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INVERSE CLOSED ALGEBRA CONVEX-CONES IN ORDERED BANACH ALGEBRAS

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ABSTRACT

Let A be a Banach algebra, with identity 1 and C be an algebra convex-cone in ordered Banach algebra (A, C). We investigate some results in ordered Banach algebra (A, C) with a proper, closed and inverse closed algebra convex-cone C and a∈A such that σ(a)={1}.

***Keywords**: Banach Algebras; Spectral Radius; Spectrum; Algebra Cones.*

***MSC (2010)**: 47B60; 47A10; 46H35; 47B65; 47L07.*

1. INTRODUCTION

A complex Banach algebra A with identity 1 is classified in paper [12] by H. Raubenheimer and S. Rode as an ordered Banach algebra (OBA), if there is distinguished a positive cone $C \subseteq A$ which is closed under non negative real linear combinations and norm limits, contains the identity, and is also closed under multiplication. In paper [3], R. Harte defined a partially ordered Banach algebra as a Banach algebra ordered by a cone that contains the unit and is closed under addition, positive scalar multiplication and multiplication by commuting positive elements.

In this paper we investigate some results about an element $a \in A$ such that $\sigma(a) = \{1\}$ and $a^N \in C$ for some $N \in \mathbb{N}$, where C is an inverse closed algebra convex-cone of an ordered Banach algebra (A, C). The ordering that we introduce is via an algebra convex-cone. For basic properties of ordered Banach algebras see [9], [12] and [13].

In section 2, we provide the definitions and basic properties of elements in Banach algebras. In section 3, we define an algebra convex-cone C of a unital complex Banach algebra A and ordered Banach algebra (OBA). Some results on OBA are also proved. In section 4, we prove our main results in inverse closed algebra convex-cone C of an OBA (A, C).

2. PRELIMINARY

Throughout A (or B) will be a complex unital Banach algebra and the field of complex numbers $\mathbb{C}$. A homomorphism φ from a Banach algebra A into a Banach algebra B is a linear map $\varphi: A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$ and $\varphi(1) = 1$. The spectrum of an element a in A will be denoted by $\sigma(a)$ and is defined by $\sigma(a, A)$ or $\sigma(a) = \{\alpha \in \mathbb{C}: a - \alpha \text{ is not invertible}\}$, the spectral radius of a in A will be denoted by $r(a)$ and is defined by $r(a, A)$ or $r(a) = sup\{|\lambda|: \lambda \in \sigma(a)\}$. By spectral radius formula we have

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf_n \|a^n\|^{\frac{1}{n}}.$$

By ([10], Theorem 1.11), $\sigma(a)$ is a closed subset of $\mathbb{C}$. The resolvent set of a is defined by $\rho(a, A)$ or $\rho(a) = \mathbb{C}\backslash\sigma(a)$. The function $\lambda \to (a - \lambda)^{-1}$ defined in the open set $\rho(a, A)$ is called the resolvent of a. Also by Theorem 1.16 in paper [10], the resolvent function $\lambda \to (a - \lambda)^{-1}$ is analytic in $\mathbb{C}\backslash\sigma(a)$.

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An element a in a unital Banach algebra A is said to be left invertible in A if there is some $z \in A$ such that $za = 1$. Also a is said to be right invertible if there is some $z \in A$ such that $az = 1$. An element a in A is said to be invertible (or non-singular) in A if there is some $z \in A$ such that $az = za = 1$. Note that if such a z exists, then it is unique; for if $z^{'}a = az^{'} = 1$, then $z = z1 = zaz^{'} = 1z^{'} = z^{'}$. z is called the inverse of a, and as usual it is written a^{-1}. The set of all invertible elements will be denoted by $Inv(A)$. Non-invertible elements are also called singular.

Theorem 2.1: ([1], Theorem 3.2.1) Suppose that A is a Banach algebra and $a \in A$ such that $\|a\| \le 1$. Then $1 - a$ is invertible and $(1-a)^{-1} = \sum_{k=0}^{\infty} a^k$.

The series in Theorem 2.1 is called the Neumann series for $(1-a)^{-1}$.

Theorem 2.2: ([1], Theorem 3.2.3) Suppose that A is a Banach algebra and that a is invertible. If $\|x - a\| < \frac{1}{\|a^{-1}\|}$, then x is invertible. Moreover the mapping $x \to x^{-1}$ is a homeomorphism from $Inv(A)$ onto $Inv(A)$.

Theorem 2.3: ([1], Lemma 3.1.2 (N. Jacobson)). Let A be a Banach algebra with unit 1 and let $a, b \in A$, $\lambda \in \mathbb{C}$, with $\lambda \neq 0$. Then $\lambda - ab$ is invertible in A if and only if $\lambda - ba$ is invertible in A.

