International Journal of Mathematical Archive-5(8), 2014, 56-61
Available online through www.ijma.info ISSN 2229 – 5046

GENERALIZED CONE METRIC SPACES AND FIXED POINT THEOREM FOR CONTRACTIVE MAPPING

S. K. Tiwari*[1], Bhudhar Tripathi[2] and R. P. Dubey[3]

[1]Department of Mathematics, Dr. C. V. Raman University, Kota, Bilaspur (C.G), India.

(Received On: 11-08-14; Revised & Accepted On: 31-08-14)

ABSTRACT

In this paper, we prove some common fixed point theorems for contractive condition on generalized cone metric spaces (G-cone metric spaces). Our results are generalization of the results of I. Beg, M. Abbas and T. Nazir [14].

Keywords: *Fixed Point, common fixed point, contractive condition, Generalized Cone Metric Space.*

Mathematics Subject Classification: *Primary 47H10; Secondary 54H25.*

1. INTRODUCTION

In 2005, Mustafa and Sims [6] introduced a new structure of generalized metric spaces which are called G- metric spaces as a generalization of metric spaces. Afterwards Mustafa *et.al.* [(7), (a)]. Obtained several fixed point theorems for mappings satisfying different contractive condition in G – metric spaces.

Later on, Huang and Zhang [10] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive condition. The normality property of cone was an important ingredient in their result (see also [11] and [12]. After wards Rezapour and Hamlbarani [13] omitting the assumption of normality of cone generalized some result [10].

Recently, I. Beg, M. Abbas and T. Nazir [14] introduced G-cone metric spaces which are generalization of G-metric spaces and cone metric space. They proved some topological properties of these spaces such as convergence properties of sequences and completeness. Some fixed point theorems satisfying certain contractive conditions have been also obtained.

Some theorem which given with φ- maps have been proved by Critina Di Bari and Pasquale Vetro [15] in cone metric spaces and W. Shatanawi[16] also obtained some fixed point result in G- metric spaces. Ramakant Bhardwaj [17] proved some common fixed point theorems for two mapping s satisfying the integral type contractive mapping in the setting of generalized cone metric spacer. In the present paper, we study the existence of a unique common fixed point theorem for contractive condition in generalized cone metric spaces. Our results have generalized of comparable results in the literature given by [14].

2. PRELIMINARY NOTES

First we recall the definition of generalized cone metric spaces and some properties of theirs [14].

Definition: 2.1 [14] Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if:
(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in \boldsymbol{P} \Rightarrow ax+by \in \boldsymbol{P}$;

Given a cone $P \subset E$, we define a partial ordering$\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in X$,

$$0 \leq x \leq y \text{ Implies that } \|x\| \leq K\,\|y\|.$$

Corresponding author: S. K. Tiwari*[1]
[1]Department of Mathematics, Dr. C. V. Raman University, Kota, Bilaspur (C.G), India.

The least positive number satisfying the above inequality is called the normal constant of P,while $x \ll y$ stands for $y - x \in intP$. Rezapour and Hamlbarani [13] proved that there are no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.

Definition 2.2[14] Let X be a non empty set. Suppose a mapping $G: X \times X \times X \to E$ satisfies
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, y, z)$: whenever$x \neq y$, for all $x, y \in X$,
(G_3) $G(x, x, y) \leq G(x, y, z)$: whenever $y \neq z$,
(G_4) $G(x, y, z) = G(x, y, z) = (G(x, y, z) =$………. (Symmetric in all three variables),
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all$x, y, z, a \in X$.

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or more specifically a $G -$cone metric space.

The concept of a $G -$ cone metric space or generalized cone metric space is more general than that of a $G -$metric spaces and cone metric spaces. For the definition of $G -$ metric, cone metric spaces and related concepts we refer the reader to [7, 8, 10 and 13].

Definition 2.3[14] A generalized cone metric space X is symmetric if
$G(x, y, z) = G(y, x, x)$ for all $x, y \in X$.

Example 2.4[14] Let (X, d) be a cone metric space. Define $G: X \times X \times X \to E$, by
$G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Example 2.5[14] Let $X = \{a, b\}$,$= R^3$, $P = \{(x, y, z) \in E: x, y, z \geq 0\}$. Define $G: X \times X \times X \to E$, by
$G(a, a, a) = (0, 0.0) = G(b, b, b)$,
$G(a, b, b) = (0, 1, 1) = G(b, b, a) = G(b, b, a)$,

$G(b, a, a) = (0, 1, 0) = G(a, b, a) = G(a, a, b)$

Note that X is non symmetric G-cone metric spaces as $G(a, a, b) \neq G(a, b, b)$.

