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On the (p, q)th order and Lower (p, q)th order of entire harmonic functions in R³

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ABSTRACT

In this paper, we study the growth properties of entire harmonic function $H(r,\theta,\phi)$ of $(p,q)^{th}$ order and $(p,q)^{th}$ type and also define cofficient characterization of order and type.

Keywords: *Entire Harmonic function; order; type.*

1. INTRODUCTION

If $H(r,\theta,\phi)$ is harmonic in a neighborhood of origin in R^3, then $H(r,\theta,\phi)$ has the following expansion in spherical coordinates

$$H(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{a_{mn}^{(1)}\cos m\phi + a_{mn}^{(2)}\sin m\phi\right\}r^n P_n^m(\cos\theta), \tag{1.1}$$

where $a_{mn}^{(1)}$ & $a_{mn}^{(2)}$ are two different coefficients.

The series (1.1) converges absolutely and uniformly on compact set of the largest open ball centered at the origin which omits singularities of $H(r,\theta,\phi)$.

The associated Legendre function of first kind, n^{th} degree and order m denoted by $P_n^m(x)$ are defined as

$$P_n^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}\left(P_n(x)\right). \tag{1.2}$$

For entire function $H(r,\theta,\phi)$, we define

$$M(r)\equiv M(r,H)=\max_{\theta,\phi} H(r,\theta,\phi), \tag{1.3}$$

Following the usual definitions of order and type of an entire function of a complex variables z, given by Srivastava [1], the $(p,q)^{th}$ order ρ_q^p and $(p,q)^{th}$ type T_q^p of $H(r,\theta,\phi)$ are defined as

$$\rho_q^p=\limsup_{r\to\infty}\frac{\log^{[p]}M(r)}{\log^{[q]}r}, \tag{1.4}$$

$$T_q^p=\limsup_{r\to\infty}\frac{\log^{[p-1]}M(r)}{r^{\rho_q^p}}. \tag{1.5}$$

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For $p = q = 1$, the above definitions coincide with the classical definition of order and type. We define lower $(p,q)^{th}$ order λ_q^p and lower $(p,q)^{th}$ type τ_q^p are defined by [3] as.

$$\lambda_q^p = \lambda_q^p(H) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r)}{\log r}, \tag{1.6}$$

$$\tau_q^p = \tau_q^p(H) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M(r)}{r^{\rho_q^p}}, \tag{1.7}$$

For $p = 2,3,4,...$ where $\log^{[0]} x = x$ and $\log^{[p]} x = \log\left(\log^{[p-1]} x\right)$, we consider lower $(p,q)^{th}$ order and lower $(p,q)^{th}$ type of an entire harmonic function $H(r,\theta,\phi)$ and obtain their various characterization of in terms of (α_n), which is defined by Srivastva [1] as

$$\alpha_n = \max_{m,i} \left\{ \frac{(n+m)!}{(n-m)!} \right\}^{\frac{1}{2}} \left| a_{mn}^{(i)} \right| \quad (i = 1,2). \tag{1.8}$$

We also write

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \left(1 + n^{-\frac{1}{2}} \right)^n z^n, \quad g(z) = \sum_{n=0}^{\infty} \alpha_n (1+2n)^{-\frac{1}{2}} z^n. \tag{1.9}$$

2. AUXILIARY RESULTS

The use the following lemmas to prove our theorems.

Lemma 1[5]: The harmonic function $H(r,\theta,\phi)$ having expansion (1.1) is entire if and only if

$$\lim_{n\to\infty} (\alpha_n)^{\frac{1}{n}} = 0, \tag{2.10}$$

Lemma 2[5]: If H is an entire harmonic function, then for all $r > 0$,

$$\left(2\sqrt{2n+1}\right)^{-1} \left(\alpha_n r^n\right) \le M(r) \le 2\sum_{n=0}^{\infty} \alpha_n r^n \left(1 + n^{-\frac{1}{2}} \right)^n. \tag{2.2}$$

Lemma 3[5]: If H is entire harmonic function, then f and g are also entire functions. Further

$$2^{-1} m(r,g) \le M(r) \le 2M(r,f), \tag{2.3}$$

where $m(r,g) = \max_n \left[\alpha_n (1+2n)^{-1} r^n \right]$ and $M(r,f) = \max_{|z|\le r} |f(z)|$.

For the proof of Lemmas 1, 2 and 3, one can see [5, pp. 27-28].

Lemma 4: Let $f(z)$ and $g(z)$ be entire functions. Then the $(p,q)^{th}$ order and $(p,q)^{th}$ type of $f(z)$ are $g(z)$ are the same.

