International Journal of Mathematical Archive-5(7), 2014, 255-262
Available online through www.ijma.info ISSN 2229 – 5046

WEAKLY RADICAL FORMULA AND WEAKLY PRIMARY SUBMODULES

Pairote Yiarayong* and Phakakorn Panpho

**Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanuloke -65000, Thailand.*

Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanuloke -65000, Thailand.

(Received On: 04-11-13; Revised & Accepted On: 28-07-14)

ABSTRACT

In this paper, we study the weakly radical of modules over commutative ring with identity. Furthermore we prove that if N_j is a weakly prime submodule of M_j, then N_j is to satisfy the weakly radical formula in M_j if and only if

$$M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$$

is to satisfy the weakly radical formula in M.

Keywords: *weakly submodule, prime submodule, radical, weakly prime radical, weakly radical formula.*

AMS Subject Classification: *13A15, 13F05, 13A10.*

1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. A submodule N of an R-module M is a weakly prime submodule of M if for each submodule K of M and $a,b \in R$ such that $abK \subseteq N$, then $aK \subseteq N$ or $bK \subseteq N$.

Recently, this notion of weakly prime submodule has been extensively studied by Behboodi and Koohi in (2004). An R-module M is a weakly prime module if every proper submodule N of M is a weakly prime submodule of M. It is easy to show that if N. is a prime submodule of M, then N is a weakly prime submodule of M.

Let N be a proper submodule R-module M. The weakly prime radical of N in M, denoted by $w.rad_M(N)$, is defined to be the intersection of all weakly prime submodules containing N. If there is no weakly prime submodule containing N, then $w.rad_M(N) = M$ (see, for example,[5, 14]).

In this note, we shall need the notion of the envelope of a submodule introduced by R. L. McCasland and M. E. Moore in [11]. For a submodule N of an R-module M, the envelope of N in M, denoted by $E_M(N)$, is defined to be the subset $\{rm : r \in R$ and $m \in M$ such that $r^k m \in N$ for some $k \in \mathbb{Z}^+\}$ of M. Note that, in general, $E_M(N)$ is not an R-module. With the help of envelopes, the notion of the radical formula is defined as follows: A submodule N of an R-module M is said to satisfy the radical formula in M, if $\langle E_M(N) \rangle = rad_M(N)$. Also, an R-module M is said to satisfy the radical formula, if every submodule of M satisfies the radical formula in M. The radical formula has been studied extensively by various authors (see [7], [12] and [13]).

*Corresponding author: Pairote Yiarayong**
**Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanuloke -65000, Thailand.*

In this paper is to introduce the notion of a weakly radical formula, we study the weakly prime radical of modules over commutative ring with identity. Furthermore we prove that if N_j is a weakly prime submodule of M_j, then N_j is to satisfy the weak radical formula in M_j if and only if $M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$ is to satisfy the weakly radical formula in M.

2. PRELIMINARIES

Let $R = \prod_{i=1}^{n} R_i$, where each R_i is a commutative ring with identity. Then an ideal $I = \prod_{i=1}^{n} I_i$ of P is prime if and only if I_i is equal to the corresponding ring R_i and the other is prime. Moreover, any R-module M can be uniquely decomposed into a direct product of modules, i.e. $M = \prod_{i=1}^{n} M_i$, where

$$M_i = (0,0,0,\ldots,0,1,0,\ldots 0)M$$

is an R_i-module with action

$$(r_1, r_2,\ldots,r_n)(m_1,m_2,\ldots,m_n) = (r_1m_1, r_2m_2,\ldots,r_nm_n),$$ where $r_i \in R_i$ and $m_i \in M_i$ [7].

Lemma 2.1: [7] Let $N = N_1 \times N_2$ be a submodule of M. Then

$$\langle E_M(N) \rangle = \langle E_{M_1}(N_1) \rangle \times \langle E_{M_2}(N_2) \rangle.$$

Corollary 2.2: [7] Let $N = \prod_{i=1}^{n} N_i$ be a submodule of M. Then $\langle E_M(N) \rangle = \prod_{i=1}^{n} \langle E_{M_i}(N_i) \rangle$.

Lemma 2.3: [14] If N is a weakly prime submodule, then $\langle E_M(N) \rangle = N$.

Lemma 2.4: [14] Let N be a semiprime submodule of an R-module M. Then $\langle E_M(N) \rangle = N$.

3. WEAKLY PRIME SUBMODULES

In this section, we give some characterizations for weakly prime submodule of R-module M.

