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ON LIE RING OF GENERALIZED DERIVATIONS

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ABSTRACT

Let $gD(R)$ be the Lie ring of generalized derivations of a ring R. In this article we show that the ring $gD(R)$ is a prime (semiprime) Lie ring if the ring R is a prime (semiprime) of characteristic not equal to 2. Also we show that the Lie ring of generalized inner derivation $g_iD(R)$ is not a prime Lie ring for any ring R. Further, examples are given to show that the conditions "characteristic not equal to 2" and "primeness" of the ring R are not superfluous.

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1. INTRODUCTION

Throughout this article R will denote an associative ring with center $Z(R)$. An additive mapping $d : R \to R$ is said to be a derivation of R if it satisfies $d(xy) = (dx)y + xdy$, for all $x, y \in R$. We denote by $D(R)$, the set of all derivations on R. Note that if $d_1, d_2 \in D(R)$ and their composition $d_1 d_2 \in R$, where R is a prime ring of characteristic not equal to 2, then either $d_1 = 0$ or $d_2 = 0$, Posner, E. C. [8]. But for any ring R, the Lie product of d_1 and d_2, $[d_1; d_2] = d_1 d_2 - d_2 d_1$ is a derivation of R. Thus $(D(R), +, [\,;\,])$ form a ring and it is said to be the Lie ring of derivations. In [5, 6, 7], Jordan, C. R. and Jordan, D. A. have given the structure of $D(R)$ by assuming different condition on R. In particular, they proved that if ring R is prime (semiprime) of characteristic not equal to 2 then $D(R)$ is a prime (semiprime) Lie ring. Recall that an additive subgroup I of a Lie ring L is called an Lie ideal of L if $[I; L] \subseteq I$. An ideal I of a Lie ring L is said to prime Lie ideal if A and B are two Lie ideals of L such that $[A; B] \subseteq I$ then either $A \subseteq I$ or $B \subseteq I$ and I is said to be semiprime Lie ideal of L if $[A; A] \subseteq I$ then $A \subseteq I$. The Lie ring L is said to be prime (semiprime) Lie ring if $\{0\}$ is the prime (semiprime) Lie ideal of L. An additive mapping $g_d: R \to R$ is said to be a generalized derivation on R, associated with a derivation d, if $g_d(xy) = g_d(x)y + xdy$, for all $x, y \in R$. As in the case of derivations, the product of two nonzero generalized derivations is not a generalized derivation, but the Lie product of any two generalized derivations is a generalized derivation. The collection of all generalized derivations, denoted by $gD(R)$, form a Lie ring. If g_{d_1} and g_{d_2} are two generalized derivations associated with derivations d_1 and d_2, then $[g_{d_1}; g_{d_2}]$ is a generalized derivation associated with the derivation $[d_1; d_2]$. An additive mapping $T: R \to R$ is said to be a left multiplier if it satisfies $T(xy) = T(x)y$, for all $x, y \in R$. Generalized derivations associated with the derivation 0 are given by $g_0(xy) = g_0(x)y$, for all $x, y \in R$. Obviously these are left multipliers of R. Note that if a ring R has the unity then $T : R \to R$ is a left multiplier if and only if $T(x) = ax$, for all $x \in R$, where a is a fixed element of R, see [11].

Let $a, b \in R$. Define $g_{a,b}: R \to R$ by $g_{a,b}(x) = ax + xb$, for all $x \in R$. Then $g_{a,b}$ satisfies $g_{a,b}(xy) = g_{a,b}(x)y + xI_{-b}(y)$, for all $x, y \in R$. Thus $g_{a,b}$ is a generalized derivation of R associated with an inner derivation $I_{-b} \in I(R) \subseteq D(R)$, induced by $-b \in R$, where $I_{-b}(y) = -by + yb$ and $I(R)$ is the collection of all inner derivations of R. The map $g_{a,b}$ is called the generalized inner derivation of R induced by a and b. We denote by $g_iD(R)$, the collection of all generalized inner derivations of R.

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2. PRELIMINARY RESULTS

The following results are used to prove our main results and to give examples.

Lemma 2.1: Let P be a Lie ideal of $D(R)$. Suppose $gP = \{g_d \in gD(R) \,|d \in P\}$. Then gP is a Lie ideal of $gD(R)$. Further, gP contains all left multipliers of R.

