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ON GENERALIZATION OF S-CLOSEDNESS FOR FUZZY TOPOLOGICAL SPACES THROUGH FUZZY IDEALS

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ABSTRACT

The paper deals with the concept of fuzzy $\mathfrak{I}_S$–closedness in the generalized setting of a fuzzy topological spaces X. Fuzzy $\mathfrak{I}_S$–closedness is shown to be a generalization of the well known concept of fuzzy S–closedness defined by D. Coker and A. Haydar Es in 1987. Fuzzy filter-base (ffb) is used to characterize the said concept to some extent. Concept of $\mathfrak{I}$ –FIP (finite intersection property modulo $\mathfrak{I}$) is used to characterize and study the concept of fuzzy $\mathfrak{I}_S$–closedness for fuzzu extremally disconnected (FED) spaces. Restriction of the notion of fuzzy $\mathfrak{I}_S$–closedness to the arbitrary fuzzy sets in X and invariance of the property under suitable maps are also taken into consideration.

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Key Words: *fuzzy ideals, fuzzy S–closedness, fuzzy $\mathfrak{I}_S$ –closedness, $\mathfrak{I}$ -FIP, s-convergence of fuzzy filter-base, s-accumulation of fuzzy filter-base.*

1. INTRODUCTION

Fuzzyness is one of the most important and useful concept in the modern scientific studies.The concept of fuzzy sets and fuzzy set operations was first introduced by L.A. Zadeh in his classical paper [25]. Subsequently several authors have applied various basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Kuratowski [13], Jancović [10] and several other authors studied on the importance of *ideal* in general topology. Mahmoud [15] and Sarkar [22] independently presented some of the *ideal* concepts in fuzzy trend and studied many other propreties. The concept of *filters* in fuzzy set theory was introduced by Lowen [14] at the same time by Katsaras who studied in his work [11] fuzzy *filters, ultrafilters, clusters* and *convergence* of *filters* in fuzzy setting. Zlata Petričević in his work [20] developed the theory of *filters* to some extent and introduce *s-convergence* and *s-accumulation* of a fuzzy *filter* (*filter-base*). The concept of *S*-closed space was first introduced by Thompson [24] and the concept has further been investigated extensively by Cameron [5], Noiri [18, 19] and many others. The concept of fuzzy S-closed space was given in [6]. The purpose of the present article is to employ the notion of fuzzy *ideals* to introduce and investigate a new class of fuzzy topological space strictly containing that of fuzzy S-closed spaces. In other words, such a space, termed as $\mathfrak{I}_S$ –closed space ($\mathfrak{I}$ being an fuzzy *ideal* on the underlying space X), serves as a generalization of an fuzzy S-*closed* space. In **Section 2** we shall characterize the introduced class of fuzzy $\mathfrak{I}_S$–closed space in terms of fuzzy *filters* (*filter-base*). In **Section 3** we shall consider $\mathfrak{I}_S$–closedness for fuzzy *extremally disconnected* (FED) topological spaces. Our aspiration, in **Section 4**, would be to generalize the idea of $\mathfrak{I}_S$–closedness for arbitrary fuzzy sets in X. The question of preservation of $\mathfrak{I}_S$–closedness under suitable map will also be addressed.