Lemma 3.1: Let $M = M_1 \times M_2$, where M_i is an R_i-module. A submodule $N_1 \times M_2$ is a weakly prime submodule of M if and only if N_1 is a weakly prime submodule of M_1.

Proof: Suppose that $N_1 \times M_2$ is a weakly prime submodule of R-module M. We will show that N_1 is a weakly prime submodule of M_1. Clearly, N_1 is a proper submodule of R_1-module M_1. To show that weakly prime submodule properties of N_1 hold, let K be a submodule of R_1-module M_1 and $a,b \in R_1$ such that $abK \subseteq N_1$.

Then
$(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times M_2$.

Since $N_1 \times M_2$ is a weakly prime submodule of R-module M, it follows that
$(aK \times \{0\}) = (a,0)(K \times \{0\}) \subseteq N_1 \times M_2$
or
$(bK \times \{0\}) = (b,0)(K \times \{0\}) \subseteq N_1 \times M_2$;
that is, $aK \subseteq N_1$ or $bK \subseteq N_1$. Therefore N_1 is a weakly prime submodule of R_1-module M_1.

Conversely, suppose that N_1 is a weakly prime submodule of R_1-module M_1. We will show that $N_1 \times M_2$ is a weakly prime submodule of R-module M. Clearly, $N_1 \times M_2$ is a proper submodule of R-module M. To show that weakly prime submodule properties of $N_1 \times M_2$ hold, let $K \times L$ be a submodule of R-module M and $(a_1, a_2), (b_1, b_2) \in R$ such that

$$a_1 b_1 K \times a_2 b_2 L = (a_1, a_2)(b_1, b_2)(K \times L) \subseteq N_1 \times M_2.$$

Since N_1 is a weakly prime submodule of R_1-module M_1 and $a_1 b_1 K \subseteq N_1$, we have $a_1 K \subseteq N_1$ or $b_1 K \subseteq N_1$.

Therefore

$(a_1, a_2)(K \times L) = a_1 K \times a_2 L \subseteq N_1 \times M_2$

or

$(b_1, b_2)(K \times L) = b_1 K \times b_2 L \subseteq N_1 \times M_2$

and hence $N_1 \times M_2$ is a weakly prime submodule of R-module M.

Corollary 3.2: Let $M = M_1 \times M_2$, where M_i is an R_i-module. A submodule $M_1 \times N_2$ is a weakly prime submodule of R-module M if and only if N_2 is a weakly prime submodule of R_2-module M_2.

Proof: This follows from Lemma 3.1.

Corollary 3.3: Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. A submodule

$$M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$$

is a weakly prime submodule of R-module M if and only if N_j is a weakly prime submodule of R_j-module M_j.

Proof: This follows from Lemma 3.1 and Corollary 3.2.

Lemma 3.4: Let $M = M_1 \times M_2$, where M_i is an R_i-module. If $N_1 \times \{0\}$ is a weakly prime submodule of M, then N_1 is a weakly prime submodule of M_1.

Proof: Suppose that $N_1 \times \{0\}$ is a weakly prime submodule of R-module M. We will show that N_1 is a weakly prime submodule of M_1. Clearly, N_1 is a proper submodule of R_1-module M_1. To show that weakly prime submodule properties of N_1 hold, let K be a submodule of R_1-module M_1 and $a, b \in R_1$ such that $abK \subseteq N_1$.

Then

$(a,0)(b,0)(K \times \{0\}) = abK \times \{0\} \subseteq N_1 \times \{0\}$.

Since $N_1 \times M_2$ is a weakly prime submodule of R-module M, it follows that

$(aK \times \{0\}) = (a,0)(K \times \{0\}) \subseteq N_1 \times \{0\}$

or

$(bK \times \{0\}) = (b,0)(K \times \{0\}) \subseteq N_1 \times \{0\}$;

that is, $aK \subseteq N_1$ or $bK \subseteq N_1$. Therefore N_1 is a weakly prime submodule of R_1-module M_1.

Corollary 3.5: Let $M = M_1 \times M_2$, where M_i is an R_i-module. If $\{0\} \times N_2$ is a weakly prime submodule of R-module M, then N_2 is a weakly prime submodule of R_2-module M_2.

Proof: This follows from Lemma 3.1.

Corollary 3.6: Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If $\{0\} \times \{0\} \times \ldots \times N_j \times \ldots \times \{0\}$ is a weakly prime submodule of R-module M, then N_j is a weakly prime submodule of R_j-module M_j.