Definition 2.6: [14] Let X be a generalized cone metric space and $\{x_n\}$ be a sequence in X.Then,
- (i) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $o \ll c$, there is a natural number N such that $G(x_n, x_m, x_l) \ll c$ for all$l$ $n, m > N$.
- (ii) $\{x_n\}_{n \geq 1}$ is convergent sequence if for every c in E with $0 \ll c$,there is N such that for all $m, n > N, G(x_m, x_n, x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
- (iii) (X, d) is called a complete generalized cone metric space if every Cauchy sequence in X is convergent in X.

Proposition 2.6: [14] Let X be a generalized cone metric space, define $d_G: X \times X \to E$ by
$$d_G(x, y) = G(x, y, y) + G(y, x, x).$$
Then (X, d_G) is a cone metric space.

Proposition 2.6: [14] Let X be a generalized cone metric space then the following are equivalent
(i) $\{x_n\}$ is converges to x.
(ii) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(iii) $G(x_n, x, x) \to 0$, as $n \to \infty$.
(iv) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Lemma 2.7: [14]
- (i) Let X be a generalized cone metric space and let $\{x_m\}, \{y_n\}$ and $\{z_l\}$ be sequence in Xsuch that
  $x_m \to x, y_n \to y$ and $z_l \to z$, then
  $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.
- (ii) Let $\{x_n\}$ be sequence in generalized cone metric space in $X$and $x \in X$, then if $\{x_n\}$ converges to x, and $\{x_n\}$ converges to y , then $x = y$.
- (iii) Let $\{x_n\}$ be sequence in generalized cone metric space in $X$and if $\{x_n\}$ converges to x, for $\in X$, then
  a. $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.
- (iv) Let $\{x_n\}$ be sequence in generalized cone metric space in $X$and $x \in X$ if $\{x_n\}$ converges to x, for $\in X$, then. $\{x_n\}$ is a Cauchy sequence.
- (v) Let $\{x_n\}$ be sequence in generalized cone metric space in $X$and $x \in X$ if $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, y_n, z_l) \to 0$ as $m, n, l \to \infty$.

3. MAIN RESULT

The results which will give are generalization of theorem 3.1, 3.3 and 3.4 of [14].

Theorem 3.1: Let (X, d_G) be a complete symmetric generalized cone metric space (G-cone metric space) and $T_1, T_2 : X \to X$ be a mapping satisfying one of the following conditions

$$G(T_1x, T_2y, T_2z) \leq aG(x, y, z) + bG(x, T_1x, T_1x) + c\,G(y, T_2y, T_2y) + dG(z, T_2z, T_2z) \quad (1)$$

OR

$$G(T_1x, T_2y, T_2z) \leq aG(x, y, z) + bG(x, T_1x, T_1x) + cG(y, T_2y, T_2y) + dG(z, T_{2,} T_2) \quad (2)$$

for all $x, y, z \in X$, where $0 \leq a + b + c + d < 1$. Then T_1 and T_2 have a unique common fixed point in X.

Proof: Suppose that T_1 and T_2 satisfies condition (1), then for all $x, y \in X$

$$G(T_1x, T_2y, T_2y) \leq aG(x, y, z) + bG(x, T_1x, T_1x) + (c + d)G(y, T_2y, T_2y) \quad (3)$$

and

$$G(T_2y, T_1x, T_1x) \leq aG(x, y, z) + bG(x, T_2y, T_2y) + (c + d)G(x, T_1x, T_1x) \quad (4)$$

Since X is a symmetric generalized cone metric space (G- Cone metric space), Therefore by adding (3) and (4), we have

$d_G(T_1x, T_2y) \leq a d_G(x, y) + \frac{b+c+d}{2}\, d_G(x, T_1x) + \frac{b+c+d}{2}\, d_G(y, T_2)$ for all $x, y \in X$.

If $\alpha = a,\ \beta = \gamma = \frac{b+c+d}{2}$. Then

$d_G(T_1x, T_2y) \leq \alpha d_G(x, y) + \beta d_G(x, T_1x) + \gamma\, d_G(y, T_2)$ for all $x, y \in X$,

where $\alpha + \beta + \gamma < 1$.