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2. PRELIMINARIES

Throughout the present paper, by (X,τ) we mean an fuzzy topological space(fts, shortly) in Chang`s [4] sense. The class of all fuzzy sets in X is denoted by I^X. The notations ClA, $IntA$ and $1-A$ will stand respectively for the fuzzy closure [4], interior [4] and complement [23] of a fuzzy set A in an fts. X. The support of a fuzzy set A in an fts. X will be denoted by $suppA$ and defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [21] in X with the singletone $supp\{x\} \subset X$ and the value α ($0 < \alpha \leq 1$) will be denoted by x_α. The fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X respectively. For two fuzzy sets A and B in X, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$ [25], while we write AqB if A is quasi-coincident ($q-coincident$, for short) with B [21], ie, if $A(x) + B(x) > 1$, for some $x \in X$. The negation of AqB is written as $A\bar{q}B$. A fuzzy set B is said to be a q-neighbourhood (q-nbd, for short) of A if there is a fuzzy open set U in X such that $A\ qU \leq B$ [21]. A fuzzy set A in an fts. X is said to be a q-nbd of a fuzzy point x_α if and only if there exists a fuzzy open set B such that $x_\alpha qB \leq A$ [3,21]. We shall denote the set of all q-nbd of a fuzzy point x_α by $N(x_\alpha)$. A fuzzy set A in an fts. X is said to be fuzzy regular open [1], (semiopen [1]), pre-open set [2]) if $Int(Cl(A) = A$ (respt. $A \leq Cl(Int(A))$, $A \leq Int(Cl(A))$,). The set of all fuzzy semiopen sets(regular open sets and preopen sets) in X will be denoted by FSO(X) [1] (respt.FRO(X) [1] and FPO(X)).The complement $1-A$ of a fuzzy semiopen set A is called fuzzy semiclosed. The semiclosure of a fuzzy set A in X, to be denoted by $SclA$, is the union of all those fuzzy points x_α such that for any fuzzy semiopen set U with $U(x) + \alpha > 1$, there exists $y \in X$ with $U(y) + A(y) > 1$.

3. FUZZY S-CLOSED SPACES MODULO FUZZY IDEAL (FUZZY $\mathfrak{T}_S$– CLOSED SPACE)

Let us recall the following definitions.

Definition 3.1 [15,22] A non empty collection $\mathfrak{T}$ of fuzzy sets of a set X ($\mathfrak{T} \subseteq I^X$) is called a fuzzy *ideal* on X if and only if (i) $A \in \mathfrak{T}$ and $B \leq A \Rightarrow B \in \mathfrak{T}$ [heredity] (ii) $A \in \mathfrak{T}$ and $B \in \mathfrak{T} \Rightarrow A \vee B \in \mathfrak{T}$ [finite additivity].
$\{0_X\}$ and I^X are the simplest examples of fuzzy *ideals* on a space X.

Definition 3.2: [22] Let (X,τ) be an fts. and $\mathfrak{T}$ be an fuzzy *ideal* on a space X. Let $A \in I^X$. Then the fuzzy local function $A^*(\mathfrak{T},\tau)$ of A is the union of all fuzzy points x_α such that if $B \in N(x_\alpha)$ and $I \in \mathfrak{T}$ then there is at least one $y \in X$ for which $B(y) + A(y) - 1 > I(y)$.

$A^*(\mathfrak{T},\tau)$ is also denoted by $A^*(\mathfrak{T})$ or A^*.

Clearly, $\mathfrak{T} = \{0_X\} \Leftrightarrow A^*(\mathfrak{T},\tau) = clA$, for any fuzzy set A of X and $\mathfrak{T} = I^X \Leftrightarrow A^*(\mathfrak{T},\tau) = 0_X$. An fts. (X,τ) with an fuzzy *ideal* $\mathfrak{T}$ on X is denoted by $(X,\tau,\mathfrak{T})$.

Definition 3.3: [7] A family $\mathcal{U}$ of fuzzy sets in an fts. (X,τ) is called a fuzzy cover of X if $sup\{U(x): U \in \mathcal{U}\} = 1$ for each $x \in X$.

Definition 3.4: [6, 16] An fts. X is said to be fuzzy S-closed if for every fuzzy cover $\mathcal{U}$ of X by fuzzy semiopen sets in X, there exists a finite subcollection $\mathcal{U}_0 = \{U_1, U_2, U_3, \ldots\ldots\ldots, U_n\}$ of $\mathcal{U}$ such that $sup\{ClU(x): U \in \mathcal{U}_0\} = 1$ for each $x \in X$. That is, $\bigvee_{U \in U_0} ClU = 1_X$.

