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GENERALIZED COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

This paper introduces generalized common fixed point theorems in complete fuzzy metric spaces. Here, the concept of R-weak commutativity in fuzzy metric spaces is also introduced with few related results and illustrative examples.

Key words: *fuzzy metric space, fixed point, R-weakly commuting mappings, common fixed point theorem.*

AMS subject classifications: *47H10, 54H25.*

1. INTRODUCTION

Zadeh development of mathematics when the notion of fuzzy set. which laid the foundation of fuzzy mathematics. Consequently the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities for fixed point theorists.

Deng[4], Erceg [5], Kaleva and Seikkala [11] and Kramosil and Michalek [12] have introduced the concept of fuzzy metric spaces in various ways. George and Veeramani [8] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [12] and defined Hausdorff topology of metric spaces which is later proved to be metrizable. every metric induces a fuzzy metric Recently, Chugh and Kumar [3] proved a Pant type theorem for two pairs of R-weakly commutingma ppings satisfyinga Boyd and Wong [1] type contraction condition which in turn, generalizes a fixed point theorem of Vasuki [18]

2. PRELIMINARIES

Definition 2.1: (cf. [20]) A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2.(cf. [16]) A binary operation * : [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if {[0, 1], *} is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$, a, b, c, d ∈ [0, 1].

Definition 2.3: (cf. [12]) The triplet (X,M,*) is a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t) \neq 0$ for $t \neq 0$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(v) $M(x, y, .) : [0,\infty) \to [0, 1]$ is left continuous for all x, y, z ∈ X and s, t > 0.

Example 2.1: (cf. [8]) Every metric space induces a fuzzy metric space. Let (X, d) be a metric space. Define a * b = ab and $M(x,y,t) = \dfrac{kt^n}{kt^n + md(x,y)}$, k, m, n, t$\in \Re^+$. Then (X, M, *) is a fuzzy space. If we put k = m = n = 1, we get

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

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The fuzzy metric induced by a metric d is referred to as a standard fuzzy metric.

Definition 2.4: (cf. [10]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M,*) is convergent to $x \in X$ if

$$\lim_{n\to\infty} m(x_n, x, t) = 1 \text{ for each } t > 0.$$

Recently, Song [17] and Vasuki and Veeramani [19] again critically reviewed the existingdefinitio ns of Cauchy sequence in a fuzzy metric space. Vasuki and Veeramani [19] suggested that the definition of Cauchy sequence due to Grabiec [10] is weaker than that contained in [17, 19] and called it a G-Cauchy sequence.

Definition 2.5: (cf. [10]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M,*) is called Cauchy if $\lim_{n\to\infty} M(x_n + x_n, t) = 1$ for every t > 0 and each p > 0. (X, M,*) is complete if every Cauchy sequence in X converges in X.

Definition 2.6: A pair of self-mappings (f, g) of a fuzzy metric space (X, M,*) is said to be
(i) weakly commuting (cf.[18]) if M(fgx, gfx, t)≥ M(fx, gx, t),
(ii) R-weakly commuting (cf.[18]) if there exists some R > 0 such that M(fgx, gfx, t) ≥ M(fx, gx, t/R),
(iii) R-weakly commuting mappings of type (Af) if there exists some R > 0 such that
M(fgx, ggx, t) ≥ M(fx, gx, t/R),
(iv) R-weakly commuting mappings of type (Ag) if there exists some R > 0 such that
M(gfx, ffx, t) ≥ M(fx, gx, t/R),
(v) R-weakly commuting mappings of type (P) if there exists some R > 0 such that
M (ffx, ggx, t) ≥ M(fx, gx, t/R), for all $x \in X$ and t > 0.

.

Example 2.2: (cf. [18]) Let $X = \Re$, the set of real numbers. Define a * b = ab and

$$M(x, y, t) = \begin{cases} \left(e^{\frac{|x-y|}{t}}\right)^{-1}, & for\ all\ x, y \in X\ and\ t > 0 \\ 0, & for\ all\ x, y \in X\ and\ t = 0 \end{cases}$$

Then it is well known (cf. [18]) that (X, M,*) is a fuzzy metric space. Define fx = 2x − 1 and gx = x^2. Then by a straightforward calculation, one can show that

$$M(fgx, gfx, t) = \left(e^{\frac{2|x-1|^2}{t}}\right) = M(fx, gx, t/2)$$

which shows that the pair (f, g) is R-weakly commuting for R=2. Note that the pair (f, g) is not weakly commuting due to a strict increasing property of the exponential function.

However, various kinds of above mentioned 'R-weak commutativity' notions are independent of one another and none implies the other. The earlier example can be utilized to demonstrate this inter-independence.

To demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (Af) notice that

$$M(fgx, ggx, t) = \left(e^{\frac{|x^4-2x^2+1|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x+1)^2}{R}}\right)^{-1}$$

$$< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ when } x > 1$$

which shows that 'R-weak commutativity' does not imply 'R-weak commutativity' of type (Af).