Proof: This follows from Lemma 3.4 and Corollary 3.5.

4. RADICAL OF WEAKLY PRIME SUBMODULES

A submodule N of an R-module M is said to satisfy the weakly radical formula in M, if $\langle E_M(N) \rangle = w.rad_M(N)$.

Lemma 4.1: Let $M = M_1 \times M_2$, where M_i is an R_i-module. If W is a weakly prime submodule of R-module M and $P = \{x \in M_1 : (x,0) \in W\}$, then $P = M_1$ or P is a weakly prime submodule of R_1-module M_1.

Proof: Suppose that $P \neq M_1$. We will show that P is a weakly prime submodule of R_1-module M_1. It is clear that, P is a proper submodule of R_1-module M_1. To show that weakly prime submodule properties of P, let $a,b \in R_1$ and K be submodule of M_1 such that $abK \subseteq P$. Let $k \in K$. Then $abk \in P$ so that $(a,0)(b,0)(k,0) = (abk,0) \in W$. Thus $(a,0)(b,0)(K \times \{0\}) \subseteq W$. Since W is a weakly prime submodule of M, we have

$(a,0)(K \times \{0\}) \subseteq W$

or

$(b,0)(K \times \{0\}) \subseteq W$.

Thus

$(ak,0) = (a,0)(k,0) \in W$

or

$(bk,0) = (b,0)(k,0) \in W$.

It follows that $ak \in P$ or $bk \in P$. Therefore $aK \subseteq P$ or $bK \subseteq P$ and hence P is a weakly prime submodule of M_1.

Corollary 4.2: Let $M = M_1 \times M_2$, where M_i is an R_i-module. If W is a weakly prime submodule of R-module M and $P = \{x \in M_2 : (0,x) \in W\}$, then $P = M_2$ or P is a weakly prime submodule of R_2-module M_2.

Proof: This follows from Lemma 4.1.

Corollary 4.3: Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If W is a weakly prime submodule of R-module M and $P = \{x \in M_j : (0,0,\ldots,x,0\ldots,0) \in W\}$, then $P = M_j$ or P is a weakly prime submodule of R_j-module M_j.

Proof: This follows from Lemma 4.1 and Corollary 4.2.

Lemma 4.4: Let $M = M_1 \times M_2$, where M_i is an R_i-module and let N be a submodule of R_1-module M_1. Then $m \in w.rad_{M_1}(N)$ if and only if $(m,0) \in w.rad_{M_1}(N \times \{0\})$.

Proof: Suppose that $M = M_1 \times M_2$, where M_i is an R_i-module. Let N be a submodule of R_1-module M_1 and let $m \in w.rad_{M_1}(N)$.

If there is no weakly prime submodule of M containing $N\times\{0\}$, then $w.rad_M\left(N\times\{0\}\right)=M$. Therefore $(m,0)\in w.rad_{M_1}\left(N\times\{0\}\right)$.

If there is weakly prime submodule of M containing $N\times\{0\}$, then there exists a weakly prime submodule W with $N\times\{0\}\subseteq W$. By Lemma 4.1 and $P=\{x\in M_1:(x,0)\in W\}$, we have $P=M_1$ or P is a weakly prime submodule of R_1-module M_1.

Case - 1: $P=M_1$. Since $m\in w.rad_{M_1}(N)$, we have $m\in P$. Then $(m,0)\in W$. Therefore if W is a weakly prime submodule of M containing $N\times\{0\}$, then $(m,0)\in W$.

Case - 2: $P\neq M_1$. Since $P\neq M_1$, we have P is a weakly prime submodule of R_1-module M_1. Let $x\in N$. Then $(x,0)\in N\times\{0\}$ so that $x\in P$. It follows that $N\subseteq P$. We have

$$\begin{aligned} w.rad_{M_1}(N) &\subseteq w.rad_{M_1}(P)\\ &=P \end{aligned}$$

so that $m\in P$. Therefore if W is a weakly prime submodule of M containing $N\times\{0\}$, then $(m,0)\in W$ and hence $(m,0)\in w.rad_{M_1}\left(N\times\{0\}\right)$.

Corollary 4.5: Let $M=M_1\times M_2$, where M_i is an R_i-module and let N be a submodule of R_2-module M_2. Then $m\in w.rad_{M_2}(N)$ if and only if $(0,m)\in w.rad_M(\{0\}\times N)$.