Our definition of the proposed class of spaces goes as follows:

Definition 3.5: Let (X,τ) be an fts. and $\mathfrak{T}$ be an fuzzy *ideal* on X. X is said to be fuzzy $\mathfrak{T}_S$–closed if for every fuzzy cover $\mathcal{U}$ of X by fuzzy semiopen sets, there exists a finite subcollection $\mathcal{U}_0 = \{U_1, U_2, U_3, \ldots\ldots, U_n\}$ of $\mathcal{U}$ such that $\{1 - \bigvee_{U \in U_0} ClU\} \in \mathfrak{T}$.

Let us define finite intersection property in our setting as follows:

Definition 3.6: Let $\mathfrak{T}$ be an fuzzy *ideal* on a space X and let $\mathcal{U} \subset I^X$. Then $\mathcal{U}$ is said to have the finite intersection property modulo $\mathfrak{T}$ (to be abbreviated as $\mathfrak{T} - FIP$), if for every finite subfamily $\{U_1, U_2, U_3, \ldots., U_n\}$ of $\mathcal{U}$, we have $\bigwedge_{i=1}^{n} U_i \notin \mathfrak{T}$.

We now establish the following theorem:

Theorem 3.1: Let (X,τ) be an fts. and $\mathfrak{T}$ an fuzzy *ideal* on X. Then the following are equivalent.

(i). (X,τ) is fuzzy $\mathfrak{T}_S$–closed.

(ii). For every family $\{F_\alpha : \alpha \in \Lambda\}$ of fuzzy semiclosed sets with $\bigwedge_{\alpha \in \Lambda} F_\alpha = 0_X$, then there exists a finite subcollection $\{F_{\alpha_i} : i = 1,2,\ldots,n\}$ such that $\bigwedge_{i=1}^{n} IntF_{\alpha_i} \in \mathfrak{T}$.

(iii). For every fuzzy cover $\{ F_\alpha : \alpha \in \Lambda \}$ of X by fuzzy regular closed sets, there exists a finite subcollection $\{ F_{\alpha_i} : i = 1,2,\dots\dots,n \}$ such that $(1 - \vee_{i=1}^{n} F_{\alpha_i}) \in \mathfrak{T}$.

(iv). For every family$\{ U_\alpha : \alpha \in \Lambda \}$ of fuzzy regular open sets having $\mathfrak{T} - FIP$, $\wedge_{\alpha\in\Lambda} U_\alpha \neq 0_X$.

Proof:

(i) ⇒ (ii): Let $\{ F_\alpha : \alpha \in \Lambda \}$ be a family of fuzzy semiclosed sets with $\wedge_{\alpha\in\Lambda} F_\alpha = 0_X$. Then $\vee_{\alpha\in\Lambda}\{1 - F_\alpha\} = 1_X$ and hence $\{ 1 - F_\alpha : \alpha \in \Lambda \}$ is a fuzzy cover of X by fuzzy semiopen sets of X and hence by (i) there exists a finite subcollection$\{1 - F_{\alpha_1}, 1 - F_{\alpha_2},\dots, 1 - F_{\alpha_n} \}$ such that $\{1 - \vee_{i=1}^{n} Cl\,(1 - F_{\alpha_i})\} \in \mathfrak{T}$. That is, $\wedge_{i=1}^{n} IntF_{\alpha_i} \in \mathfrak{T}$.

(ii) ⇒ (i): Let $\{ U_\alpha : \alpha \in \Lambda\}$ be a fuzzy *cover of* X by fuzzy semiopen sets in X. Then $\{1 - U_\alpha : \alpha \in \Lambda \}$ is a family of fuzzy semiclosed sets with $\wedge_{\alpha\in\Lambda}\{1 - U_\alpha\} = 0_X$. Thus by (ii) there exists finite subcollection$\{1 - U_{\alpha_1}, 1 -$ *Uα2,……, 1−Uαn*$\}$ such that *i=1nInt(1−Uαi)∈ 𝔗*. That is, *(1 – i=1nClUαi) ∈ 𝔗* and hence *X* is fuzzy $\mathfrak{T}_S$ – closed.