Theorem 2.4: ([1], Theorem 3.2.8) Let A be a Banach algebra and $a \in A$. Then the function $\lambda \to (\lambda 1 - a)^{-1}$ is analytic on $\mathbb{C} - \sigma(a)$ and goes to 0 at infinity.

Theorem 2.5: ([1], Corollary 3.2.9) (I. M. Gelfand-S. Mazur). If A is a Banach algebra in which every non-zero element is invertible then A is isometrically isomorphic to $\mathbb{C}$.

Let $p(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with coefficients $\alpha_i \in \mathbb{C}$ for all $i = 1, 2, \dots, n$. For $a \in A$ we write $p(a) = \sum_{i=0}^{n} \alpha_i a^i$. The mapping $p \to p(a)$ is homomorphism from the algebra of all polynomials to Banach algebra A.

Theorem 2.6: ([10], Proposition 1.1.34) (Spectral mapping theorem) Let a be an element of a Banach algebra A and let $p(z)$ be a polynomial. Then $\sigma(p(a)) = p(\sigma(a))$.

Let A be a Banach algebra. An element $a \in A$ is said to be idempotent if $a^2 = a$. An element $a \in A$ is said to be nilpotent if there is a natural number n such that $a^n = 0$. The set of nilpotent elements of Banach algebra A will be denoted by $N(A)$. If $a \in A$ such that $\sigma(a) = 0$, then a is said to be quasinilpotent. The set of quasinilpotent elements of A will be denoted by $QN(A)$. It is well known that in a finite-dimensional Banach algebra all quasinilpotents are nilpotent. We notice that if $a \in A$ is nilpotent, then it is quasinilpotent, since

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|0\|^{\frac{1}{n}} = 0.$$

Theorem 2.7: ([2], Theorem 1.2.9). Let A be a Banach algebra. If $a \in A$ satisfies $r(a) < 1$, then $1 - a$ is invertible and $(1-a)^{-1} = 1 + \sum_{k=1}^{\infty} a^k$.

3. ORDERED BANACH ALGEBRAS

In ([12], section 3), we defined an algebra cone C of a complex Banach algebra A and showed that C induced on A an ordering that was compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra (OBA). We recall those definitions now and also the additional properties that C may have.

Let A be a complex Banach algebra with unit 1. We call a nonempty subset C of A a cone of A if C satisfies the following properties:

1. $C + C \subseteq C$
2. $\lambda C \subseteq C$ (for $0 \le \lambda \in R$).

   If in addition C satisfies $C \cap -C = \{0\}$, then C is called a proper cone.

   Any cone C of A induces an ordering $\le$ on A in such that $a \le b$ if and only if $b - a \in C$ for $a, b \in A$.

   It can be shown that this ordering is a partial order on A, i.e., for every $a, b, c \in A$,
   (a) $a \le a$ ($\le$ is reflexive),
   (b) if $a \le b$ and $b \le c$, then $a \le c$ ($\le$ is transitive).

   Furthermore, C is proper if and only if this partial order has the additional property of being antisymmetric, i.e. if $a \le b$ and $b \le a$, then $a = b$.

Considering the partial order that C induces we find that $C = \{a \in A: a \geq 0\}$ and therefore we call the elements of C positive.

A cone C of a Banach algebra A is called an algebra cone of A if C satisfies the following conditions:

3. $C.C \subseteq C$,
4. $1 \in C$.

Motivated by this concept we call a complex Banach algebra with unit element 1 an ordered Banach algebra (OBA) if A is partially ordered by a relation " $\leq$ " in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$,

(1) $a, b \geq 0 \ \Rightarrow a + b \geq 0$,
(2) $a \geq 0, \lambda \geq 0 \ \Rightarrow \lambda a \geq 0$,
(3) $a, b \geq 0 \Rightarrow ab \geq 0$,
(4) $1 \geq 0$.

Therefore, if A is ordered by an algebra cone C, then A, or more specifically (A, C), is an OBA.

A cone C of a Banach algebra A is called *algebra convex-cone* if it satisfies the following:

(i) $ab \in C$ for all $a, b \in C$ such that $0 \leq \lambda \leq 1$ implies λa+(1-λ)b∈C.
(ii) $1 \in C$, where 1 is the unit of A.

An algebra convex-cone C of A is called proper if C is a proper convex-cone of A, and closed if it is a closed subset of A. Furthermore, C is said to be normal if there exists a constant $\alpha > 0$ such that it follows from $0 \leq a \leq b$ in A that $\|a\| \leq \alpha\|b\|$.

Note that if $a \in C \cap -C$, then it follows that $-a \in C$ i.e., $a \leq 0$. Hence if C is normal, with normality constant $0 < \alpha \in \mathbb{R}$, then $0 \leq \|a\| \leq \alpha\|0\| = 0$ and so $a = 0$. Therefore every normal cone is proper.

