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STRONG GOLDIE DIMENSION AND KRULL DIMENSION OF MODULES

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ABSTRACT

Let R be an associative ring with identity. A unital right R-module M is said to be of strong Goldie dimension n (written SG.dim M = n) if sup {G.dim (M/N) | $N \leq M$} = n. Otherwise, we set SG.dimM = $+\infty$. A module M is called strongly finite dimensional if SG.dim M < $+\infty$, where G.dim denotes the Goldie dimension of a module. In this paper we attempt to provide a new insight to characterize strongly finite dimensional module. Some equivalent conditions are obtained regarding finite Goldie dimension and Krull dimension. It has been proved that if M be a serial module having no 0-critical submodule , then M is strongly finite dimensional if and only if M has Krull dimension. Using the concepts of strong Goldie dimension and Krull dimension, extending modules are characterized.

Keywords: *Strong Goldie dimension, Krull dimension, Serial module, Critical module, Extending module.*

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1. INTRODUCTION

Throughout this article R denotes an associative ring with identity and all modules are unital right R-module. Let M be a module and N be a submodule of M. Then N is said to be an essential submodule of M if every non-zero submodule of M has non-zero intersection with N and is denoted by $N \leq_e M$. In this case M is called essential extension of N. For any module M, the sum of all simple submodules of M is called the socle of M and is denoted by SocM. Equivalently SocM is the intersection of all essential submodules of M. A module M is called semisimple if SocM = M.

A submodule N of a module M is said to be a closed submodule of M if N has no proper essential extension inside M . In other words, N is a closed submodule of M if $N \leq_e L \leq M$ gives L = N for any submodule L of M.

A uniform module is a non-zero module M such that any two non-zero submodules of M have non-zero intersection. Equivalently, a module M is uniform if and only if M = 0 and every non-zero submodule of M is essential in M.

A module M has Goldie dimension n(written G.dim M = n)if there is an essential submodule $V \leq_e M$ that is a direct sum of n uniform submodules. If on the other hand, no such integer n exists, we write G.dimM = $+\infty$. A module M is called finite dimensional if G.dimM < $+\infty$.

A ring R is called right finite dimensional if it is finite dimensional as a right R-module. Left finite dimensional rings can be defined similarly.

The notion of Goldie dimension is further generalized to various new concepts. Shen and Chen [8] use the concept of Goldie dimension to define strong Goldie dimension of Modules. A module M is said to be of strong Goldie dimension n (written SG.dim M = n) if sup {G.dim (M/N) | $N \leq M$} = n. Otherwise, we set SG.dimM = $+\infty$. A module M is called strongly finite dimensional if SG.dimM < $+\infty$. A ring R is called right strongly finite dimensional if it is strongly finite dimensional as a right R- module. Left strongly finite dimensional rings can be defined similarly.

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Quotient finite dimensional (q.f.d) modules have been studied by several authors [1],[2],[3] and [6].It is obvious that every strongly finite dimensional module is q.f.d. As a consequence the results for q.f.d modules necessarily hold for strongly finite dimensional modules. A strongly finite dimensional module is Noetherian if it satisfies the ascending chain condition(ACC) on subdirectly irreducible submodules [6].

In Section 2, a characterization of Krull dimension of module is given. Some equivalent conditions regarding finite Goldie dimension and Krull dimension of module have been explored. A relation between finite strong Goldie dimension and Krull dimension of module has been established.

In Section 3, extending modules have been characterized using finite strong Goldie dimension and Krull dimension.

2. STRONG GOLDIE DIMENSION AND KRULL DIMENSION OF MODULES

The purpose of this section is to characterize strongly finite dimensional modules with the help of Krull dimension. It has been shown that a serial module M having no 0-critical submodule is strongly finite dimensional if and only if M has Krull dimension. We begin with the following definition.

Definition 2.1: The Krull dimension of a module M denoted by K.dimM is defined by a transfinite induction. First, K.dimM = -1 if and only if M = 0.

Second, consider an ordinal $\alpha \geq 0$; assuming that we have already defined which modules have Krull dimension β for ordinals $\beta < \alpha$, we now define K.dimM = α if and only if

(1) we have not already defined K.dimM = β for some ordinal $\beta < \alpha$, and
(2) for every (countable)descending chain $M_0 \geq M_1 \geq M_2 \geq$ ....of submodules of M, we have K.dim(M_i /M_{i+1}) $< \alpha$ for all but finitely many indices i.