(i) ⇒ (iii): It is trivial because of the fact that each fuzzy regular closed sets is fuzzy semiopen.

(iii) ⇒ (iv): Let $\{U_\alpha: \alpha \in \Lambda\}$ be a family of fuzzy regular open sets of X having $\mathfrak{T} - FIP$.If possible, let $\wedge_{\alpha\in\Lambda} U_\alpha = 0_X$. Then $\{ 1 - U_\alpha : \alpha \in \Lambda \}$ is a fuzzy cover of X by fuzzy regular closed sets and hence by (iii) there exists a finite subcollection$\{1 - U_{\alpha_1}, 1 - U_{\alpha_2},\dots\dots, 1 - U_{\alpha_n} \}$ such that $\{1 - \vee_{i=1}^{n}(1 - U_{\alpha_i})\} \in \mathfrak{T}$. That is, $\wedge U_{\alpha_i} \in \mathfrak{T}$ …… a contradiction. Hence $\wedge_{\alpha\in\Lambda} U_\alpha \neq 0_X$.

(iv) ⇒ (i) : Let (X,τ) is not fuzzy $\mathfrak{T}_S$ –closed and $\{ U_\alpha : \alpha \in \Lambda \}$ be a fuzzy cover of X by fuzzy semiopen sets in X such that for every finite subcollection $\{U_{\alpha_1}, U_{\alpha_2},\dots\dots, U_{\alpha_n} \}$ of $\{ U_\alpha : \alpha \in \Lambda \}$, we have $(1 - \vee_{i=1}^{n} Cl\, U_{\alpha_i}) \notin \mathfrak{T}$. That is, $\wedge_{i=1}^{n}(1 - ClU_{\alpha_i}) \notin \mathfrak{T}$. Thus $\{1 - ClU_\alpha: \alpha \in \Lambda\}$ is a family of fuzzy regular open sets having $\mathfrak{T} - FIP$. Therefore, by (iv) $\wedge_{\alpha\in\Lambda}(1 - Cl(U_\alpha)) \neq 0_X$. That is,$(1 - \vee_{\alpha\in\Lambda} Cl(U_\alpha)) \neq 0_X$. So $(\vee_{\alpha\in\Lambda} Cl(U_\alpha)) \neq 1_X$ which implies $\vee_{\alpha\in\Lambda} U_\alpha \neq 1_X$ ----- a contradiction [since $\{U_\alpha: \alpha \in \Lambda\}$ be a fuzzy cover of X]. Hence (X,τ) is fuzzy $\mathfrak{T}_S$ –closed.

Theorem 3.2: Let (X,τ) be an fts. and $\mathfrak{T}$ an fuzzy ideal on X.If X is fuzzy $\mathfrak{T}_S$–closed then every cover $\{U_\alpha: \alpha \in \Lambda\}$ of X by fuzzy preopen sets in X has a finite subcollection $\{U_{\alpha_i}: i = 1,2, \dots, n\}$ such that $\{1 - \vee_{i=1}^{n} Cl(U_{\alpha_i})\} \in \mathfrak{T}$.

Proof: Follows from the fact that the closure of a fuzzy preopen set is fuzzy semiopen.

Before going to characterize fuzzy $\mathfrak{T}_S$–closedness in terms of fuzzy *filters* (*filter-base*) let us recall the following definitions and results.

Definition 3.7: [11] Let be $\mathcal{B}$ a non-empty family of fuzzy subsets of X. Then is $\mathcal{B}$ called a *base* for a fuzzy *filter* (or a fuzzy *filter-base*) on X if the following two conditions are satisfied.

(i) $0_X \notin \mathcal{B}$.

(ii) If $A, B \in \mathcal{B}$, then $A \wedge B \in \mathcal{B}$.