Let $x_0 \in X$ be an arbitrary point. We define a sequence

$x_{2n+1} = T_1 x_{2n} = T_1^{2n+1} x_0$ and

$x_{2n+2} = T_1 x_{2n+1} = T_1^{2n+2} x_0$.

Now we consider

$$\begin{aligned} d_G(x_{2n+1,} x_{2n}) &= d_G(T_1 x_{2n+1,}\, T_2 x_{2n}) \\ &\leq \alpha d_G(x_{2n+1}, x_{2n}) + \beta d_G(x_{2n+1}, T_1 x_{2n+1}) + \gamma d_G(x_{2n}, T_2 x_{2n}) \\ &\leq \alpha d_G(x_{2n+1}, x_{2n}) + \beta d_G(x_{2n+1}, x_{2n}) + \gamma d_G(x_{2n}, x_{2n-1}) \\ &\leq (\alpha + \beta)\, d_G(x_{2n+1}, x_{2n}) + \gamma d_G(x_{2n}, x_{2n-1}) \end{aligned}$$

$$1\text{-}(\alpha + \beta)\, d_G(x_{2n+1}, x_{2n}) \leq \gamma d_G(x_{2n}, x_{2n-1})$$

$$d_G(x_{2n+1}, x_{2n}) \leq \frac{\gamma}{1-(\alpha+\beta)}\, d_G(x_{2n}, x_{2n-1})$$

$d_G(x_{2n+1}, x_{2n}) \leq K\, d_G(x_{2n}, x_{2n-1})$, where $\frac{\gamma}{1-(\alpha+\beta)} = K < 1$. It follows that

$$d_G(x_{2n+1}, x_{2n}) \leq K^{2n}\, d_G(x_1, x_0).$$

For any $m > n$, we have

$$d_G(x_{2n+1}, x_{2n}) \leq \frac{K^{2n}}{1-K}\, d_G(x_1, x_0).$$

Let $0 \ll c$ be given, following similar argument to those given in [13, theorem 2.3] we conclude that

$$\frac{K^{2n}}{1-K}\, d_G(x_1, x_0) \ll c.$$

So we have $d_G(x_{2n}, x_{2m}) \ll c$, for all $m > n$. Therefore $\{x_{2n}\}$ is a Cauchy sequence and hence $x_{2n} \to x^*$. First prove that $x_{2n+1} \to T_1x^*$. For this, Take $x = T_1 2n$ and $y = x^*$ in (5), we have

$$\begin{aligned} d_G(T_1 x_{2n+1,}\, T_1x^*) &\leq \alpha d_G(T_1 x_{2n}, x^*) + \beta d_G(T_1 x_{2n}, T_1 x_{2n+1}) + \gamma d_G(T_1x^*, x^*) \\ &\leq \alpha d_G(T_1 x_{2n}, x^*) + \beta d_G(T_1 x_{2n}, T_1 x_{2n+1}) + [\,\gamma\, d_G(T_1 x_{2n+1}, T_1x^*) + d_G(T_1 x_{2n+1}, x^*)] \\ &\leq \alpha d_G(T_1 x_{2n}, x^*) + \beta K^{2n} d_G(x_1, x_0) + [\,\gamma\, d_G(T_1 x_{2n+1}, T_1x^*) + d_G(T_1 x_{2n+1}, x^*)] \end{aligned}$$

Thus $d_G(T_1x_{2n+1,} T_1x^*) \;\leq \frac{1}{1-\gamma}\,[\alpha d_G(T_1x_{2n}, x^*) + \beta K^{2n} d_G(x_1, x_0) + \gamma d_G(T_1x_{2n+1}, x^*)]]$
$\ll c$

For any $c \in E$, which show that $T_1x_{2n+1} \to T_1x^*$ as $n \to \infty$. Now
$d_G(x^*, T_1x^*) \;\leq\; d_G(T_1x_{2n+1}, x^*) +\; d_G(T_1x_{2n+1}, T_1x^*)$

