# International Journal of Mathematical Archive-5(11), 2014, 161-167 <br> IMA Available online through www.ijma.info ISSN 2229-5046 

## On the $(p, q)^{\text {th }}$ order and Lower $(p, q)^{\text {th }}$ order of entire harmonic functions in $R^{\mathbf{3}}$

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(Received On: 11-11-14; Revised \& Accepted On: 25-11-14)


#### Abstract

In this paper, we study the growth properties of entire harmonic function $H(r, \theta, \phi)_{\text {of }}(p, q)^{t h}$ order and $(p, q)^{\text {th }}$ type and also define cofficient characterization of order and type.


Keywords: Entire Harmonic function; order; type.

## 1. INTRODUCTION

If $H(r, \theta, \phi)$ is harmonic in a neighborhood of origin in $\mathrm{R}^{3}$, then $H(r, \theta, \phi)$ has the following expansion in spherical coordinates

$$
\begin{equation*}
H(r, \theta, \phi)=\sum_{n=0}^{\infty} \sum_{m=0}^{\infty}\left\{a_{m n}{ }^{(1)} \cos m \phi+a_{m n}{ }^{(2)} \sin m \phi\right\} r^{n} P_{n}^{m}(\cos \theta) \tag{1.1}
\end{equation*}
$$

where $a_{m n}{ }^{(1)}$ \& $a_{m n}{ }^{(2)}$ are two different coefficients.
The series (1.1) converges absolutely and uniformly on compact set of the largest open ball centered at the origin which omits singularities of $H(r, \theta, \phi)$.

The associated Legendre function of first kind, $n^{\text {th }}$ degree and order $m$ denoted by $P_{n}^{m}(x)$ are defined as

$$
\begin{equation*}
P_{n}^{m}(x)=\left(1-x^{2}\right)^{\frac{m}{2}} \frac{d^{m}}{d x^{m}}\left(P_{n}(x)\right) \tag{1.2}
\end{equation*}
$$

For entire function $H(r, \theta, \phi)$, we define

$$
\begin{equation*}
M(r) \equiv M(r, H)=\max _{\theta, \phi} H(r, \theta, \phi) \tag{1.3}
\end{equation*}
$$

Following the usual definitions of order and type of an entire function of a complex variables z, given by Srivastava [1], the $(p, q)^{t h}$ order $\rho_{q}^{p}$ and $(p, q)^{t h}$ type $T_{q}^{p}$ of $H(r, \theta, \phi)$ are defined as
$\rho_{q}^{p}=\limsup _{r \rightarrow \infty} \frac{\log ^{[p]} M(r)}{\log ^{[q]} r}$,
$T_{q}^{p}=\limsup _{r \rightarrow \infty} \frac{\log ^{[p-1]} M(r)}{r^{\rho_{q}^{p}}}$.

For $p=q=1$, the above definitions coincide with the classical definition of order and type. We define lower $(p, q)^{t h}$ order $\lambda_{q}^{p}$ and lower $(p, q)^{t h}$ type $\tau_{q}^{p}$ are defined by [3] as.

$$
\begin{equation*}
\lambda_{q}^{p}=\lambda_{q}^{p}(H)=\liminf _{r \rightarrow \infty} \frac{\log ^{[p]} M(r)}{\log r} \tag{1.6}
\end{equation*}
$$

$\tau_{q}^{p}=\tau_{q}^{p}(H)=\liminf _{r \rightarrow \infty} \frac{\log ^{[p-1]} M(r)}{r^{\rho_{q}^{p}}}$,
For $p=2,3,4, \ldots$ where $\log ^{[0]} x=x$ and $\log ^{[p]} x=\log \left(\log ^{[p-1]} x\right)$, we consider lower $(p, q)^{t h}$ order and lower $(p, q)^{t h}$ type of an entire harmonic function $H(r, \theta, \phi)$ and obtain their various characterization of in terms of $\left(\alpha_{n}\right)$, which is defined by Srivastva [1] as

$$
\begin{equation*}
\alpha_{n}=\max _{m, i}\left\{\frac{(n+m)!}{(n-m)!}\right\}^{\frac{1}{2}}\left|a_{m n}^{(i)}\right| \quad(i=1,2) \tag{1.8}
\end{equation*}
$$