If $\mathcal{B}$ has the additional property

(iii) $C \in \mathcal{B}$ and $C \leq D$ implies $D \in \mathcal{B}$ then $\mathcal{B}$ is called a fuzzy *filter* on X.

A maximal, with respect to set inclusion, fuzzy *filter* on X is called a fuzzy *ultrafilter* or a fuzzy *maximal filter(* or *filter-base*). If $\mathcal{B}$ is a base for a fuzzy *filter* on X, the collection $F_{\mathcal{B}} = \{ A \in I^X : \exists\, B \in \mathcal{B}$ with $B \leq A\}$ is the fuzzy *filter* generated by $\mathcal{B}$. We say a *filter* F_1 is finer than a *filter* F_2 if $F_2 \subset F_1$ .That is, if for each $B \in F_2$ there exists $A \in F_1$ such that $A \leq B$.

Result 3.1. [20]

(i) If F_1, F_2 be any two fuzzy *filter*-bases (ffb ,shortly) on an fts. X.Then the family $F_1 \vee F_2 = \{ A \vee B : A \in F_1, B \in F_2\}$ is an ffb on X.

(ii) If $A \wedge B \neq 0_X$ for each $A \in F_1$ and each $B \in F_2$, then $F_1 \wedge F_2 = \{ A \wedge B : A \in F_1, B \in F_2\}$ is an ffb on X.

(iii) A non-empty family $\mathcal{B} \subset I^X$ is an ffb on X *iff* for any finite collection $\{ A_i : i = 1,2, \dots\dots\dots, n \}$ of $\mathcal{B}$, $\wedge_{i=1}^{n} A_i \neq 0_X$.

(iv) Let $\mathcal{B}$ is an ffb on X and $f: X \to Y$ be a function. Then $f(\mathcal{B}) = \{ f(A): A \in \mathcal{B}\}$ is an ffb on Y. If f is onto and $\mathcal{B}$ is an ffb on Y, then $f^{-1}(\mathcal{B}) = \{ f^{-1}(C) : C \in \mathcal{B} \}$ is an ffb on X.

Definition 3.8: [20] A fuzzy point x_α is said to be a fuzzy cluster point of an ffb $\mathcal{B}$ if every q-nbd of x_α is q −coincident with each member of $\mathcal{B}$.

Result 3.2: [20] A fuzzy point $x_\alpha(0 < \alpha < 1)$ in an fts. X is a cluster point of an ffb $\mathcal{B}$ *iff* $x_\alpha \leq ClA$, for each $A \in \mathcal{B}$.

This result is equivalent to the definition 3.2 [8].

Definition 3.9: [8] An fuzzy *filter-base* $\mathcal{B}$ is said to converge to a fuzzy point $x_\alpha (\mathcal{B} \to x_\alpha)$ if every q-nbd of x_α contains a member of $\mathcal{B}$ and $x_\alpha \leq ClA$ for every$A \in \mathcal{B}$.

Result 3.3: [11] Let X and Y be two fts. and x_α be a fuzzy point in X. If f is a mapping from X to Y, continuous at x_α, then for every fuzzy *filter-base* $\mathcal{B}$, $\mathcal{B} \to x_\alpha$, implies $f(\mathcal{B}) \to f(x)_\alpha$.

Definition 3.10: [20] We say that a fuzzy point $x_\alpha \in Scl_\theta A$ ($x_\alpha \in Cl_\theta A$) if for each fuzzy semiopen set U (respt. fuzzy open set U), $x_\alpha q U$ implies $ClUqA$.

Definition 3.11: [20] An fuzzy *filter* (or an ffb) $\mathcal{F}$ is said to $s-$accumulate to x_α if $x_\alpha \in Scl_\theta A$, for each $A \in \mathcal{F}$.

Definition 3.12: [20] Let (X,τ) be an fts. Let $S(x_\alpha)$ be the *filter* generated by the family $\mathrm{S0}(x_\alpha) = \{ClA : x_\alpha q A \in \mathrm{S0}(X,\tau)\}$. We say an fuzzy *filter* (or an) $\mathcal{F}$ $s-$converges to x_α if $S(x_\alpha) \subset \mathcal{F}$.

