# MORE ON NONSTANDARD ASYMPTOTIC APPROXIMATION 

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#### Abstract

In the first part we investigate some Theorems about Nonstandard Asymptotic Approximation of integrals and the convergence of sequences and series.


Keywords: Asymptotic Approximation, Asymptotic Approximation of Integrals, Asymptotic Approximation of Series, Approximations.

## INTRODUCTION AND MAIN RESULTS

Asymptotic analysis is a useful mathematical tool which provides analytical insight and numerical information about the solutions of complicated problems in applied mathematics, engineering, physics and many other sciences, which require a mathematical framework for describing and modeling scientific problems. We now describe the problem at hand. First of all, let $a$ be a real number, we say that $a$ is infinitesimal if and only if $|a|<r$ for all $r \in$ R. If $a$ and $b$ are any two real numbers, then $a$ and $b$ are called infinitely near, this will be denoted by $a \cong b$ if $a-b$ is infinitesimal. For some $r \in \mathrm{R}^{+}$a real number $a$ is said to be limited if $|a| \leq r$. There is a real number $a$, called appreciable, if $a$ is limited but not infinitesimal. For any real number $a$ is said to be unlimited if $|a|>r$ for all $r \in \mathrm{R}^{+}$. Furthermore the collection of limited, unlimited real numbers of infinitesimals are said to be external sets (see, [3], [5]). We define the set of all real numbers, which are infinitely near to a standard real number $a$, will be called the monad of $a$. In particular, the external set of infinitesimal real numbers is called the monad of 0 , and denote bym $(a)$ and $m(0)$ respectively, it's the monad of $a$ and it's the monad of 0 (see, [6], see also [7]). And denote by $G$ the set of all limited real numbers called principal galaxy. In [2] for any real number $a$ the set of all real numbers $x$ such that $x-a$ limited will be called the galaxy of $a$, and denote $\mathrm{G}(a)$, Given $\propto \neq 0$, let $r \in \mathrm{R}$, we shall use the following definition

$$
\propto-\mathrm{G}(r):=\propto-\operatorname{galaxy}(r):=\left\{x \in \mathrm{R}: \frac{x-r}{\alpha} \quad \text { is limited }\right\} .
$$

And so on, we write $\mathrm{A}^{+}$the set of appreciable numbers, the following Euler formula is well known:

$$
\mathrm{e}^{-x} \cong\left(1-\frac{x}{\lambda}\right), \quad \lambda \text { be standard unlimited positive integer. }
$$

For the sake of simplifying notation, we set

$$
\Psi:[a, b] \rightarrow \mathrm{R} \text { where } \Psi:=f(x)-g(x)
$$

It is easy to see that $\Psi(x) \cong 0$ for every $x \in[a, b]$.
We begin with Asymptotic Approximation of Integrals.
In the followingproperties (Lemma 1 and 3) permit, particularly to give an approximate value to the integral of a function which is near a function with known primitive. Notice that in the proof of Lemma 3, the major function $\Psi$ interferes only in the justification, and did not interfere in the approximation's quality.

Lemma 1: Givenf and gbetwo measurable internal functions such that $f(x) \cong g(x)$ for all $x \in[a, b]$ of limited length, Then

$$
\int_{a}^{b} f(x) \cong \int_{a}^{b} g(x)
$$

Proof: Set

$$
\beta:=\sup _{x \in[a, b]}|\Psi(x)|
$$

Then

$$
\beta \cong 0
$$

We have

$$
\begin{aligned}
\left|\int_{a}^{b} f(x) d x-\int_{a}^{b} g(x) d x\right| & \leq \int_{a}^{b}|\Psi(x) d x| \\
& \leq \beta(b-a) \\
& \cong 0
\end{aligned}
$$

We get

$$
\int_{a}^{b} f(x) \cong \int_{a}^{b} g(x)
$$

This completes the proof.
Next, we state Robinson Lemma (see, [2]).
Lemma 2: There exist a non-limitedn $\in N$ such that

$$
\int_{-n}^{n} f(x) d x-\int_{-n}^{n} g(x) d x
$$

Now, we are ready to prove Lemma 3:
Lemma 3: If $f$ and gare two measurable internal functions such that $f(x) \cong g(x)$ for every limited $x$. Furthermore, if $\Psi$ be an integrable standard function such that $|f(x)|,|g(x)| \leq \Psi(x)$ for everyx $\in \mathrm{R}$, Then