Proof: Since $0 = d \in P$. All generalized derivations corresponding to the derivation 0 are the left multipliers of R which are in gP. Now, let P be a Lie ideal of $D(R)$ and $g_d \in gP$ and let $g_\delta \in gD(R)$, then for all $x, y \in R$, $[g_d; g_\delta](xy) = [g_d; g_\delta](x)y \,+ x[d;\delta](y)$ but $[d;\delta] \in P \Rightarrow [g_d;\, g_\delta] \in gP$ which implies $[gP;\, gD(R)] \subseteq gP$. Thus gP is a Lie ideal of $gD(R)$.

Lemma 2.2: Let R be a ring with unity 1 so that every derivation on R is inner. Then every generalized derivation on R is generalized inner derivation.

Proof: Let I_a be an inner derivation on R induced by $a \in\, R$. Suppose $x, y \in R$ and g be a generalized derivation on R associated with I_a then $g(xy) = g(x)y \;+\; xI_a(y)$.

Replace x by 1 we have $g(y) = g(1)y \;+\; I_a(y)$. Let b be the image of 1 under g then we have
$g(y) = by + ay - ya \;=\; (a + b)y + (-a)y$, for all $y \in\, R$. Thus g is a generalized inner derivation induced by $(a \;+\; b)$ and $(-a)$.

Lemma 2.3: [[5], Lemma 7] Let R be a 2-torsion free semiprime ring and $d \in D(R)$. If $d(R) \subseteq Z$ and $d^2 \;= 0$, then $d = 0$.

Lemma 2.4: [[9], Theorem 1] Let $\mathbb{Z}G$ be the integral group ring of finite group G, then any derivation of $\mathbb{Z}G$ is inner.

Lemma 2.5: [[3], Proposition Page 100] Let R be a simple algebra finite dimensional over its center. Then any derivation of R is inner.

3. MAIN RESULTS

The following result gives the relation between $D(R)$ and $gD(R)$.

Theorem 3.1: Let R be a prime ring and $D(R)$ and $gD(R)$ be the set of all derivations and the set of all generalized derivations of R, respectively. The mapping $\phi :\, gD(R) \rightarrow\, D(R)$ given by $\phi(g_d) = d$ for all $g_d \in gD(R)$ is a Lie epimorphism with kernel, where $\mathcal{T}$ is the set of all left multipliers of R. Consequently, $\mathcal{T}$ is an Lie ideal of $gD(R)$ and $gD(R)/\mathcal{T} \cong D(R)$.

Proof: The mapping ϕ is well defined. For, let $g_d \in gD(R)$ such that $\phi(g_d) = d_1$ and $\phi(g_d) = d_2$. Then for all $x, y \in R$, we have
$g_d(xy) = g_d(x)y + xd_1(y)$
$g_d(xy) = g_d(x)y + xd_2(y)$

Subtracting these two we get $x(d_1 - d_2)y = 0$. Now replace y by wz we have
$x(d_1 - d_2)wz = 0$
or, $x(w(d_1 -\, d_2)z + (d_1 -\, d_2)(w)z) = 0$
or, $xw(d_1 - d_2)z = 0$
or, $xR(d_1 - d_2)z = 0$; for all $x, z \in R$.

Since R is a prime ring we have $(d_1 - d_2)z = 0$, for all $z \in R$. Thus $d_1 = d_2$. It is clear that for every $d \in D(R)$, we can get a $g_d \in gD(R)$. Thus ϕ is onto. Now if $g_{d_1}, g_{d_2} \in gD(R)$, then $[g_{d_1}\,; g_{d_2}]$ is a generalized derivation associated with the derivation $[d_1; d_2] \in D(R)$. So $\phi[g_{d_1}\,; g_{d_2}\,] = [d_1; d_2] = [\phi(g_{d_1}); \phi(g_{d_2}\,)]$. Hence ϕ is a Lie homomorphism with kernel $Ker\phi = \{g_d|\phi(g_d) = 0\} = \{g_d|\; d \;=\; 0\} = \mathcal{T}$, the set of all left multipliers of R.