Secondly, in order to demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (P) note that

$$M(ffx, ggx, t) = \left(e^{\frac{|x^4-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x^2+2x+3)^2}{R}}\right)^{-1}$$

$$< \left(e^{\frac{R|x-1|^2}{t}} \right)^{-1} = M\left(fx, gx, t/R\right) \text{ for x > 1.}$$

Finally, for a change the pair (f, g) is R-weakly commutingof type (Ag) as

$$M\left(gfx,\ ffx,t\right) = \left(e^{\frac{\left|(2x-1)^2-4x+3\right|}{t}} \right)^{-1} = \left(e^{\frac{4|x-1|^2}{t}} \right)^{-1}$$

=M(fx, gx t/4)

which shows that (f, g) is R-weakly commutingof type (Ag) for R=4. This situation may also be utilized to interpret that an R-weakly commuting pair of type (Ag) need not be R-weakly commuting pair of type (Af) or type (P)

Lemma:2.3: Let A,B, S and T be mappings from a complete fuzzy metric space (X,M,*) into itself satisfying $A(X) \subset T(X)$,$B(X) \subset S(X)$ and $M(Ax, By, t) \geq r(M(Sx, Ty, t))$ for all $x, y \in X$, where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(s) > s$ for each $0 < s < 1$. Suppose that one of A,B, S and T is continuous, pairs (A, S) and (B, T) are R-weakly commuting on X. Then A, B, S and T have a unique common fixed point in X.Note that Theorem A for a pair of R-weakly commuting mappings was proved by Vasuki [18] provided one of the mapping is continuous.

Lemma.2.4: let (X,M,*) be a complete fuzzy metric space and let A,B, S and T be self-mappings of X satisfying the following conditions:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.1}$$

$$\left(M\left(Ax,By,t\right), M\left(Sx,Ty,t\right), M\left(Sx,Ax,t\right), M\left(Ty,By,t\right)\right) \geq 0 \tag{3.2}$$

for all $x, y \in X$, where $\varphi : [0, 1] \to [0, 1]$ is a continuous function such that $\varphi(s) > s$ for each $0 < s < 1$. Then for any arbitrary point $x_0 \in X$, by (3.1), we choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this point x_1, there exists a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n},\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \text{ for } n = 0, 1, 2 \dots \tag{3.3}$$

Firstly, we prove the following lemma.

Lemma 2.5: Let A, B, S and T be self-mappings of a fuzzy metric space (X,M,*) satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.

Proof: For $t > 0$,

$M(y_{2n}, y_{2n+1}, t) \geq M(Ax_{2n},Bx_{2n+1}, t)$

$M(Sx_{2n}, Tx_{2n+1}, t),M(Sx_{2n},Ax_{2n}, t),M(Tx_{2n+1},Bx_{2n+1}, t) \geq 0$

$$= \{M(y_{2n-1}, y_{2n}, t),M(y_{2n-1}, y_{2n}, t),M(y_{2n}, y_{2n+1}, t)\})$$

$$> \begin{cases} M\left(y_{2n-1}, y_{2n}, t\right), & if\ M\left(y_{2n-1}, y_{2n}, t\right) < M\left(y_{2n}, y_{2n+1,} t\right) \\ M\left(y_{2n}, y_{2n+1}, t\right), & if\ M\left(y_{2n-1}, y_{2n}, t\right) < M\left(y_{2n}, y_{2n+1,} t\right) \end{cases} \tag{3.4}$$

as $\varphi(s) > s$ for $0 < s < 1$. Thus $\{M(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $l \leq 1$. We assert that $l = 1$. If not, $l < 1$ which on letting $n \to \infty$ in (3.4) one gets $l \geq \varphi(l) > l$ a contradiction yielding thereby $l = 1$. Therefore for every $n \in N$, using analogous arguments one can show that $\{M(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a sequence of positive real numbers in [0, 1] which tends to a limit $l = 1$. Therefore for every $n \in N$

$$M\left(y_n, y_{n+1}, t\right) > M\left(y_{n-1}, y_n, t\right) \text{ and } \lim_{n\to\infty} M\left(y_n, y_{n+1}, t\right) = 1$$

Now for any positive integer p

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * \dots * M(y_{n+p-1}, y_{n+p}, t/p)$.

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ for t > 0, it follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1*1*......*1 = 1$$

which shows that $\{y_n\}$ is a Cauchy sequence in X.

3. MAIN THEOREM

Theorem 3.1: Let A, B, S and T be four self-mappings of a fuzzy metric space (X,M,*) satisfying the condition

$$(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)) \geq 0$$

for all x, y ∈ X and t > 0 where φ : [0, 1]→ [0, 1] is a continuous function with φ(s) > s whenever 0 < s < 1. If A(X) ⊂ T (X) and B(X) ⊂ S(X) and one of A(X), B(X), S(X) and T (X) is a complete subspace of X, then
(i) A and S have a point of coincidence,
(ii) B and T have a point of coincidence.