Proof: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be any entire function of $(p,q)^{th}$ order $\rho_q^p(F)$ and $(p,q)^{th}$ type $T_q^p(F)$. Then it is known from the results of Bajpai *et al.* [2], we have

$$\rho_q^p(F) = \limsup_{n\to\infty} \frac{n \log^{[p-1]} n}{\log^{[q]} |a_n|^{-1}}, \quad T_q^p(F) = \limsup_{n\to\infty} \frac{\log^{[p-2]} n}{e^{\lambda_q^p}}. \tag{2.4}$$

For the function

$$f(z)=\sum_{n=0}^{\infty}\alpha_n\left(1+n^{-\frac{1}{2}}\right)z^n,$$

We have

$$\frac{1}{\rho_q^p(f)}=\liminf_{n\to\infty}\frac{\log^{[q]}\left\{\alpha_n\left(1+n^{-\frac{1}{2}}\right)^n\right\}^{-1}}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log\left\{\alpha_n\left(1+n^{-\frac{1}{2}}\right)^n\right\}^{-1}\right]}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log(\alpha_n)^{-1}-n\log\left(1+n^{-\frac{1}{2}}\right)\right]}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}{n\log^{[p-1]}n}.$$

Similarly for

$$g(z)=\sum_{n=0}^{\infty}\alpha_n(1+2n)^{-\frac{1}{2}}z^n,$$

we have

$$\frac{1}{\rho_q^p(g)}=\liminf_{n\to\infty}\frac{\log^{[q]}\left\{\alpha_n(1+2n)^{-\frac{1}{2}}\right\}^{-1}}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log\left\{\alpha_n(1+2n)^{-\frac{1}{2}}\right\}^{-1}\right]}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log\alpha_n^{-1}+\frac{1}{2}\log(1+2n)\right]}{n\log^{[p-1]}n}$$

$$=\liminf_{n\to\infty}\frac{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}{n\log^{[p-1]}n}\quad .$$

Thus we have $\rho_q^p(f)=\rho_q^p(g)$.

Since f and g are of same order, using (2.4) we get $T_q^p(f)=T_q^p(g)$.

3. MAIN RESULTS

In this paper, we prove the following theorems.

Theorem 1: Let $H(r,\theta,\phi)$ be an entire harmonic function of $(p,q)^{th}$ order ρ_q^p, lower $(p,q)^{th}$ order λ_q^p, $(p,q)^{th}$ type T_q^p and lower $(p,q)^{th}$ type τ_q^p. If f and g are entire function as defined in (1.9), then

$$\rho_q^p(f)=\rho_q^p(g)=\rho_q^p \tag{2.5}$$

$$T_q^p(f)=T_q^p(g)=T_q^p \tag{2.6}$$

$$\lambda_q^p(f)=\lambda_q^p(g)=\lambda_q^p \tag{2.7}$$

$$\tau_q^p(g)\le\tau_q^p\le\tau_q^p(f) \tag{2.8}$$

Proof: From Srivastava [1], we have

$$2^{-1}m(r,g)\le M(r)\le 2M(r,f).$$

Also from [6],

$$\lim_{r\to\infty}\sup(\inf)\frac{\log^{[p]}m(r,g)}{\log^{[q]}r}\le\lim_{r\to\infty}\sup(\inf)\frac{\log^{[p]}M(r)}{\log^{[q]}r}\le\lim_{r\to\infty}\sup(\inf)\frac{\log^{[p]}M(r,f)}{\log^{[q]}r}$$

$$\log^{[p]}M(r,f)\approx\log^{[p]}m(r,f)\quad\text{as}\quad r\to\infty.$$

Hence from the above inequalities, we get

$$\rho_q^p(g)\le\rho_q^p\le\rho_q^p(f)\ \text{and}\ \lambda_q^p(g)\le\lambda_q^p\le\lambda_q^p(f). \tag{2.9}$$

Since $\rho_q^p(g)=\rho_q^p(f)$, we obtain (2.5) and (2.7) from (2.9).

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]}m(r,g)}{r^{\rho_q^p}}\le\lim_{r\to\infty}\sup\frac{\log^{[p-1]}M(r)}{r^{\rho_q^p}}\le\lim_{r\to\infty}\sup\frac{\log^{[p-1]}M(r,f)}{r^{\rho_q^p}}$$

Hence from Lemma 4, we obtain (2.6).

The proof of (2.8) is similar.