Definition 2.2: Let α be an ordinal, $\alpha \geq 0$. A module M is called α-critical if K.dimM = α and K.dim (M/N) $<\alpha$ for all non-zero submodules N of M. A module M is called critical if it is α-critical for some ordinal $\alpha \geq 0$.

Definition 2.3: A module M is called a Max module if every non-zero submodule of M contains a maximal submodule.

Definition 2.4: A module is called uniserial if its submodules are linearly ordered by inclusion. A module is called serial if it is a direct sum of uniserial modules.

Lemma 2.5: (**[4]**, 5.15(1)) A module M satisfies ACC on essential submodules if and only if M/SocM is Noetherian.

Lemma 2.6: (**[4]**, 6.2)

(1) Any Noetherian R-module has Krull dimension.
(2) Any R-module with Krull dimension has finite Goldie dimension.

Lemma 2.7: (**[5]**, Proposition 1.1(a)) If $A \leq B \leq C$, then $A \leq_e C$ if and only if $A \leq_e B \leq_e C$.

Lemma 2.8: ([**9]**, Theorem 3.8(1), (2)) The following conditions for a module are equivalent:

(1) A module M is Noetherian.
(2) Every factor module of M is a Max module and has a finitely generated socle.

Proposition 2.9: Let R be an associative ring with identity and M be any right R-module. Then the following statements are equivalent:

(1) M satisfies ACC on essential submodules.
(2) M/K is Noetherian for every essential submodule K of M.
(3) M/K has finitely generated socle for every essential submodule K of M.
(4) M/K has finite Goldie dimension for every essential submodule K of M.
(5) M/SocM has Krull dimension.
(6) M/SocM is Noetherian.

Proof:

(6)⇒ (5) ⇒ (4): Assume (6).Then by Lemma 2.6(1), M/SocM has Krull dimension. Hence (5). Again by Lemma 2.6(2) we have M/SocM has finite Goldie dimension. Now, for every essential submodule K of M we have SocM $\leq$ K which implies M/K $\leq$ M/SocM and so G.dim(M/K) $\leq$ G.dim(M/SocM). Therefore M/K also has finite Goldie dimension. Hence (4).

(4)⇔ (2): Assume (4). Let $A_1/K \leq A_2/K \leq A_3/K \leq$ ....be an ascending chain of submodules of M/K. By (4) there is an integer $t \geq 0$ such that $A_t/K \leq_e A_{t+n}/K$ for all $n \geq 0$. By Lemma 2.7, $K \leq_e M$ gives $K \leq_e A_1 \leq_e A_2 \leq_e A_3 \leq_e$ .... $\leq_e A_t \leq_e A_{t+1} \leq_e$ .... But $K \leq A_t \leq_e A_{t+1}$ and $A_t/K \leq e\, A_{t+1}/K$ will give K is closed in A_{t+1}. Therefore $A_t = A_{t+1}$ which implies $A_t/K = A_{t+1}/K$.
Hence (2).

Again assume (2). Then by Lemma 2.6(1) we have M/K has Krull dimension for every essential submodule K of M. Therefore by Lemma 2.6(2) it follows that M/K has finite Goldie dimension for every essential submodule K of M.
Hence (4).

(2) ⇒ (1): Assume (2). Let $A_1 \leq A_2 \leq A_3 \leq$ .... be an ascending chain of essential submodules of M. Since $K \leq_e M$, so by Lemma 2.7, $K \leq_e A_1 \leq_e A_2 \leq_e A_3 \leq_e$ ....which gives an ascending chain $A_1/K \leq A_2/K \leq A_3/K \leq$ .... of essential submodules of M/K for every essential submodule K of M. By (2) there is a maximal t such that $A_t/K = A_{t+1}/K$ which implies
$A_t = A_{t+1}$.

Hence (1).

(1) ⇔ (6): It follows directly from Lemma 2.5.

(2) ⇔ (3)

(2) ⇒ (3) is obvious.

(3) ⇒ (2): Assume (3). Let K be an essential submodule of M. By hypothesis, submodules of factor mod- ules of M/K have maximal submodules. By (3) and Lemma 2.8, M/K is Noetherian. Hence (2).

Proposition 2.10: If a module M is strongly finite dimensional then M/SocM has Krull dimension.

Proof: Let M be a strongly finite dimensional module. This implies G.dim $(M/N) < +\infty$ for any submodule N of M. In particular, M/K has finite Goldie dimension for every $K \leq_e M$. By Proposition 2.9, we have M/SocM has Krull dimension.