$$
\int_{-\infty}^{\infty} f(x) d x \cong \int_{-\infty}^{\infty} g(x) d x
$$

Proof: For every limited $s \in \mathrm{~N}$,
In view of Lemma 1 we get

$$
\int_{-S}^{S} f(x) d x-\int_{-S}^{S} g(x) d x
$$

it follows by Lemma 2 that there exist a non-limited $n \in \mathrm{~N}$ such that

$$
\int_{-n}^{n} f(x) d x-\int_{-n}^{n} g(x) d x
$$

Now, let us consider the case, $\Psi(x)$ is integrable standard function, we obtain

$$
\int_{|x|>n} \Psi(x) d x \cong 0
$$

Consequently,

$$
\int_{|x|>n} f(x) d x \cong 0
$$

and

$$
\int_{|x|>n} g(x) d x \cong 0
$$

This, in turn, implies

$$
\int_{-n}^{n} f(x) d x=\int_{-\infty}^{\infty} f(x) d x
$$

and
Readily yields

$$
\int_{-n}^{n} g(x) d x=\int_{-\infty}^{\infty} g(x) d x
$$

This concludes our proof.
Note that Lemma 3, we form a condition in order that the approximate equality of Lemma1 holds for unbounded intervals; however, since we still have:

Lemma 4: Iff andg are two measurable internal functions such that

$$
f(x) \cong g(x) \text { for } x \in \mathrm{~A}^{+}
$$

Let $\Psi$ be an integrable standard function such that $|f(x)|,|g(x)| \leq \Psi(x)$ for everyx $\in \mathrm{R}^{+}$, then

$$
\int_{0}^{\infty} f(x) d x \cong \int_{0}^{\infty} g(x) d x
$$

We give Example of Asymptotic Approximation of Integrals.

Example 1: Suppose that $\theta>0$ and $\ell$ be standard, and $\lambda$ be unlimited positive integer,
Thus, we set

$$
f(Z):=\left\{\begin{array}{rcc}
\frac{(\theta-Z)^{\lambda-1}}{(1+Z)^{\lambda-\ell}} & \text { if } & 0 \leq Z \leq \theta \\
0 & \text { if } & z>\theta
\end{array}\right.
$$

Also, define

$$
\Gamma(\theta):=\int_{0}^{\theta} f(Z)
$$

First the maximum of the integrand attains for $Z=0$, using this maximum as a factor, we have

$$
\Gamma(\theta)=\int_{0}^{\theta} \frac{(\theta-z)^{\lambda-1}}{(1+Z)^{\lambda-\ell}} d Z
$$

and applying simple calculations yields

$$
\Gamma(\theta)=\theta^{\lambda-1} \int_{0}^{\theta} \frac{\left(1-\frac{Z}{\theta}\right)^{\lambda-1}}{(1+Z)^{\lambda-\ell}} d Z=\theta^{\lambda-1} \int_{0}^{\theta} g(Z) d Z
$$

The new integrand is limited everywhere.
Now, by Euler formula, we believe that the passage from the appreciable to the infinitesimal for the integrand effected when leaving the

$$
\frac{\theta}{\lambda}-\text { galaxy } .
$$

We get an approximately standard integrand, taking

$$
z:=\frac{\theta y}{\lambda}
$$

Thus,

$$
\Gamma=\frac{\theta^{\lambda}}{\lambda} \int_{0}^{\lambda} \frac{\left(1-\frac{y}{\lambda} \lambda^{\lambda-1}\right.}{\left(1+\frac{\theta y}{\lambda}\right)^{\lambda-1}} d y
$$

Set

$$
\varphi(y):=\left\{\begin{array}{ccc}
\frac{\left(1-\frac{y}{\lambda}\right)^{\lambda-1}}{\left(1+\frac{\theta y}{\lambda}\right)^{\lambda-1}} & \text { if } & 0 \leq y \leq \lambda \\
0 & \text { if } & y>\lambda
\end{array}\right.
$$

Thus, for every $y \geq 0$,

$$
\begin{aligned}
\varphi(y) \leq\left(1-\frac{y}{\lambda}\right)^{\lambda-1} & =\mathrm{e}^{(\lambda-1) \log \left(1-\frac{y}{\lambda}\right)} \\
& \leq \mathrm{e}^{\frac{(\lambda-1)}{\lambda} y} \\
& \leq \mathrm{e}^{-\frac{\lambda}{2}}
\end{aligned}
$$

While for every limited $y \geq 0$, we have $\varphi(y)=\mathrm{e}^{(1+\theta) y}$
Applying Lemma3we obtain

$$
\begin{aligned}
\int_{0}^{\infty} \varphi(y) d y & \cong \int_{0}^{\infty} \mathrm{e}^{-(1+\theta) y} d y \\
& =\frac{1}{1+\theta}
\end{aligned}
$$

It follows that

$$
\Gamma(\theta)=(1+a) \frac{\theta^{\lambda}}{\lambda(1+\theta)}, \text { when ce } a \cong 0
$$

Comments on the strategy that emerges from Example 1 the example above is an illustration of the situation for which Lemma3 is applied particularly, and of the procedure to follow.