$\leq\; \frac{C}{2} + \frac{C}{2} = \text{c}$

Whenever $n > N$. Thus $d_G(x^*, T_1x^*) \;\leq \frac{C}{M}$, for all $m \geq 1$. So, $\frac{C}{M} - d_G(T_1x^*, x^*) \in P$, for all m$\geq$1. Since $\frac{C}{M} \to 0$ as $m \to \infty$ and P is closed, Therefore, - $d_G(T_1x^*, x^*) \in P$ gives $d_G(T_1x^*, x^*)$ = 0 and hence, $T_1x^* =\; x^*$ is a fixed point of T_1. If y^* is another fixed point of T_1. Then we have
$d_G(x^*, y^*) \;\leq \alpha d_G\;(x^*, y^*) + \beta d_G(x^*, T_1x^*) +\; \gamma d_G(x^*\, T_1y^*)$

Thus $d_G(x^*, y^*) \leq \alpha d_G\;(x^*, y^*) + \beta d_G(x^*, x^*) +\; \gamma d_G(x^*x^*)$
$\leq 0.$

Implies that $x^* =\; y^*$ is a unique fixed point of T_1. Similarly, we can prove that $T_2x^* = x^*$. Thus $T_1x^* =\; x^* = T_2x^*$. Hence x^* is the common Fixed point of pair maps of T_1 and T_2.

Remark 3.2: If X is not a Symmetric G- Cone Metric Space, then as in above theorem, adding (3) and (4) we obtain the following
$d_G(T_1x, T_2y) \leq\; ad_G(x, y) + \frac{2(b+c+d)}{3}\; d_G(x, T_1x) + \frac{2(b+c+d)}{3}\; d_G(y, T_2)$, for all $x, y \in X$.

Here $0 \leq a + \frac{2(b+c+d)}{3} + \frac{2(b+c+d)}{3}$ which may not be less than 1. So above theorem gives no information.

Theorem 3.3: Let (X, d) be a complete G- cone metric space and $T_1, T_2\colon X \to X$ be a mapping satisfying one of (1) or (2). Then T_1 and T_2 have a unique common fixed point in *X.*

Proof: Let $x_0 \in X$ be an arbitrary point. We define a sequence
$x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_0$ and

$x_{2n+2} = T_1x_{2n+1} = T_1^{2n+2}x_0$.

From (1) we have
$G\big(x_{2n,}x_{2n+1,}\, x_{2n+1}\big) \leq aG\big(\,x_{2n-1,}x_{2n,}x_{2n}\big) + bG\big(x_{2n-1,}\, x_{2n}, x_{2n}\big) + (c + d)G(x_{2n,}x_{2n+1}x_{2n+1})$

It implies that
$G\big(x_{2n,}x_{2n+1,}\, x_{2n+1}\big) \leq rG\;\big(\,x_{2n-1,}x_{2n,}x_{2n}\big)$,

Where $= \frac{a+b}{1-c-d}$. Obviously $0 \leq r < 1$. Continue this process to obtain
$G\big(x_{2n,}x_{2n+1,}\, x_{2n+1}\big) \leq\, r^{2n}G\;\big(\,x_{0,}x_{1,}x_2\big)$, Moreover for all $n, m \in N$ with $m > n$, we have

$$G\big(x_{2n,}x_{2m,}\, x_{2m}\big) \;\leq\; G\big(x_{2n,}x_{2n+1,}\, x_{2n+1}\big) + G\big(x_{2n,}x_{2n+2,}\, x_{2n+3}\big) + \ldots\ldots\ldots + G\big(x_{2m-1}\, x_{2m,}\, x_{2m}\big)$$

$$\leq\; (r^{2n} + r^{2n-1} + \cdots \ldots \ldots . + r^{2m-1})\; G\big(x_{0,}x_{1,}\, x_1\big)$$

$$\leq \frac{r^{2n}}{1-r}\, G\big(x_{0,}x_{1,}\, x_1\big).$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_{\delta(0)} \subseteq P$, where $N_{\delta(0)} = \{y \in E\colon \|y\| < \delta\}$. Also, choose a natural number N_1 such that $\frac{r^{2n}}{1-r}\; G\big(x_{0,}x_{1,}\, x_1\big) \in N_{\delta(0)}$, for all $m \geq N_1$. Then $\frac{r^{2n}}{1-r}\, G\big(x_{0,}x_{1,}\, x_1\big) \ll c$, for all $m \geq N_1$. So we have $G\big(x_{0,}x_{1,}\, x_1\big) \ll c$, for all $m \geq n$. Thus $\{x_{2n}\}$ is a Cauchy sequence, so there exist $u \in X$ such that $\{x_{2n}\}$ converges to x^*.