Theorem 3.2: Let $D(R)$ and $gD(R)$ be the Lie ring of derivations and the Lie ring of generalized derivations, respectively, of a ring R. Then any Lie ideal P of $D(R)$ is a prime Lie ideal if and only if $gP = \{g_d \in gD(R)|d \in P\}$ is a prime Lie ideal of $gD(R)$.

Proof: Let P be a prime Lie ideal of $D(R)$. That is, A, B are two Lie ideals of $D(R)$ such that $[A; B] \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$. Let $gP = \{g_d \in gD(R)|d \in P\} \subseteq gD(R)$. By Lemma 2.1, gP is a Lie ideal of $gD(R)$. Next we show that gP is a prime Lie ideal in $gD(R)$. Let gC and gD are two Lie ideals of $gD(R)$ such that $[gC;\; gD] \subseteq gP$. Suppose

$gD \nsubseteq gP$. [Note that gC and gD both should contain a generalized derivation other than left multipliers, otherwise $gC \subseteq gP$ or $gD \subseteq gP$ is trivially true by Lemma 2.1]. Let $D = \{d | g_d \in gD\} \neq \{0\}$. Since $gD \nsubseteq gP$, there exists $d \in D$ such that $d \notin P$. Let $g_\delta \in gC$ and $g_d \in gD$ then $[g_\delta; g_d] \in gP$ where $d \in D$. Since $d \notin P$ we get for all $x\, y \in R$, $[g_\delta; g_d](xy) = [g_\delta; g_d](x)y + x[\delta; d](y)$. Since $[\delta; d] \subseteq P$ and $d \notin P$ this implies that $\delta \in P$. Thus for every $\delta \in C$, we have $\delta \in P \Rightarrow C \subseteq P \Rightarrow gC \subseteq gP$. Hence gP is a prime Lie ideal of $gD(R)$.

Conversely, Let gP be a prime Lie ideal of $gD(R)$. Two cases arises:

Case-1: Suppose gP contains atleast one generalized derivation other than left multipliers. Then $P = \{d | g_d \in gP\} \neq \{0\}$. Since gP is Lie ideal of $gD(R)$ we have $[gP;\, gD(R)] \subseteq gP$. Let $g_d \in gP$ and $g_\delta \in gD(R)$, then for all $x, y \in R$ we have $[g_d; g_\delta](xy) = [g_d; g_\delta](x)y + x[d; \delta](y)$. Thus $[g_d; g_\delta] \subseteq gP \;\Rightarrow [d; \delta] \subseteq P$ and so $[P; D(R)] \subseteq P$. Hence P is a Lie ideal of $D(R)$. Now it remains to show that P is a prime Lie ideal. Let A and B are two Lie ideals of $D(R)$ such that $[A; B] \subseteq P$ and let $B \nsubseteq P$. This implies that there exists $d \in B$ such that $d \notin P \;\Rightarrow\; g_d \in gB$ but $g_d \notin gP \Rightarrow gB \nsubseteq gP$ where $gB = \{g_d | d \in B\}$. Let $gA = \{g_\delta | \delta \in A\}$. Let $\delta \in A$ and $d \in B$ then $g_\delta \in\; gA$ and $g_d \in gB$. Now for all $x, y \in R$, we have $[g_\delta; g_d](xy) = [g_\delta; g_d](x)y + x[\delta;\, d](y)$. Since $[\delta;\, d] \in [A; B] \subseteq P \Rightarrow [g_\delta; g_d] \in gP \Rightarrow gA \subseteq gP$ or $gB \subseteq gP$, but $gB \nsubseteq gP \Rightarrow gA \subseteq gP$. Thus $A \subseteq P$. Hence P is a prime Lie ideal of $D(R)$.