Proof: This follows from Lemma 4.4.

Corollary 4.6: Let $M=\prod_{i=1}^{n}M_i$, where M_i is an R_i-module and let N be a submodule of R_j-module M_j. Then $m\in w.rad_{M_j}(N)$ if and only if

$$(0,\ldots,m,0,\ldots,0)\in w.rad_M(\{0\}\times\{0\}\times\ldots\times N\times\{0\}\times\ldots\times\{0\}).$$

Proof: This follows from Lemma 4.4 and Corollary 4.5.

Lemma 4.7: Let $M=M_1\times M_2$, where M_i is an R_i-module. If N_i be a submodule of R_i-module M_i, then $w.rad_{M_1}(N_1)\times w.rad_{M_2}(N_2)\subseteq w.rad_M(N_1\times N_2)$.

Proof: Suppose that $M=M_1\times M_2$, where M_i is an R_i-module. Let N_i be a submodule of R_i-module M_i. We will show that $w.rad_{M_1}(N_1)\times w.rad_{M_2}(N_2)\subseteq w.rad_M(N_1\times N_2)$. Let

$$(x,y)\in w.rad_{M_1}(N_1)\times w.rad_{M_2}(N_2).$$

Then $x\in w.rad_{M_1}(N_1)$ and $y\in w.rad_{M_2}(N_1)$. By Lemma 4.1 and Lemma 4.4, we have

$$(x,0)\in w.rad_M(N_1\times\{0\})\subseteq w.rad_M(N_1\times N_2)$$

and

$$(0,y)\in w.rad_M(\{0\}\times N_2)\subseteq w.rad_M(N_1\times N_2).$$

Then $(x,y)=(x,0)+(0,y)\in w.rad_M(N_1\times N_2)$ and hence $w.rad_{M_1}(N_1)\times w.rad_{M_2}(N_2)\subseteq w.rad_M(N_1\times N_2)$.

Corollary 4.8:. Let $M=\prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If N_i be a submodule of R_i-module M_i, then $\prod_{i=1}^{n} w.rad_{M_i}(N_i) \subseteq w.rad_M(\prod_{i=1}^{n} N_i)$.

Proof: This follows from Lemma 4.7.

Theorem 4.9: Let $M=M_1\times M_2$, where M_i is an R_i-module. If N is a submodule of R_1-module M_1, then $w.rad_{M_1}(N_1)\times w.rad_{M_2}(M_2)=w.rad_M(N_1\times M_2)$.

Proof: Suppose that $M=M_1\times M_2$, where M_i is an R_i-module. Let N be a submodule of R_1-module M_1. By Lemma 4.7, we have $w.rad_{M_1}(N)\times w.rad_{M_2}(M_2)\subseteq w.rad_M(N\times M_2)$. We show that $w.rad_M(N_1\times M_2)\subseteq w.rad_{M_1}(N_1)\times w.rad_{M_2}(M_2)$. If there is no weakly prime submodule of M containing N, then $w.rad_{M_1}(N)=M_1$. Then

$$w.rad_M(N_1\times M_2)\subseteq w.rad_{M_1}(N_1)\times w.rad_{M_2}(M_2).$$

If there is weakly prime submodule of M containing N, then there exists W is a weakly prime submodule of M_1 containing N. Then $W\times M_2$ is a weakly prime submodule of M containing $N\times M_2$. Let P be a weakly prime submodule of M containing $N\times M_2$. Then

$$\begin{aligned} N\times M_2 &\subseteq w.rad_{M_1}(N)\times M_2 \\ &= w.rad_{M_1}(N)\times w.rad_{M_2}(M_2). \end{aligned}$$

Therefore $w.rad_M(N_1\times M_2)\subseteq w.rad_{M_1}(N_1)\times w.rad_{M_2}(M_2)$ and hence

$$w.rad_M(N_1\times M_2)= w.rad_{M_1}(N_1)\times w.rad_{M_2}(M_2).$$

Corollary 4.10: Let $M=M_1\times M_2$, where M_i is an R_i-module. If N is a submodule of R_2-module M_2, then $w.rad_M(M_2\times N)=w.rad_{M_1}(M_2)\times w.rad_{M_2}(N)$.

Proof: This follows from Lemma 4.9.

Corollary 4.11: Let $M=\prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If N_i be a submodule of R_i-module M_i, then $\prod_{i=1}^{n} w.rad_{M_i}(N_i) = w.rad_M(\prod_{i=1}^{n} N_i)$.