Now from (1), we get
$G\big(x_{2n,}T_1\, x^*, T_1\, x^*\big) \;\leq\; aG(\,x_{2n-1,}x^*, x^*) + bG\big(x_{2n-1,}x_{2n}, x_{2n}\big) + (c + d)G(x^*, T_1\, x^*, T_1\, x^*)$

Taking limit$n \to \infty$, we get
$G(x^*, T_1\, x^*, T_1\, x^*) \le (c+d)G(x^*, T_1\, x^*, T_1\, x^*)$,

This implies that $T_1\, x^* = x^*$. To prove uniqueness, suppose that $x^* \neq y^* = T_1 y^*$. Then
$G(x^*,\ y^*, y^*) \le aG(\,x^*, y^*, y^*) + bG(x^*, T_1\, x^*, T_1\, x^*) + (c+d)G(y^*, T_1\, y^*, T_1\, y^*)$

$= aG(\,x^*, y^*, y^*)$

Implies that $x^* = y^*$ is a unique fixed point of T_1. Similarly, we can prove that $T_2 x^* = x^*$. Hence $T_1 x^* = x^* = T_2 x^*$.Thus x^* is the common Fixed point of pair maps of T_1 and T_2.

Theorem 3.4: Let (X, d_G) be a complete symmetric generalized cone metric space (G-cone metric space) and $T_1, T_2 \colon X \to X$ be a mapping satisfying one of the following conditions
$G(T_1 x, T_2 y, T_2 y) \le a\{G(x\, T_2 y, T_2 y) + G(y, T_1 x, T_1 x)\ \}$ (6)
OR
$G(T_1 x, T_2 y, T_2 y) \le a\{G(x, x, T_2 y) + G(y, y, T_1 x)\ \}$ (7)
for all $x, y, \in X$, where $0 \in [0, \frac{1}{2}\,]$, Then T_1 and T_2 have a unique common fixed point in *X*.

Proof: Suppose that T_1 and T_2 satisfies condition (6), then for all $x, y \in X$

$G(T_1 x, T_2 y, T_2 z) \le a\{G(y, T_1 x, T_1 x) + G(x, T_2 y, T_2 y)\ \}$
and
$G(T_2 y, T_1 x, T_1 x) \le a\{G(x, T_2 y, T_2 y) + G(y, T_1 x, T_1 x)\ \}$

Now if X is a symmetric G-cone metric space, then above two inequalities give
$d_G(T_1 x, T_2 y) \le a\{d_G(x, T_2 y) + d_G(y, T_1 x)\}$, for all $x, y \in X$. Since$0 \le a < \frac{1}{2}$,

Therefore result follows from [13, theorem 2.3]. Now, if X is not a symmetric G-cone metric space. Then adding (6) and (7) we obtain
$d_G(T_1 x, T_2 y) = G(T_1 x, T_2 y, T_2 y) + G(T_2 y, T_1 x, T_1 x)$

$\le 2a\{\ G(y, T_1 x, T_1 x) + G(x, T_2 y, T_2 y)\}$

$\le \frac{4a}{3}\{d_G(y, T_1 x) + d_G(x, T_2 y)\ \}$,

For all $x, y \in X$. here, contractively factor $\frac{4a}{3}$ may not be less than 1. Therefore cone metric gives no information. In this case, let $x_0 \in X$ be an arbitrary point. We define a sequence
$x_{2n+1} = T_1 x_{2n} = T_1^{2n+1} x_0$

and $x_{2n+2} = T_1 x_{2n+1} = T_1^{2n+2} x_0$.

From (6) we have
$G\big(x_{2n,} x_{2n+1,}\, x_{2n+1}\big) \le a\{G\big(\,x_{2n-1,} x_{2n,} x_{2n}\big) + G(x_{2n,}\, x_{2n}, x_{2n})\}$

$= aG\big(x_{2n-1,} x_{2n+1,}\, x_{2n+1}\big)$.

But $G\big(x_{2n-1,} x_{2n+1,}\, x_{2n+1}\big) \le G\big(\,x_{2n-1,} x_{2n,} x_{2n}\big) + G(x_{2n,}\, x_{2n}, x_{2n})$

Thus we have
$G\big(x_{2n,} x_{2n+1,}\, x_{2n+1}\big) \le k\ G\big(\,x_{2n-1,} x_{2n,} x_{2n}\big)$,

Where $k = \frac{a}{1-a}$, and $0 \le k < 1$. Continuing above process, we obtain

$G\big(x_{2n,} x_{2n+1,}\, x_{2n+1}\big) \le k^{2n}\ G\big(\,x_{0,} x_1\, x_1\big)$.