Result 3.4: [20] An fuzzy *filter* (or an ffb) $\mathcal{F}$ in an fts. X $s-$converges to x_α *iff* for each fuzzy semiopen q-nbd U of x_α, there is a $A \in \mathcal{F}$ such that $A \leq ClU$ and $x_\alpha \in Scl_\theta A$ for each $A \in \mathcal{F}$.

Theorem 3.3: [20] An fuzzy *filter*-base $\mathcal{B}$ (or an fuzzy *filter*) on X $s-$accumulates iff *there* exists a fuzzy *filter* $\mathcal{F}$ finer than $\mathcal{B}$ which $s-$converges.

We shall characterize the notion of fuzzy $\mathfrak{I}_S$–closedness in terms of fuzzy *filter-bases.*

Theorem 3.4: For an fuzzy topological space (X,τ) with fuzzy *ideal* $\mathfrak{I}$ the following are equivalent.
(i) (X,τ) is fuzzy $\mathfrak{I}_S$–closed.
(ii) Every maximal fuzzy *filter*-base on $I^X \setminus \mathfrak{I}$ $s-$converges.
(iii) Each fuzzy *filter*-base on $I^X \setminus \mathfrak{I}$ $s-$accumulates.

Proof:
(i) ⇒ (ii): Let (X,τ) is fuzzy $\mathfrak{I}_S$ –closed. And let $\mathcal{F}$ be a maximal fuzzy *filter*-base on $I^X \setminus \mathfrak{I}$. If possible let $\mathcal{F}$ is not $s-$convergent to any fuzzy point in X. As $\mathcal{F}$ is maximal, $\mathcal{F}$ does not $s-$accumulate to any fuzzy point in X. That is, $x_\alpha \notin Scl_\theta U_{x_\alpha}$ for an $U_{x_\alpha} \in \mathcal{F}$ (definition 2.10). This implies that for each fuzzy point x_α there exists a fuzzy semiopen set $V_{x_\alpha} \in S(x_\alpha)$ and an $U_{x_\alpha} \in \mathcal{F}$ such that $x_\alpha q V_{x_\alpha}$ and $ClV_{x_\alpha} \bar{q} U_{x_\alpha}$ (definition 2.9). Without loss of generality, we may assume that $\{ V_{x_\alpha} : x_\alpha \in I^X \}$ is a cover of X by fuzzy semiopen sets in X. Since X is fuzzy $\mathfrak{I}_S$ –closed, there exists a finite subcollection$\{ V_{x_{\alpha_i}} : i = 1,2,\dots\dots, n \}$ of $\{V_{x_\alpha} : x_\alpha \in I^X \}$ such that $(1 - \bigvee_{i=1}^{n} ClV_{x_{\alpha_i}}) \in \mathfrak{I}$. As $\mathcal{F}$ is maximal fuzzy *filter-base* there exists $F \in \mathcal{F}$, $F \neq 0_X$ such that $F \leq \bigwedge_{i=1}^{n} U_{x_{\alpha_i}}$ and $U_{x_{\alpha_i}} \bar{q} ClV_{x_{\alpha_i}}$ for each $i = 1,2,\dots\dots, n$. This implies that $F \bar{q} ClV_{x_{\alpha_i}}$ for each $i = 1,2,\dots\dots, n$. That is, $F \,\bar{q} \bigvee_{i=1}^{n} ClV_{x_{\alpha_i}}$. So $F \leq (1 - \bigvee_{i=1}^{n} ClV_{x_{\alpha_i}}) \in \mathfrak{I}$ ………. a contradiction. Hence $\mathcal{F}$ $s-$converges.