If an algebra cone C has the property that if $a \in C$ and a is invertible, then $a^{-1} \in C$, then C is said to be inverse-closed.

Example 3.1: Let L be a non-zero complex Banach lattice and let$= \{x \in L: x = \|x\|\}$. $If\ K = \{T \in L(L): TC \subseteq C\}$, then K is a closed, normal algebra cone of $L(L)$. Therefore $(L(L), K)$ is an OBA.

Definition 3.2: Let (A, C) be an OBA. If $0 \leq a \leq b$ relative to the algebra convex-cone C implies $r(a) \leq r(b)$, then the spectral radius (function) is monotone w. r. t. the algebra convex-cone C.

Theorem 3.3: Let $(A\ C)$ be an OBA with a normal algebra convex-cone C. Then the spectral radius is monotone w. r. t. C.

Proof: Let $0 \leq a \leq b$, then by using Principal of Mathematical Induction, we see that $0 \leq a^n \leq b^n$. Let α be the constant of normality, then $\|a^n\| \leq \alpha\|b^n\|$ for all $n \in \mathbb{N}$, so

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} (\alpha\|b\|^{\frac{1}{n}})$$
$$= \lim_{n\to\infty}(\alpha^{\frac{1}{n}}) . \lim_{n\to\infty}(\|b\|^{\frac{1}{n}}) = r(b).$$

The converse of theorem 3.3 is in general not true. Also if the algebra convex-cone is not normal, the spectral radius may not be monotone. See Example 4.2 in paper [12].

Theorem 3.4: Let (A, C) be an OBA with algebra convex-cone C such that the spectral radius is monotone. Let $a, b \in A$ be such that $0 \leq a \leq b$ relative to C. If b is quasinilpotent, then a is quasinilpotent.

Proof: If b is quasinilpotent, then $r(b) = 0$. So from Theorem 3.3, we have $0 \leq r(a) \leq 0$, which gives $r(a) = 0$. Hence a is quasinilpotent.

Theorem 3.5: ([12], Theorem 4.4)**.** Let (A, C) be an OBA with normal algebra cone C and $a, b \in C$. If $ab \leq ba$, then $r(ab) \leq r(b)r(a)$ and $r(ab) \leq r(a)r(b)$.

By using above theorem, it can be easily proved that $r(a + b) \leq r(a) + r(b)$.

Theorem 3.6: ([12], Proposition 5.1) Let A be an OBA with a closed normal algebra cone C and $a \in C$. Then $r(a) \in \sigma(a)$.

Theorem 3.7: ([12], Theorem 5.2)**.** Let (A, C) be an OBA with a closed algebra cone C such that the spectral radius function is monotone. If $a \in C$, then $r(a) \in \sigma(a)$.

4. INVERSE CLOSED ALGEBRA CONVEX-CONES

Let A be a Banach algebra with unit 1. Let C be an inverse closed algebra convex-cone. Let an element a with unit spectrum belongs to an ordered Banach algebra A. We prove that if $a^N \in C$ for some $N \in \mathbb{N}$, then a is unit element in A.

Theorem 4.1: Let (A, C) be an ordered Banach algebra with a proper, closed and inverse closed algebra convex-cone C and $a \in A$ such that $\sigma(a) = \{1\}$. If $a^N \in C$ for some $N \in \mathbb{N}$, then $a = 1$.

Proof: Since $\sigma(a) = \{1\}$, therefore from Theorem 3.6 and Theorem 2.7, for $|\lambda| > 1$ we have

$$(\lambda 1 - a^N)^{-1} = \sum_{k=0}^{\infty} \frac{a^{Nk}}{\lambda^{k+1}}$$

Let $\lambda > 1$. Since $a^N \in C$ and C is closed and convex-cone, so $a^N \geq 0$ and it follows that $(\lambda 1 - a^N)^{-1} \in C$. Also C is inverse closed, therefore we have $\lambda 1 - a^N \in C$ for $\lambda > 1$. Let us take the limit $\lambda \to 1^+$ and since C is closed, therefore it again follows that $1 - a^N \in C$.

Since $a^N \in C$ and C is inverse closed, so $a^{-N} \in C$. From the similar argument as above we conclude that $1 - a^{-N} \in C$.

Again C is closed algebra convex-cone, so it is algebraically closed under multiplication, we have
$a^N - 1 = a^N(1 - a^{-N}) \in C$.

Now $1 - a^N \in C$, $a^N - 1 \in C$ and C is proper, we must have $a^N - 1 = 0$ or $a^N = 1$.