Now following similar arguments as those given in theorem 3.3.Thus $\{x_{2n}\}$ is a Cauchy sequence, so there exist $u \in X$ such that $\{x_{2n}\}$ converges tox^*. Now we show that $T_1 x^* = x^*$. From (6) we have
$G\big(x_{2n,} T_1 x^*, T_1 x^*\big) \le a\{G\big(\,x_{2n-1,} T_1 x^*, T_1 x^*\big) + G(x^*, x_{2n}, x_{2n})$.

Which on taking limit$n \to \infty$, implies that $G(x^*, T_1x^*, T_1x^*) \leq aG(x^*, T_1x^*, T_1x^*)$, Thus $T_1x^* = x^*$. To prove uniqueness, suppose that $x^* \neq y^* = T_1y^*$. Then

$$G(x^*, y^*, y^*) \leq a\{G(x^*, y^*, y^*) + bG(y^*, x^*, x^*)$$

$$\leq kG(x^*, y^*, y^*)$$

Again we have $G(x^*, y^*, y^*) \leq k^2 G(x^*, y^*, y^*)$, which implies that $x^* = y^*$ is a unique fixed point of T_1.

Similarly, we can prove that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$.Thus x^* is the common Fixed point of pair maps of T_1 and T_2. This completes the proof of the theorem.

REFERENCES

1. S. Gahler, 2- metrics Raume ihre topologische structure, Math. Nachr, 26, 115-148 (1963)
2. S. Gahler, Zur geometric 2- Metrische Raume, Revue Roumain de Mathematiques Pures et appliqués 40, 664-669 (1969).
3. B. C. Dhage Generalized metric space and Mapping with fixed point, Bull. Cal. Math. Soc. 84(192), 329-336.
4. B. C. Dhage, Generalized Metric spaces and topological structure, I. Analele stintifice Universities Al. I. Cuza din, Iasi, Serie Naua. Mathematica, Vol.46, no.1, pp.3-24,(2000)
5. Ha. K.S., Cho, Y.J., and White A, Strictly convex and strictly 2-convex, 2-normed space, Mathematica Japonica, Vol. 33(3), pp. 375-384.(1.984)
6. Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point Theory, Ph. D thesis, the University of New castle, Australia (2005).
7. Z. Mustafa and B. Sims, some remarks concerning D- metric spaces, proceeding of the int.conference on fixed Point Theory and Applications. Yokohama, Yokohama, Japan (2004), 189 -198.
8. Z. Mustafa and B. Sims, A New Approach to generalized metric spaces, J. Nonlinear and convex, Anal. 7(2), 289- 297.(2006).
9. Z. Mustafa, O. Hamed and F. Awawdeh, some fixed point theorem for mappings on complete G-Metric spaces, Fixed point Theory and Application, 2008(2008), 1-12.
10. Huang, L. G.,and Zhang, X., cone m,etric spaces and fixed point theorems of contractive mappings, J. Math. Anal. and Appl. 332(2), 1468-1476, (2007).
11. M. Abbas and G. Jungck, common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341, 416-420. (2008).
12. M. Abbas and B. E. Rhoades fixed and periodic point results in cone metric spaces, APPL. Math. Lecture (2008).
13. S. Rezapour and R. Hamlbarani, Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings" J. Math. Anal. Appl. 345(2008), 719-724.
14. I. Beg, M. Abbas, and T. Nazir, generalized cone metric spaces, J. of Non-linear scienc and Appl. 1(2010), 21-31.
15. C. Di Bari and P.Vetro, φ-Pairs and common fixed points in cone metric space, Rendiconti del cir colo mathematic di Palermo 57(2008), 279-285.
16. W. Shatanawi, fixed point theory for contractive mappings satisfying φ- maps in G- metric spaces, Fixed Point Theory and Applications 2010 (20100, 1-9.
17. R. Bhardwaj, common fixed point theorem for two mappings in generalized cone metric spaces, South Assain J. of Math., 2012, 2(2), 175-181.

Source of support: Nil, Conflict of interest: None Declared