Proof: This follows from Lemma 4.9 and Corollary 4.10.

Theorem 4.12: Let $M=M_1\times M_2$, where M_i is an R_i-module. If N_1 is a weakly prime submodule of M_1, then N_1 is to satisfy the weakly radical formula in M_1 if and only if $N_1\times M_2$ is to satisfy the weakly radical formula in M.

Proof: Suppose that N_1 is a weakly prime submodule of M_1 and N_1 is to satisfy the weakly radical formula in M_1. We will show that $N_1\times M_2$ is to satisfy the weakly radical formula in M. Since N_1 is a weakly prime submodule of M_1, it follows that

$$\begin{aligned} w.rad_M(N_1 \times M_2) &= w.rad_{M_1}(N_1) \times w.rad_{M_2}(M_2) \\ &= \left\langle E_{M_1}(N_1) \right\rangle \times M_2 \\ &= \left\langle E_M(N_1 \times M_2) \right\rangle. \end{aligned}$$

Therefore $N_1 \times M_2$ is to satisfy the weakly radical formula in M. Conversely, suppose that N_1 is a weakly submodule of M_1 and $N_1 \times M_2$ is to satisfy the weakly radical formula in M. We will show that N_1 is to satisfy the weakly radical formula in M_1. Since $N_1 \times M_2$ is a weakly prime submodule of M, it follows that

$$\begin{aligned} \left\langle E_{M_1}(N_1) \right\rangle \times M_2 &= \left\langle E_M(N_1 \times M_2) \right\rangle \\ &= w.rad_{M_1}(N_1) \times w.rad_{M_2}(M_2). \end{aligned}$$

Then $w.rad_{M_1}(N_1) = \left\langle E_{M_1}(N_1) \right\rangle$ and hence N_1 is to satisfy the weakly radical formula in M_1.

Corollary 4.13: Let $M = \prod_{i=1}^{n} M_i$, where M_i is an R_i-module. If N_j is a weakly prime submodule of M_j, then N_j is to satisfy the weakly radical formula in M_j if and only if

$$M_1 \times M_2 \times \ldots \times M_{j-1} \times N_j \times M_{j+1} \times \ldots \times M_n$$

is to satisfy the weakly radical formula in M.

Proof: This follows from Theorem 4.12.

ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

REFERENCES

1. Abd El-Bast Z. and Smith PF, Multiplicaion modules**,** Comm. Algebra., 16(4) (1988), 755-779.

2. Ameri R., On the prime submodules of multiplication modules, Inter. J. of Math. and Math-ematical Sciences., 27(2003), 1715-1724.

3. Anderson D.D. and Smith E., Weakly prime ideals, Houston J. of Math., 29(2003), 831-840.

4. Azizi A., Weakly prime submodules and prime submodules, Glasgow Math. J., 48(2003), 343–346.

5. Behbooid M., On weakly prime radical of modules and semi-compatible modules, Acta Math. Hungar., 113(3)(2003), 243-254.

6. Behboodi M. and Koohi K., Weakly prime submodules, Vietnam J. Math., 32(2) (2004), 185–195.

7. Ebrahimi Atani S. and Esmaeili Khalil Saraei F., Modules which Satisfy the Radical Formula.Int.J.Contemp. Math. Sci., 2(1) (2007), 13 - 18

8. S. Ebrahimi Atani and F. Farzalipour, On weakly prime submodules, Tamkang J. Math., 38(3) (2007), 247-252.

9. Jenkins J. and Smith P. F., On the prime radical of a module over a commutative ring, Comm. Algebra, 20(12) (1992), 3593–3602.

10. Leung K. H. and Man S. H., On commutative noetherian rings which satisfy the radical formula, Glasgow Math. J. 39(3) (1997), 285–293.

11. McCasland R. L. and Moore M. E., On radicals of submodules. Comm. Algebra, 19(5) (1991), 1327–1341.

12. Pusat-Yilmaz D. and Smith P. F., Modules which satisfy the radical formula, Acta Math. Hungar, 95(1-2) (2002), 155–167.

13. Sharif H., Sharifi Y.and Namazi S., Rings satisfying the radical formula, Acta Math. Hungar, 71(1-2) (1996), 103–108.

14. Yilmaza E. and Cansu S., Envelopes and weakly primadicals of submodules, rXiv: 1205.2983v1 [math.AC] 14 May-2012.

Source of support: Nil, Conflict of interest: None Declared