First of all we observe that Lemma3 is mainly concerned with the functions $f$, which are:

1. Limited everywhere.
2. [2] Noninfinitesimals on a subset of R, which contains at least one interval of appreciable length, and does not exceed the principal galaxy.

Indeed, Let $\{x: f(x) \cong 0\} \supseteq \mathrm{G}$, we cannot find a standard majoration of $f$, As an example,

Let $\{x: f(x) \cong 0\} \subset m(0)$ we obtain

$$
\int_{0}^{\infty} f(x) d x \cong 0
$$

Therefore

$$
\int_{0}^{\infty} g(x) d x \cong 0
$$

Now such approximation of an infinitesimal number by another infinitesimal number is not meaningful. In order to be able to treat the case of an integrand, which does not satisfy the conditions 1 and 2.

We examine the reason in the Example 1 Departure function $f(Z)$ is not necessarily limited everywhere, and $\{x: f(Z) \cong 0\} \subset m(0)$
a) We get the integr and $g(Z)$ which is limited everywhere by putting maximum of $f(Z)$ in factor.
b) Evidently, $\{z f(Z)=0\}$ is equal to the positive part of $\frac{\theta}{\lambda}-$ galaxy

This observation leads us to choose the variable changement $Z:=\frac{\theta y}{\lambda}$ after which we obtain the integrand $\varphi(y)$ satisfies 1 and 2.

Now, we give some Asymptotic Approximation Lemmas on the series.
Lemma 5: If $f$ is a positive, decreasing functions such that $f(0) \cong 0$ then for every $q \in \mathrm{~N}$
a) $\sum_{i=0}^{q} f(i) \cong \int_{0}^{q} f(x) d x$
b) $\sum_{i=0}^{\infty} f(i)$ is convergent if and only if $\int_{0}^{\mathcal{q}} f(x) d x$ is convergent

In the case of convergence we have $\sum_{i=0}^{\infty} f(i) \cong \int_{0}^{\infty} f(x) d x$.
Proof: a) We used the fact that

$$
\sum_{i=0}^{q} f(i) \geq \int_{0}^{q} f(x) d x \geq \sum_{i=1}^{q+1} f(i)
$$

and

$$
\sum_{i=0}^{q} f(i)-\sum_{i=0}^{q+1} f(i)=f(0)-f(q+1) \cong 0
$$

readily yields

$$
\sum_{i=0}^{q} f(i) \cong \int_{0}^{q} f(x) d x .
$$

b) It follows by

$$
\sum_{i=0}^{\infty} f(i) \geq \int_{0}^{\infty} f(x) d x \geq \sum_{i=1}^{\infty} f(i)
$$

that
$\sum_{i=0}^{\infty} f(i)$ is convergent if and only if $\int_{0}^{\infty} f(x) d x$ is convergent yields

$$
\sum_{i=0}^{\infty} f(i) \cong \int_{0}^{\infty} f(x) d x
$$

Which proves the result.
We emphasize that, Lemma 5 links the series and integrals. where as Lemma 6is a versionof Lemma4.
Lemma 6: If $\left\{S_{i}\right\}_{i \in \mathrm{~N}}$ and $\left\{X_{i}\right\}_{i \in \mathrm{~N}}$ are two internal sequences such that $S_{i} \cong X_{i}$ for every $i \in \mathrm{~N}$, Let $\sum_{i=0}^{\infty} C_{i}$ be a convergent standard series such that $\left|S_{i}\right|,\left|X_{i}\right| \leq C_{i}$ for every $i \in \mathrm{~N}$, Then

$$
\sum_{i=0}^{\infty} S_{i} \cong \sum_{i=0}^{\infty} X_{i} .
$$

Next, We shall proveLemma 7
Lemma 7: Let $\left\{S_{i}\right\}_{i \in \mathrm{~N}}$ is an internal sequences of nonzero terms such that $\frac{S_{i+1}}{S_{i}} \cong 0$ for every $i \in \mathrm{~N}$, Then

$$
\sum_{i=0}^{\infty} S_{i} \cong S_{0}(1+\alpha)
$$

where $\alpha \cong 0$.