We also write

$$
\begin{equation*}
f(z)=\sum_{n=0}^{\infty} \alpha_{n}\left(1+n^{-\frac{1}{2}}\right)^{n} z^{n}, \quad g(z)=\sum_{n=0}^{\infty} \alpha_{n}(1+2 n)^{-\frac{1}{2}} z^{n} \tag{1.9}
\end{equation*}
$$

## 2. AUXILIARY RESULTS

The use the following lemmas to prove our theorems.
Lemma 1[5]: The harmonic function $H(r, \theta, \phi)$ having expansion (1.1) is entire if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\alpha_{n}\right)^{\frac{1}{n}}=0 \tag{2.10}
\end{equation*}
$$

Lemma 2[5]: If H is an entire harmonic function, then for all $r>0$,

$$
\begin{equation*}
(2 \sqrt{2 n+1})^{-1}\left(\alpha_{n} r^{n}\right) \leq M(r) \leq 2 \sum_{n=0}^{\infty} \alpha_{n} r^{n}\left(1+n^{-\frac{1}{2}}\right)^{n} \tag{2.2}
\end{equation*}
$$

Lemma 3[5]: If H is entire harmonic function, then $f$ and $g$ are also entire functions. Further

$$
\begin{equation*}
2^{-1} m(r, g) \leq M(r) \leq 2 M(r, f) \tag{2.3}
\end{equation*}
$$

where $m(r, g)=\max _{n}\left[\alpha_{n}(1+2 n)^{-1} r^{n}\right] \quad$ and $\quad M(r, f)=\max _{|z| \leq r}|f(z)|$.
For the proof of Lemmas 1, 2 and 3, one can see [5, pp. 27-28].
Lemma 4: Let $f(z)$ and $g(z)$ be entire functions. Then the $(p, q)^{t h}$ order and $(p, q)^{t h}$ type of $f(z)$ are $g(z)$ are the same.

Proof: Let $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ be any entire function of $(p, q)^{\text {th }}$ order $\rho_{q}^{p}(F)$ and $(p, q)^{\text {th }}$ type $T_{q}^{p}(F)$. Then it is known from the results of Bajpai et al. [2], we have

$$
\begin{equation*}
\rho_{q}^{p}(F)=\lim _{n \rightarrow \infty} \sup \frac{n \log ^{[p-1]} n}{\log ^{[q]}\left|a_{n}\right|^{-1}}, \quad T_{q}^{p}(F)=\limsup _{n \rightarrow \infty} \frac{\log ^{[p-2]} n}{e^{\lambda_{q}^{p}}} \tag{2.4}
\end{equation*}
$$

For the function

$$
f(z)=\sum_{n=0}^{\infty} \alpha_{n}\left(1+n^{-\frac{1}{2}}\right) z^{n}
$$

We have

$$
\begin{aligned}
\frac{1}{\rho_{q}^{p}(f)} & =\liminf _{n \rightarrow \infty} \frac{\log ^{[q]}\left\{\alpha_{n}\left(1+n^{-\frac{1}{2}}\right)^{n}\right\}^{-1}}{n \log ^{[p-1]} n} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[q-1]}\left[\log \left\{\alpha_{n}\left(1+n^{-\frac{1}{2}}\right)^{n}\right\}^{-1}\right]}{n \log ^{[p-1]} n} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}-n \log \left(1+n^{-\frac{1}{2}}\right)\right]}{n \log ^{[p-1]} n} \\
& =\lim _{n \rightarrow \infty} \inf \frac{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]}{n \log { }^{[p-1]} n} .
\end{aligned}
$$

Similarly for

$$
g(z)=\sum_{n=0}^{\infty} \alpha_{n}(1+2 n)^{-\frac{1}{2}} z^{n}
$$

we have

$$
\begin{aligned}
\frac{1}{\rho_{q}^{p}(g)} & =\liminf _{n \rightarrow \infty} \frac{\log ^{[q]}\left\{\alpha_{n}(1+2 n)^{-\frac{1}{2}}\right\}^{-1}}{n \log ^{[p-1]} n} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[q-1]}\left[\log \left\{\alpha_{n}(1+2 n)^{-\frac{1}{2}}\right\}^{-1}\right]}{n \log ^{[p-1]} n} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[q-1]}\left[\log \alpha_{n}^{-1}+\frac{1}{2} \log (1+2 n)\right]}{n \log ^{[p-1]} n} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]}{n \log { }^{[p-1]} n} .
\end{aligned}
$$

Thus we have $\rho_{q}^{p}(f)=\rho_{q}^{p}(g)$.
Since $f$ and $g$ are of same order, using (2.4) we get $T_{q}^{p}(f)=T_{q}^{p}(g)$.