Moreover, if the pairs (A, S) and (B, T) are coincidentally commuting, then A, B, S and T have a unique common fixed point.

Proof: Suppose condition (3.2) holds, i.e. the pair (A, S) is occ and the pair (B, T) is owc. Then, as (B, T) is owc, there exist some point p ∈ X such that BTp = TBp whenever Bp = Tp = z (say) in X. So that for a given p ∈ X, Bz = Tz whenever Bp = Tp = z. Next, since (A; S) is occasionally converse commuting (occ). Then, by definition, there exist some such that ASu = SAu implies Au = Su = w (say). So that for a given u, Aw = Sw implies that Au = Su = w (say) in X. We claim that AAu = Bz. If not, then putting x = Au and y = z in (3.1), and using ASu = SAu = AAu and Tz = Bz, we obtain

$$(M(AAu, Bz, t), M(SAu, Tz, t), M(SAu, AAu, t), M(Tz, Bz, t)) \geq 0$$

$$(M(AAu, Bz, t), M(AAu, Bz, t), 1, 1) \geq 0$$

a contradiction .Thus AAu = Bz. Therefore Aw = Bz = Sw = Tz. We claim Au = Bz. If not, then putting x = u and y = z in (3.1), we get

$$(M(Au, Bz, t), M(Su, Tz, t), M(Su, Au, t), M(Tz, Bz, t)) \geq 0$$

$$(M(Au, Bz, t), M(Au, Bz, t), 1, 1) \geq 0$$

a contradiction. Thus Au = Bz. Therefore, Au = Bz = Tz = Su = AAu = SAu. It follows that Au is a common fixed point of A and S. Next, we claim that Bz = z. If not, take x = u and y = p in (3.1), we obtain

$$(M(Au, Bp, t), M(Su, Tp, t), M(Su, Au, t), M(Tp, Bp, t)) \geq 0$$

$$(M(Bz, z, t), M(Bz, z, t), 1, 1) \geq 0$$

a contradiction .Thus Bz = z. Therefore, Bz = z = Tz = Au = Su = AAu = SAu. Hence z is a common fixed point of A, B, S and T. For uniqueness, let w be another common fixed point of A, B, S and T. We show that w = z, suppose not, then by (3.1) take x = z, y = w, we obtain

$$(M(Az, Bw, t), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, t)) \geq 0$$

$$\Delta\ (M(z, w, t), M(z, w, t), 1, 1) \geq 0$$

A contradiction Thus A, B, S and T have a unique common fixed point. The proof is same if condition (3.3) holds.

Next, we prove the following result for both pairs occasionally converse commuting:

Theorem 3.2: Let A, B, S and T be self mappings of an fuzzy metric space (X, M, *;) satisfying the condition (3.1). If both the pairs (A, S) and (B, T) are occasionally converse commuting (occ), then A; B; S and T have a unique common fixed point in X.

Proof: As the pair (A, S) is occasionally converse commuting, by definition, there exist some such that ASu = SAu implies Au = Su. It follows that AAu = ASu = SAu. Also, the occasionally converse commuting for the pair (B, T) implies that there exist such that BTv = TBv implies Bv = Tv. Hence BBv = BTv = TBv. First, we show that Au = Bv. If not, then putting x = u and y = v in (3.1), we obtain

$$\left(M\left(Au,Bv,t\right),M\left(Su,Tv,t\right),M\left(Su,Au,t\right),M\left(Tv,Bv,t\right)\right)\geq 0$$

$$\left(M\left(Au,Bv,t\right),M\left(Au,Bv,t\right),1,1\right)\geq 0$$

a contradiction Thus, Au = Bv. Next, we show that AAu = Au. Suppose not, then, by putting x = Au and y = v in (3.1), we have

$$\left(M\left(AAu,Bv,t\right),M\left(SAu,Tv,t\right),M\left(SAu,AAu,t\right),M\left(Tv,Bv,t\right)\right)\geq 0$$

$$\left(M\left(AAu,Au,t\right),M\left(AAu,Au,t\right),1,1\right)\geq 0$$

a contradiction. Thus Au = AAu. Similarly, Bv = BBv. Since Au = Bv, we have Au = Bv = AAu = ASu = SAu = BBv = BTv = TBv. Therefore Au = z (say), is a common fixed point of A, B, S and T.

For uniqueness, let w be another common fixed point of A, B, S and T. We show that w = z, suppose not, then by (3.1) take x = z, y = w, we obtain

$$\left(M\left(Az,Bw,t\right),M\left(Sz,Tw,t\right),M\left(Sz,Az,t\right),M\left(Tw,Bw,t\right)\right)\geq 0$$

$$\left(M\left(z,w,t\right),M\left(z,w,t\right),1,1\right)\geq 0$$

a contradiction Therefore z = w = Au. Hence, Au is a unique common fixed point of A, B, S and T. This completes the proof.

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