Proof: First we observe that

$$
\sum_{i=0}^{\infty} S_{i}=S_{0} \sum_{i=0}^{\infty} \frac{s_{i}}{S_{0}^{\prime}}
$$

Put $X_{i}:=\frac{s_{i}}{s_{0}}$ For every $i \in \mathrm{~N}$ we obtain $X_{0}=1$ and $X_{i} \cong 0$.

We only need to prove $\sum_{i=0}^{\infty} S_{i} \cong S_{0}(1+\alpha)$ for every $i \in \mathrm{~N}$.
To this end, by

$$
X_{i}:=\frac{s_{i}}{s_{i-1}} \frac{s_{i-1}}{s_{i-2}} \ldots \frac{s_{1}}{s_{0}}<\beta
$$

where $\beta$ is a standard number greater than zero. and by

$$
\sum_{i=0}^{\infty} X_{i} \cong 1
$$

readily yields

$$
\sum_{i=0}^{\infty} S_{i} \cong S_{0}(1+\alpha), \quad \alpha \cong 0
$$

Thus the lemma is proved.
Also, we will prove the results below:
Lemma 8: If $\left\{S_{i}\right\}_{i \in \mathrm{~N}}$ is an internal sequence such that $S_{0}$ is limited, $S_{i} \cong 0$ for everyi $\geq 1$, and such that the ratio of two nonzero successive terms is infinitesimally zero. Then

$$
\sum_{i=0}^{\infty} S_{i} \cong S_{0}
$$

Proof: First of all, assume that $\ell$ be the minimal index such that $S_{\ell} \neq 0$.
In view of Lemma7, we obtain

$$
\sum_{i=0}^{\infty} S_{i}=S_{\ell}(1+\alpha)
$$

For $\ell>0$,
for every $i \geq 1, \quad S_{\ell}(1+\alpha) \cong 0$ follows from $S_{i} \cong 0$, since $S_{0} \cong 0$, then $S_{\ell}(1+\alpha) \cong S_{0}$.
Furthermore, if $\ell=0$, then

$$
S_{\ell}(1+\alpha)=(1+\alpha) S_{0} \cong S_{0}
$$

This concludes the proof of Lemma8.
We remark that the statement of bothLemmas6 and 7remains true if the sum is taken only up to certain index or if we mitigate the conditions on the ratio $\frac{S_{i+1}}{S_{i}}$.

For the sake of completeness, we have;
Lemma 9: Given $\left\{S_{i}\right\}_{i \in N}$ be an internal sequence and $q \in N$. Suppose that $S_{0}$ is limited, $\frac{S_{\ell}}{S_{0}} \cong 0$, when $\ell>0$ is the smallest index for which $S_{\ell} \neq 0$, and that the ratio of two nonzero successive terms is of indexes included between $\ell$ and qis $<(\cong) 1$. Then

$$
\sum_{i=0}^{q} S_{i} \cong S_{0}
$$

Notice that Lemma 9 as an example for the above remark. However, these properties (Lemma 7 and 8) are particular cases for the above lemma. They facilitate the calculations of the theoretical characterization concerning the shadow developments.

## REFERENCES

1. Copson, E. T.: Asymptotic Expansions. Cambridge University, England, (1976)
2. Diener, M., Van Denberg, I.: Halos et galaxiesune extension du lemma de Robinson. CompteRendus de I'acadimie de Science de Paris, t. 293 Serie, pp. 385-388 (1983)
3. Hrbacek, K.: Nonstandard set theory. Amer. Math. Monthly, Vol.86, pp. 659-677(1979)
4. Ismail, T. H.: A characterization of a good approximation. J. Ed., Sc., Vol.42, (2000)
5. Lutz, G.A., Goze, M.: Nonstandard analysis. Practical Guide With Applications. Lecture Notes in Mathematics. No.881, Springer-Verlage (1981)
6. Nelson, E.: Internal set theory: Anew approach to nonstandard analysis. Bull. Amer. Math. Soc. Vol83, No.6, pp.1165-1189 (1977)
7. Robinson, A.: Nonstandard analysis $2^{\text {Ed }}$. North-Holland, Amsterdam: Pub. Comp. (1974)
8. Stroyan, K.D., Luxemburg, W.A.: Introduction to the Theory of Infinitesimals. New York: Academic Press (1976).