But the converse of the above proposition is not true which can be seen in case of the ring of integers $\mathbb{Z}$ by considering as a module over itself.

Lemma 2.11: (**[8]**, Corollary 2.8) If M is a serial module, then SG.dimM = G.dimM.

Proposition 2.12: If M is a serial module having no 0-critical submodule, then the following statements are equivalent:

(1) M is strongly finite dimensional.
(2) M has Krull dimension.

Proof:
(1) ⇒ (2): Assume (1). Then by Proposition 2.10 above, M/SocM has Krull dimension. Since M has no 0-critical submodule, so M has no simple submodule. Therefore we have SocM = 0. Thus M has Krull dimension.

(2) ⇒ (1): Assume (2). Then by Lemma 2.6(2) we have G.dimM$< \infty$. Since M is a serial module, so by Lemma 2.11, SG.dimM $< \infty$.

Proposition 2.13: If a critical module M is serial then SG.dimM = 1.

Proof: Let M be a critical module. Then M is uniform, therefore G.dimM = 1. Since M is serial, so by Lemma 2.11 we have SG.dimM = 1.

Above result does not hold if M is not serial which can be seen in case of the ring of integers $\mathbb{Z}$ by considering as a module over itself.

3. STRONGLY FINITE DIMENSIONAL EXTENDING MODULE

In this section we characterize strongly finite dimensional extending modules and investigate their various direct sum decomposition properties. We begin with the following definition.

Definition 3.1: A module M is called an extending module if every closed submodule of M is a direct summand of M. Equivalently; M is an extending module if and only if every submodule of M is essential in a direct summand of M. M is called uniform-extending if every uniform submodule of M is essential in a direct summand of M.

Lemma 3.2: (**[4]**, Corollary 18.6) Let M be an extending module.

(1) Assume M/SocM has finite Goldie dimension. Then M is a direct sum of a semisimple module and a module with finite Goldie dimension.
(2) If M has ACC on essential submodules, then $M = S \oplus N$ where S is semisimple and N is Noetherian.
(3) If M has DCC on essential submodules, then $M = S \oplus A$ where S is semisimple and A is Artinian.

Proposition 3.3: If M is a strongly finite dimensional extending module, then $M = S \oplus G$ where S is a semisimple module and G is a module with finite Goldie dimension.

Proof: Assume M be strongly finite dimensional. By Proposition 2.10, M/SocM has Krull dimension. Then by Lemma 2.6(2), M/SocM has finite Goldie dimension. Since M is extending, therefore by Lemma 3.2(1) we have $M = S \oplus G$.

Corollary 3.4: If M is a strongly finite dimensional extending module, then $M = L \oplus G$ where L is an extending module and G is a module with finite Goldie dimension.

Proof: Since every semisimple module is extending, so it is obvious by Proposition 3.3.

Proposition 3.5: If M is a strongly finite dimensional extending module, then $M = S \oplus K$ where S is a semisimple module and K is a module with Krull dimension.

Proof: Assume M be strongly finite dimensional. By Proposition 2.10, M/SocM has Krull dimension. Then by Proposition 2.9, M has ACC on essential submodules. Since M is extending, by Lemma 3.2(2) and 2.6(1) we have $M = S \oplus K$.

Lemma 3.6: (**[7]**, Lemma 7.1) Any direct summand of a (uniform) extending module is also (uniform) extending.

Lemma 3.7: (**[8]**, Corollary 2.6) Let $M_1, M_2, \ldots, M_n$ be modules. Then $SG.\dim (M_1 \oplus M_2 \oplus \ldots \oplus M_n) = SG.\dim M_1 + SG.\dim M_2 + \ldots + SG.\dim M_n$.

Proposition 3.8: If M is a strongly finite dimensional (uniform) extending module, then every direct summand of M is also a strongly finite dimensional (uniform) extending module.

Proof: Assume M be a strongly finite dimensional (uniform) extending module. Assume M_1 be a direct summand of M. Then by Lemma 3.6 we have M_1 is also (uniform) extending. But $M = M_1 \oplus M_2$ for some submodule M_2 of M. Therefore by Lemma 3.7 we have $SG.\dim M = SG.\dim M_1 + SG.\dim M_2$. Thus, M_1 is also strongly finite dimensional.

Remark 3.9: The proposition 2.12 in section 2 also holds if we consider a finitely generated extending module in place of a module having no 0-critical submodule and at the same time an alternative proof of the same can be given.

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