Theorem 2: Let $H(r,\theta,\phi)$ be an entire function of $(p,q)^{th}$ order ρ_q^p, lower $(p,q)^{th}$ order λ_q^p and lower $(p,q)^{th}$ type τ_q^p. If $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ is non decreasing sequence for $n>n_0$ then

$$\lambda_q^p=\lim_{n\to\infty}\inf\frac{n\log^{[p]}n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}. \tag{2.10}$$

Proof: For the entire function $F(z)=\sum_{n=0}^{\infty}a_nz^n$, if $\left|\frac{a_n}{a_{n+1}}\right|$ forms a non decreasing sequence for $n>n_0$, then we have

$$\lambda_q^p=\lim_{n\to\infty}\inf\frac{n\log^{[p]}n}{\log^{[q]}|a_n|^{-1}}. \tag{2.11}$$

If $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ be a non decreasing sequence for $n>n_0$, we obtain

$$\lambda_q^p(f) = \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q]}\left\{\alpha_n\left(1+n^{-\frac{1}{2}}\right)^n\right\}^{-1}}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log\left\{\alpha_n\left(1+n^{-\frac{1}{2}}\right)^n\right\}^{-1}\right]}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1} - n\log\left(1+n^{-1/2}\right)\right]}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}.$$

Similarly for

$$g(z) = \sum_{n=0}^{\infty}\alpha_n\left(1+2n^{-\frac{1}{2}}\right)z^n$$

We have

$$\lambda_q^p(g) = \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q]}\left\{\alpha_n\left(1+2n^{-\frac{1}{2}}\right)\right\}^{-1}}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log\left\{\alpha_n\left(1+2n^{-1/2}\right)\right\}^{-1}\right]}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1} + \frac{1}{2}\log(1+2n)\right]}$$

$$= \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}$$

From (2.5), we have

$$\lambda_q^p = \liminf_{n\to\infty} \frac{n\log^{[p]} n}{\log^{[q-1]}\left[\log(\alpha_n)^{-1}\right]}.$$

Theorem 3: Let $H(r,\theta,\phi)$ be an entire harmonic function of lower $(p,q)^{th}$ order λ_q^p and let $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ forms a non decreasing sequence for $n > n_0$. Then

$$\lambda_q^p = \liminf_{n\to\infty} \frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left(\frac{\alpha_n}{\alpha_{n+1}}\right)\right]}. \tag{2.12}$$

Proof: For an entire function $F(z)=\sum_{n=0}^{\infty} a_n z^n$,

$$\lambda_q^p(f)=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q]}\left|\frac{a_n}{a_{n+1}}\right|} \tag{2.13}$$

Provided $\left|\frac{a_n}{a_{n+1}}\right|$ is non decreasing sequence for $n>n_0$.

Using the condition on (α_n), we can easily shows as above theorem.

$$\lambda_q^p(f)=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q]}\left|\left\{\frac{\alpha_n}{\alpha_{n+1}}\frac{\left(1+n^{-1/2}\right)^n}{\left(1+(n+1)^{-1/2}\right)^{n+1}}\right\}^{-1}\right|}$$

$$=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left|\left\{\frac{\alpha_n}{\alpha_{n+1}}\frac{\left(1+n^{-1/2}\right)^n}{\left(1+(n+1)^{-1/2}\right)^{n+1}}\right\}^{-1}\right|\right]}$$

$$=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\frac{\alpha_n}{\alpha_{n+1}}+n\log\left(1+n^{-1/2}\right)-(n+1)\log\left(1+(n+1)^{-1/2}\right)\right]}.$$

Since $n\log\left(1+n^{-1/2}\right)-(n+1)\log\left(1+(n+1)^{-1/2}\right)\to 0$ as $n\to\infty$, we have

$$\lambda_q^p(f)=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left(\frac{\alpha_n}{\alpha_{n+1}}\right)\right]}.$$

Also for $g(z)=\alpha_n(1+2n)^{-1/2}z^n$, we have

$$\lambda_q^p(g)=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q]}\left|\left\{\frac{\alpha_n}{\alpha_{n+1}}\frac{(1+2n)^{-1/2}}{(1+2(n+1))^{-1/2}}\right\}^{-1}\right|}$$

$$=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left\{\frac{\alpha_n}{\alpha_{n+1}}\frac{(1+2n)^{-1/2}}{(1+2(n+1))^{-1/2}}\right\}^{-1}\right]}$$

$$=\liminf_{n\to\infty}\frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left(\frac{\alpha_n}{\alpha_{n+1}}\right)+\frac{1}{2}\log\left(\frac{1+2(n+1)}{1+2n}\right)\right]}$$

$$= \liminf_{n\to\infty} \frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left(\frac{\alpha_n}{\alpha_{n+1}}\right)\right]}.$$

Thus by using (2.3), we get

$$\lambda_q^p = \liminf_{n\to\infty} \frac{\log^{[p]} n}{\log^{[q-1]}\left[\log\left(\frac{\alpha_n}{\alpha_{n+1}}\right)\right]}.$$

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