(ii) ⇒ (iii): Let $\mathcal{F}$ be any fuzzy *filter-base* on $I^X \setminus \mathfrak{I}$. Then there exists a maximal fuzzy *filter-base* $\mathcal{F}^*$ on $I^X \setminus \mathfrak{I}$ containing $\mathcal{F}$ and hence by (ii) $\mathcal{F}^*$ $s-$converges to some fuzzy point x_α in X. Then by result 2.4 for each fuzzy semiopen q-nbd U of x_α, there is a $A \in \mathcal{F}^*$ such that $A \leq ClU$ and $x_\alpha \in Scl_\theta A$ for each $A \in \mathcal{F}^*$. This implies that $\mathcal{F}^*$ $s-$accumulates (definition 2.10). As $\mathcal{F}^*$is maximal fuzzy *filter-base* containing $\mathcal{F}$, $\mathcal{F}$ $s-$accumulates.

(iii) ⇒ (i): Let (X,τ) is not fuzzy $\mathfrak{I}_S$ –closed. Then by theorem 2.1 there exists a family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of fuzzy semiclosed sets in with $\bigwedge_{\alpha \in \Lambda} F_\alpha = 0_X$ such that for every finite subcollection $\mathcal{F}_0$ of $\mathcal{F}$, $\bigwedge_{F_\alpha \in F_0} IntF_\alpha \notin \mathfrak{I}$. Let $\mathcal{B} = \{ \bigwedge_{F_\alpha \in F_0} IntF_\alpha : \mathcal{F}_0$ is a finite subcollection of $\mathcal{F}\}$. Then obviously $\mathcal{B}$ is a fuzzy *filter-base* on $I^X \setminus \mathfrak{I}$. So by (iii) $\mathcal{B}$ $s-$accumulates at some fuzzy point x_α in X. This implies that for each fuzzy semiopen set U_{x_α} threre exists $B_{x_\alpha} \in \mathcal{B}$, $x_\alpha q U_{x_\alpha}$implies $ClU_{x_\alpha} q B_{x_\alpha}$ (definition 2.9 & 2.10)…………(1). As $x_\alpha \notin \wedge \mathcal{F}$ and $\wedge \mathcal{F} = 0_X$, there exists an $V_{x_\alpha} \in \mathcal{F}$ such that $x_\alpha \notin V_{x_\alpha}$and $V_{x_\alpha}(x_\alpha) = 0$ which implies that $intV_{x_\alpha}(x) = 0$ and $1 - intV_{x_\alpha}(x) = 1$. So $Cl(1 - V_{x_\alpha})(x) = 1$. This implies that $Cl(1 - V_{x_\alpha}) \wedge IntV_{x_\alpha} = 0_X$. Let $(1 - V_{x_\alpha}) = V$, a fuzzy semiopen set(as V_{x_α} is fuzzy semiclosed set) and let $intV_{x_\alpha} = B \in \mathcal{B}$. So $ClV \wedge B = 0_X$. This implies that $ClV \bar{q} B$ which contradicts (1). Hence X is fuzzy $\mathfrak{I}_S$ –closed.

4. FUZZY $\mathfrak{I}_S$–CLOSEDNESS FOR FUZZY EXTREMALLY DISCONNECTED (FED) TOPOLOGICAL SPACES

In this Section we shall characterize fuzzy $\mathfrak{I}_S$–closedness for fuzzy *extremally disconnected* spaces.

Theorem 4.1: Let an fts. (X,τ) with an fuzzy *ideal* $\mathfrak{I}$ be fuzzy $\mathfrak{I}_S$–closed. Then for every family of fuzzy open sets $\{ U_\alpha : \alpha \in \Lambda \}$ of X with $\mathfrak{I} - FIP$ it holds $\wedge_{\alpha\in\Lambda} ClU_\alpha \neq 0_X$.