Case-2: Suppose gP contains only left multipliers of R. Then $P = \{0\}$. Let A and B are two Lie ideals of $D(R)$ such that $[A; B] \subseteq P = \{0\}$ and let $B \neq \{0\}$, then there exists $0 \neq d \in B$. Let $gB = \{g_d | d \in B\}$ and $gA = \{g_\delta | \delta \in A\}$. Then for all $x, y \in R$, we have $[g_\delta; g_d](xy) = [g_\delta; g_d](x)y \,+\, x[\delta; d](y)$, where $\delta \in A$ and $d \in B$. Thus $[\delta; d] \in [A; B] \subseteq P = \{0\}$. This implies $[g_\delta; g_d] \in gP$. Hence $[gA; gB] \subseteq gP$, which implies $gA \subseteq gP$ or $gB \subseteq gP$ but $gB \nsubseteq gP$ because gB contains generalized derivation other than left centralizer so $gA \subseteq\; gP$ and hence $A \subseteq P = \{0\}$. Thus $A = \{0\}$.

Corollary 3.3: $D(R)$ is a prime (semiprime) Lie ring if and only if $gD(R)$ is prime (semiprime) Lie ring

For any noncommutative ring R, the set of all left multipliers form a Lie ring. We can use this ring to prove the following result.

Theorem 3.4: If R is a noncommutative ring with unity then the Lie ring of R denoted by $(L(R), +, [\,;\,])$ can not be a prime Lie ring.

Proof: Since R has the unity, $\mathcal{T} = \{T_a | a \in R, T_a x = ax, \forall x \in R\}$. Define $\phi : L(R) \to \mathcal{T}$ by $\phi(a) = T_a$, for all $a \in R$. Then ϕ is a Lie isomorphism. Let $A = \{T_a \,| a \in Z(R)$, the center of $R\}$, then $[A; \mathcal{T}] = \{0\} \subseteq A$, i.e. A is a nonzero Lie ideal of $\mathcal{T}$. Let B be any other non zero Lie ideal of $\mathcal{T}$. Then $[A; B] \,=\, 0$. Thus $\mathcal{T}$ is not a prime Lie ring and hence $L(R)$ is not a prime Lie ring.

Lemma 3.5: [[5], Theorem 2] Let R be a noncommutative prime ring of characteristic not equal to 2. Then $D(R)$ is a prime Lie ring.

Theorem 3.6: Let R be a noncommutative prime (semiprime) ring of characteristic not equal to 2. Then $gD(R)$ is a prime (semiprime) Lie ring.

Proof: It follows from Corollary 3.3 and Lemma 3.5.

In [5], it is proved that if the ring R is prime (semiprime) of characteristic not equal to 2, then $I(R)$, the collection of all inner derivations, is a prime (semiprime) Lie ring. Here we show that $g_i D(R)$, the collection of all generalized inner derivations is not a prime Lie ring.

Theorem 3.7: For any ring R, the following hold:

1. If $g_{a,b}, g_{c,d} \in g_i D(R)$, for $a, b, c \in R$ then $[g_{a,b}; g_{c,d}] = g_{[a,c],[d,b]}$ which implies that $g_i D(R)$ is Lie subring of $gD(R)$.
2. $g_i D(R)$ is not a prime Lie ring.

Proof: First is easy to compute. For second, let $A = \{g_{a,b} | a \in Z(R)\}$ and $B = \{g_{c,d} | d \in Z(R)\}$, then A and B are non zero Lie ideals of $g_i D(R)$, but $[A; B] = 0$ by 1.

For a commutative ring R we have the following:

Lemma 3.8: Let R be a commutative ring. Let $g_\delta \in gD(R)$ and $rg_\delta : R \to\, R$ given by $(rg_\delta)(s) = rg_\delta(s)$ then $rg_\delta \in gD(R)$ and hence Rg_δ is a subring of $gD(R)$.

Theorem 3.9: Let R be a commutative ring and $0 \neq \delta \in D(R)$, then $R\delta$ and Rg_δ are Lie isomorphic.

Proof: Define $\phi: Rg_\delta \to R\delta$ by $\phi(rg_\delta) = r\delta$. Let $x, y \in R$ then $(rg_\delta)(xy) = (rg_\delta)(x)y + x(r\delta)(y)$, means that (rg_δ) is a generalized derivation associated with the derivation $(r\delta)$. The function ϕ is a well defined, one-one and onto. Let $rg_\delta, sg_\delta \in R\delta$ then $[rg_\delta; sg_\delta]$ is a generalized derivation associated with the derivation $[r\delta; s\delta]$. So, $\phi[rg_\delta; sg_\delta] = [r\delta; s\delta] = [\phi(rg_\delta); \phi(sg_\delta)]$. Hence Rg_δ and $R\delta$ are Lie isomorphic.