Again by factorization, we have

$$a^N - 1 = (a - 1)(a^{N-1} + a^{N-2} + \cdots + 1) = 0$$

Since $\sigma(a) = \{1\}$ and by Theorem 2.6, we have $a^{N-1} + a^{N-2} + \cdots + 1$ is invertible. Therefore $a = 1$.

The next theorem states that the elements of a closed, inverse closed algebra convex-cone, are dominated by their spectral radii.

Theorem 4.2: Let (A, C) be an ordered Banach algebra with closed and inverse closed algebra convex-cone C. If $a \in C$, then $0 \leq a \leq r(a)1$.

Proof: Let $|\lambda| > r(a)$, so from Theorem 2.7, we have

$$(\lambda 1 - a)^{-1} = \sum_{k=0}^{\infty} \frac{a^k}{\lambda^{k+1}}$$

Since $a \in C$ and C is closed algebra convex-cone, so it is algebraically closed under multiplication and addition, it follows that for all $|\lambda| > r(a)$, $\sum_{k=0}^{\infty} \frac{a^k}{\lambda^{k+1}} \in C$. Hence for all $|\lambda| > r(a)$, $(\lambda 1 - a)^{-1} \in C$. Again since the algebra convex-cone C is inverse closed, it follows that for all $|\lambda| > r(a)$ we have $\lambda 1 - a \in C$. Let us take the limit $\lambda \to r(a)^+$ and since C is closed, therefore it again follows that $r(a)1 - a \in C$. Since C is algebra convex-cone, so its elements are positive and hence $0 \leq a \leq r(a)1$.

Theorem 4.3: Let (A, C) be an ordered Banach algebra with a proper, closed and inverse closed algebra convex-cone$C$. Then $QN(A) \cap C = \{0\}$.

Proof: It is obvious that $0 \in QN(A) \cap C$.

Let $a \in QN(A) \cap C$. Therefore $a \in QN(A)$ and so $r(a) = 0$.

From Theorem 4.2, we have $0 \leq a \leq 0.1 = 0$. Hence $a \leq 0$ and $0 \leq a$ and since C is proper, therefore $a = 0$.

One of the simplest examples of a proper, closed and inverse closed algebra convex-cone is obtained if we consider the cone of all $n \times n$ diagonal matrices with nonnegative real entries. It is clear that the spectrum of an element belonging to the cone is the set of points on the diagonal. Thus, a quasinilpotent element of the cone has zeros on the main diagonal, and is therefore the zero matrix.

5. CONCLUSIONS

Let A be a Banach algebra, with identity 1. Let (A, C) be an ordered Banach algebra. We proved some results in OBA (A, C) and investigated some results in (A, C) with a proper, closed and inverse closed algebra convex-cone C and $a \in A$ such that $\sigma(a) = \{1\}$.

REFERENCES

1. Aupetit, B.: A primer on spectral theory, Springer-Verlag, New York, 1991.
2. Bonsall, F.F. and Duncan, J.: Complete normed algebras, Springer Verlag, Berlin, 1973.
3. Harte, R.: Tim Starr, Trevor West and a positive contribution to operator theory, Filomat, 21, 1 (2007), 129-135.
4. Herzog, G. and Schmoeger, C.: A note on a theorem of Raubenheimer and Rode, Proceedings of the American Mathematical Society, Vol. 131, 11 (2003), 3507-3509.
5. Kinani, A. E., Nejjari, M. A. and Oudadess, M.: Normal cones and strictly real algebra structure, Rev. Real Academia de Ciencias, Zaragoza, 60 (2005), 91–97.
6. Mouton, S.: A condition for spectral continuity of positive elements, Proceedings of the American Mathematical Society, Vol. 137, 5 (2009), 1777-1782.
7. Mouton, S.: On spectral continuity of positive elements, Studia Mathematica, Vol. 174, 1(2006), 75-84.
8. Mouton, S.: On the boundary spectrum in Banach algebras, Bull. Austral. Math. Soc., Vol. 74 (2006), 239-246.
9. Mouton, S. and Raubenheimer, H.: More spectral theory in ordered Banach algebras, Positivity, 1 (1997), 305-317.
10. Muller, V.: Spectral theory of linear operators and spectral system in Banach algebras, (2nd ed.). Birkhäuser Verlag, Basel, (2007).
11. Nagler, J.: On the spectrum of positive finite-rank operators with a partition of unity property, arXiv: 1403.4522v1 [math.CA] 18 Mar 2014.
12. Raubenheimer, H. and Rode, S.: Cones in Banach algebras, Indag. Math., 7(4) (1996), 489-502.
13. Render, H.: Lattice structures of ordered Banach algebras, Illinois Journal of Mathematics, Vol. 36, No. 2 (1992), 238-250.
14. Rudin, W.: Functional analysis, Tata McGraw-Hill, New Delhi, 2006.

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