## 3. MAIN RESULTS

In this paper, we prove the following theorems.
Theorem 1: Let $H(r, \theta, \phi)$ be an entire harmonic function of $(p, q)^{t h}$ order $\rho_{q}^{p}$, lower $(p, q)^{\text {th }}$ order $\lambda_{q}^{p},(p, q)^{\text {th }}$ type $T_{q}^{p}$ and lower $(p, q)^{t h}$ type $\tau_{q}^{p}$. If $f$ and $g$ are entire function as defined in (1.9), then
$\rho_{q}^{p}(f)=\rho_{q}^{p}(g)=\rho_{q}^{p}$
$T_{q}^{p}(f)=T_{q}^{p}(g)=T_{q}^{p}$
$\lambda_{q}^{p}(f)=\lambda_{q}^{p}(g)=\lambda_{q}^{p}$
$\tau_{q}^{p}(g) \leq \tau_{q}^{p} \leq \tau_{q}^{p}(f)$
Proof: From Srivastava [1], we have

$$
2^{-1} m(r, g) \leq M(r) \leq 2 M(r, f)
$$

Also from [6],
$\limsup _{r \rightarrow \infty}($ inf $) \frac{\log ^{[p]} m(r, g)}{\log ^{[q]} r} \leq \limsup _{r \rightarrow \infty}($ inf $) \frac{\log ^{[p]} M(r)}{\log ^{[q]} r} \leq \limsup _{r \rightarrow \infty}(\inf ) \frac{\log ^{[p]} M(r, f)}{\log ^{[q]} r}$
$\log ^{[p]} M(r, f) \approx \log ^{[p]} m(r, f) \quad$ as $\quad r \rightarrow \infty$.
Hence from the above inequalities, we get

$$
\begin{equation*}
\rho_{q}^{p}(g) \leq \rho_{q}^{p} \leq \rho_{q}^{p}(f) \text { and } \lambda_{q}^{p}(g) \leq \lambda_{q}^{p} \leq \lambda_{q}^{p}(f) \tag{2.9}
\end{equation*}
$$

Since $\rho_{q}^{p}(g)=\rho_{q}^{p}(f)$, we obtain (2.5) and (2.7) from (2.9).
$\limsup _{r \rightarrow \infty} \frac{\log ^{[p-1]} m(r, g)}{r^{\rho_{q}^{p}}} \leq \limsup _{r \rightarrow \infty} \frac{\log ^{[p-1]} M(r)}{r^{\rho_{q}^{p}}} \leq \limsup _{r \rightarrow \infty} \frac{\log ^{[p-1]} M(r, f)}{r^{\rho_{q}^{p}}}$
Hence from Lemma 4, we obtain (2.6).
The proof of (2.8) is similar.
Theorem 2: Let $H(r, \theta, \phi)$ be an entire function of $(p, q)^{t h}$ order $\rho_{q}^{p}$, lower $(p, q)^{\text {th }}$ order $\lambda_{q}^{p}$ and lower $(p, q)^{\text {th }}$ type $\tau_{q}^{p}$. If $\left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)$ is non decreasing sequence for $n>n_{0}$ then

$$
\begin{equation*}
\lambda_{q}^{p}=\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]} \tag{2.10}
\end{equation*}
$$

Proof: For the entire function $F(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$, if $\left|\frac{a_{n}}{a_{n+1}}\right|$ forms a non decreasing sequence for $n>n_{0}$, then we have

$$
\begin{equation*}
\lambda_{q}^{p}=\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q]}\left|a_{n}\right|^{-1}} \tag{2.11}
\end{equation*}
$$

If $\left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)$ be a non decreasing sequence for $n>n_{0}$, we obtain

$$
\begin{aligned}
\lambda_{q}^{p}(f) & =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q]}\left\{\alpha_{n}\left(1+n^{-\frac{1}{2}}\right)^{n}\right\}^{-1}} \\
& =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left\{\alpha_{n}\left(1+n^{-\frac{1}{2}}\right)^{n}\right\}^{-1}\right]} \\
& =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}-n \log \left(1+n^{-1 / 2}\right)\right]} \\
& =\lim _{n \rightarrow \infty} \inf \frac{n \log \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]} .}{}
\end{aligned}
$$