Lemma 3.10: [[5], Theorem 6] Let R be a commutative domain with unity and characteristic not equal to 2. If $0 \neq \delta \in D(R)$, then $R\delta$ is a prime Lie ring.

Theorem 3.11: Let R be a commutative domain with unity and characteristic not equal to 2. If $0 \neq g_\delta \in gD(R)$, then Rg_δ is a prime Lie ring.

Proof: It follows from Theorem 3.9 and Lemma 3.10.

Lemma 3.12: [[5], Theorem 11] Let R be a $2-$torsion free commutative semiprime ring. Then $R\delta$ is a semiprime Lie ring, for all $\delta \in D(R)$.

Theorem 3.13: Let R be a $2-$torsion free commutative semiprime ring. Then Rg_δ is a semiprime Lie ring, for all $g_\delta \in gD(R)$.

Proof: It follows from Theorem 3.9 and Lemma 3.12.

Theorem 3.14: Let R be a $2-$torsion free commutative semiprime ring. Then $gD(R)$ is a semiprime Lie ring.

Proof: We prove the result for a ring R^1 with unity 1. If ring R does not have unity, then we can associate it with unity in a natural way and since $gD(R^1) \cong gD(R)$, we can assume all ring with unity. Now every element of $gD(R)$ is in the form rg_d, possibly the value of r is 1.

Let gP be an Lie ideal of $gD(R)$, such that $[gP;\ gP] = 0$. We note that the kernel of the epimorphism in Theorem 3.1 has no Lie structure in this case and $\mathcal{T} \subseteq gP$ for any Lie ideal of $gD(R)$. Consider $\mathcal{T} \subsetneq gP$. Let $g_d \in gP \backslash \mathcal{T}$ and $r \in R$ then$[g_d;\ rg_d] \in gP$. If $P = \{d \in D(R) | g_d \in gP\}$ then$[d;\ rd] \in P$, which gives $d(r)d \in P$, so $[d;\ d(r)d] \subseteq [P;\ P]$. Let ϕ be a Lie epimorphism as in Theorem 3.1 then we have $0 = \phi(0) = \phi[gP;\ gP] = [P;\ P]$. Thus we have $[d;\ d(r)d] \subseteq [P;\ P] = 0$ or $d^2(r)d(x) = 0$, for all $x \in R$. In particular for $x = d(r)$, we have $d^2(r)d^2(r) = 0$. Since R is commutative semiprime ring we have $d^2(r) = 0$. So by Lemma 2.3 we get $d = 0$. Since ϕ is a Lie epimorphism then our choice of g_d shows that $g_d = 0$. Hence $gD(R)$ is a semiprime Lie ring.

4. EXAMPLE

The following examples show that the primeness and characteristic of ring are crucial in our results.

Example 4.1: Consider Heisenberg group over $\mathbb{Z}_3$,
i.e. $Heis(\mathbb{Z}_3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \ \mathbb{Z}_3 \right\}$. Then $O\big(Heis(\mathbb{Z}_3)\big) = 3^3$ and $O(Z(Heis(\mathbb{Z}_3))) = 3$. Center of this group is generated by an element $x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of order 3. Now consider the group ring $R = \mathbb{Z}Heis(\mathbb{Z}_3)$, then by Lemma 2.4, every derivation on R is inner. Since this ring has the unity, so every generalized derivation is a generalized inner derivation by Lemma 2.2. Thus $gD(R) = g_iD(R)$. Hence $gD(R)$ is not a prime Lie ring by Theorem 3.7. Note that the ring R is not a prime ring. Since $(I - x)R(I + x + x^2) = (I - x)(I + x + x^2)R = (I - x^3)R = 0$, but neither $I - x = 0$ nor $I + x + x^2 = 0$.

Example 4.2: Let $\mathbb{F}$ be a field of 2 elements. Let $R = \mathbb{F}_2$, the matrix ring over $\mathbb{F}$. Then every derivation on R is an inner derivation by Lemma 2.5. So $gD(R) = g_iD(R)$ by Lemma 2.2, and hence $gD(R)$ is not a prime Lie ring by Theorem 3.7.

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