x_{2}\right]\left[\begin{array}{rr}
5 & 5 \\
-2 & -2
\end{array}\right]=[0,0] } \\
\Rightarrow & {\left[5 x_{1}-2 x_{2}, 5 x_{1}-2 x_{2}\right]=[0,0] }
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{aligned}
& 5 x_{1}-2 x_{2}=0 \\
& 5 x_{1}-2 x_{2}=0
\end{aligned}
$$

The solution to this system is $x_{1}=\frac{2}{5} x_{2}$, with $x_{2}$ arbitrary. The left eigenvectors of $A$ corresponding to $\lambda=-2$ are $\left[x_{1}, x_{2}\right]=\left[\frac{2}{5} x_{2}, x_{2}\right]=x_{2}[2 / 5,1]$ with $x_{2}$ arbitrary.

Right eigenvectors of the matrix $A^{T}$ : As the matrix $A$ and $A^{T}$ have the same eigenvalues (by property I ), therefore the eigenvalues of $A^{T}$ are $\lambda=1$ and $\lambda=-2$.

The right eigenvectors of $A^{T}$ corresponding to $\lambda=1$ are obtained by solving equation (1) for $X=\left[x_{1}, x_{2}\right]^{T}$ with this value of $\lambda$. After substituting and rearranging, we have

$$
\left(\left[\begin{array}{cc}
3 & -2 \\
5 & -4
\end{array}\right]-1\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

$$
\begin{aligned}
& \Rightarrow\left[\begin{array}{cc}
2 & -2 \\
5 & -5
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \\
& \Rightarrow\left[\begin{array}{l}
2 x_{1}-2 x_{2} \\
5 x_{1}-5 x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{aligned}
& 2 x_{1}-2 x_{2}=0 \\
& 5 x_{1}-5 x_{2}=0
\end{aligned}
$$

The solution to this system is $x_{1}=x_{2}$, with $x_{2}$ arbitrary, so the right eigenvectors of $A^{T}$ corresponding to $\lambda=1$ are

$$
X=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{l}
1 \\
1
\end{array}\right] \text { with } x_{2} \text { arbitrary. }
$$

When $\lambda=-2$, equation (1) becomes

$$
\begin{aligned}
& \left\{\left[\begin{array}{ll}
3 & -2 \\
5 & -4
\end{array}\right]-(-2)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right\}\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \\
\Rightarrow & {\left[\begin{array}{ll}
5 & -2 \\
5 & -2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] } \\
\Rightarrow & {\left[\begin{array}{l}
5 x_{1}-2 x_{2} \\
5 x_{1}-2 x_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] }
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{aligned}
& 5 x_{1}-2 x_{2}=0 \\
& 5 x_{1}-2 x_{2}=0
\end{aligned}
$$

The solution to this system is $x_{1}=\frac{2}{5} x_{2}$ with $x_{2}$ arbitrary, so the right eigenvectors of $A^{T}$ corresponding to $\lambda=-2$ are

$$
X=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]=\left[\begin{array}{l}
\frac{2}{5} x_{2} \\
x_{2}
\end{array}\right]=x_{2}\left[\begin{array}{c}
2 / 5 \\
1
\end{array}\right] \text { with } x_{2} \text { arbitrary. }
$$

Left eigenvectors of the matrix $A^{T}$ : The eigenvalues of $A^{T}$ were found in above to be $\lambda=1$ and $\lambda=-2$.
Set $X=\left[x_{1}, x_{2}\right]$.
With $\lambda=1$, equation (3) becomes

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]\left(\left[\begin{array}{ll}
3 & -2 \\
5 & -4
\end{array}\right]-1\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)=[0,0] } \\
\Rightarrow & {\left[x_{1}, x_{2}\right]\left[\begin{array}{cc}
2 & -2 \\
5 & -5
\end{array}\right]=[0,0] } \\
\Rightarrow & {\left[2 x_{1}+5 x_{2},-2 x_{1}-5 x_{2}\right]=[0,0] }
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{array}{r}
2 x_{1}+5 x_{2}=0 \\
-2 x_{1}-5 x_{2}=0
\end{array}
$$

The solution to this system is $x_{1}=-\frac{5}{2} x_{2}$, with $x_{2}$ arbitrary. The left eigenvectors of $A^{T}$ corresponding to $\lambda=1$ are thus $\left[x_{1}, x_{2}\right]=\left[-\frac{5}{2} x_{2}, x_{2}\right]=x_{2}[-5 / 2,1]$ with $x_{2}$ arbrary .

For $\lambda=-2$, equation (3) reduces to

$$
\begin{aligned}
& {\left[x_{1}, x_{2}\right]\left\{\left[\begin{array}{ll}
3 & -2 \\
5 & -4
\end{array}\right]-(-2)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right\}=[0,0] } \\
\Rightarrow & {\left[x_{1}, x_{2}\right]\left[\begin{array}{ll}
5 & -2 \\
5 & -2
\end{array}\right]=[0,0] } \\
\Rightarrow & {\left[5 x_{1}+5 x_{2},-2 x_{1}-2 x_{2}\right]=[0,0] }
\end{aligned}
$$

Which is equivalent to the set of equations

$$
\begin{array}{r}
5 x_{1}+5 x_{2}=0 \\
-2 x_{1}-2 x_{2}=0
\end{array}
$$

The solution to this system is $x_{1}=-x_{2}$, with $x_{2}$ arbitrary. The left eigenvectors of $A^{T}$ corresponding to $\lambda=-2$ are $\left[x_{1}, x_{2}\right]=\left[-x_{2}, x_{2}\right]=x_{2}[-1,1]$ with $x_{2}$ arbitrary.

## 4. CONCLUSION

In this paper, we have studied left and right eigenvectors of $A$ and $A^{T}$. In example 1, we have calculated left and right eigenvectors of $A$ and $A^{T}$ for comparison. From example 1, we have seen that all the eigenvectors of $A$ and $A^{T}$ are different but if we take the transpose of the right eigenvectors of $A$, then that eigenvectors are exactly equal to the left eigenvectors of $A^{T}$ corresponding to the same eigenvalues.

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