Similarly for

$$
g(z)=\sum_{n=0}^{\infty} \alpha_{n}\left(1+2 n^{-\frac{1}{2}}\right) z^{n}
$$

We have

$$
\begin{aligned}
\lambda_{q}^{p}(g) & =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q]}\left\{\alpha_{n}\left(1+2 n^{-\frac{1}{2}}\right)\right\}^{-1}} \\
& =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left\{\alpha_{n}\left(1+2 n^{-1 / 2}\right)\right\}^{-1}\right]} \\
& =\lim _{n \rightarrow \infty} \inf \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}+\frac{1}{2} \log (1+2 n)\right]} \\
& =\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]}
\end{aligned}
$$

From (2.5), we have

$$
\lambda_{q}^{p}=\liminf _{n \rightarrow \infty} \frac{n \log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\alpha_{n}\right)^{-1}\right]}
$$

Theorem 3: Let $H(r, \theta, \phi)$ be an entire harmonic function of lower $(p, q)^{t h}$ order $\lambda_{q}^{p}$ and let $\left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)$ forms a non decreasing sequence for $n>n_{0}$. Then

$$
\begin{equation*}
\lambda_{q}^{p}=\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)\right]} \tag{2.12}
\end{equation*}
$$

Proof: For an entire function $F(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$,

$$
\begin{equation*}
\lambda_{q}^{p}(f)=\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q]}\left|\frac{a_{n}}{a_{n+1}}\right|} \tag{2.13}
\end{equation*}
$$

Provided $\left|\frac{a_{n}}{a_{n+1}}\right|$ is non decreasing sequence for $n>n_{0}$.
Using the condition on $\left(\alpha_{n}\right)$, we can easily shows as above theorem.

$$
\begin{aligned}
\lambda_{q}^{p}(f) & =\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q]}\left\{\left.\left\{\frac{\alpha_{n}}{\alpha_{n+1}} \frac{\left(1+n^{-1 / 2}\right)^{n}}{\left(1+(n+1)^{-1 / 2}\right)^{n+1}}\right\}^{-1} \right\rvert\,\right.} \\
& \left.=\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\left.\log \left\{\frac{\alpha_{n}}{\alpha_{n+1}} \frac{\left(1+n^{-1 / 2}\right)^{n}}{\left(1+(n+1)^{-1 / 2}\right)^{n+1}}\right]^{-1} \right\rvert\,\right]}\right] \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \frac{\alpha_{n}}{\alpha_{n+1}}+n \log \left(1+n^{-1 / 2}\right)-(n+1) \log \left(1+(n+1)^{-1 / 2}\right)\right]}
\end{aligned}
$$

Since $n \log \left(1+n^{-1 / 2}\right)-(n+1) \log \left(1+(n+1)^{-1 / 2}\right) \rightarrow 0$ as $n \rightarrow \infty$, we have

$$
\lambda_{q}^{p}(f)=\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)\right]}
$$

Also for $g(z)=\alpha_{n}(1+2 n)^{-1 / 2} z^{n}$, we have

$$
\begin{aligned}
\lambda_{q}^{p}(g) & =\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q]}\left|\left\{\frac{\alpha_{n}}{\alpha_{n+1}} \frac{(1+2 n)^{-1 / 2}}{(1+2(n+1))^{-1 / 2}}\right\}^{-1}\right|} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left\{\frac{\alpha_{n}}{\alpha_{n+1}} \frac{(1+2 n)^{-1 / 2}}{(1+2(n+1))^{-1 / 2}}\right\}^{-1}\right]} \\
& =\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)+\frac{1}{2} \log \left(\frac{1+2(n+1)}{1+2 n}\right)\right]}
\end{aligned}
$$

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On the ( $p, q)^{\text {th }}$ order and Lower ( $\left.p, q\right)^{\text {th }}$ order of entire harmonic functions in $R^{3} /$ IJMA-5(11), Nov.-2014.

$$
=\liminf _{n \rightarrow \infty} \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)\right]}
$$

Thus by using (2.3), we get

$$
\lambda_{q}^{p}=\lim _{n \rightarrow \infty} \inf \frac{\log ^{[p]} n}{\log ^{[q-1]}\left[\log \left(\frac{\alpha_{n}}{\alpha_{n+1}}\right)\right]}
$$

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## Source of support: Nil, Conflict of interest: None Declared

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