Proof: Let $\{ U_\alpha : \alpha \in \Lambda \}$ be a family of fuzzy open sets of X with $\mathfrak{I} - FIP$. If possible let $\wedge_{\alpha\in\Lambda} ClU_\alpha = 0_X$. Then $\vee_{\alpha\in\Lambda}(1 - ClU_\alpha) = 1_X$ and hence $\{1 - ClU_\alpha\}_{\alpha\in\Lambda}$ is a cover of X by fuzzy semiopen sets $\{1 - ClU_\alpha\}_{\alpha\in\Lambda}$. Hence there exists a finite subcollection$\{U_{\alpha_i} : i = 1,2, \dots\dots, n\}$ such that $\{1 - \vee_{i=1}^{n} Cl(1 - ClU_{\alpha_I})\} \in \mathfrak{I}$ (as X is fuzzy $\mathfrak{I}_S$–closed). Therefore, $\wedge_{i=1}^{n} U_{\alpha_i} \leq \wedge_{i=1}^{n}\{1 - \vee_{i=1}^{n} Cl(1 - ClU_{\alpha_I})\} \in \mathfrak{I}$.So $\wedge_{i=1}^{n} U_{\alpha_i} \in \mathfrak{I}$ ---------- a contradiction (since $\{ U_\alpha : \alpha \in \Lambda \}$ has $\mathfrak{I} - FIP$). Hence $\wedge_{\alpha\in\Lambda} ClU_\alpha \neq 0_X$.

Definition 4.1: [9] An fts. X is fuzzy *extremally disconnected* (FED, for short) if the closure of every fuzzy open set is fuzzy open.

Theorem 4.2: If an fts. (X,τ) with an fuzzy *ideal* $\mathfrak{I}$ is fuzzy *extremally disconnected* and for any family of fuzzy open sets $\{U_\alpha : \alpha \in \Lambda \}$ of X with $\mathfrak{I} - FIP$ it holds $\wedge_{\alpha\in\Lambda} ClU_\alpha \neq 0_X$, then X is fuzzy $\mathfrak{I}_S$–closed.

Proof: Let $\{ U_\alpha : \alpha \in \Lambda \}$ be a cover of X by fuzzy semiopen sets in X. If possible let X is not fuzzy $\mathfrak{I}_S$–closed. Hence for each finite family $\{ U_{\alpha_i} : i = 1,2, \dots, n \}$, we have $\{1 - \vee_{i=1}^{n} ClU_{\alpha_i}\} \notin \mathfrak{I}$. Therefore, by theorem 2.1, $\wedge_{i=1}^{n} Int(1 - U_{\alpha_i}) \notin \mathfrak{I}$. That is, $\{ Int(1 - U_\alpha) : \alpha \in \Lambda \}$ is a family of fuzzy open sets with $\mathfrak{I} - FIP$. That is, $\{(1 - ClU_\alpha): \alpha \in \Lambda \}$ is a family of fuzzy open sets with $\mathfrak{I} - FIP$ (since $Int(1 - U_\alpha) = (1 - ClU_\alpha) =$ a fuzzy open set). Then by the condition, $\wedge_{\alpha\in\Lambda} Cl(1 - ClU_\alpha) \neq 0_X$. Since X is FED and since $ClU_\alpha = Cl(IntU_\alpha)$ for each fuzzy semiopen set U_α in X, it follows that ClU_α is fuzzy open for each $\alpha \in \Lambda$. That is, $(1 - ClU_\alpha)$ is fuzzy closed. Therefore, $\vee_{\alpha\in\Lambda} U_\alpha \leq \vee_{\alpha\in\Lambda} ClU_\alpha = \vee_{\alpha\in\Lambda}(1 - (1 - ClU_\alpha)) = \vee_{\alpha\in\Lambda}(1 - Cl(1 - ClU_\alpha)) < 1_X$. [since $1 - ClU_\alpha$ is fuzzy closed and $\wedge_{\alpha\in\Lambda} Cl(1 - ClU_\alpha) \neq 0_X$]. So $\vee_{\alpha\in\Lambda} U_\alpha < 1_X$.------a contradiction as $\{ U_\alpha : \alpha \in \Lambda \}$ is a fuzzy cover of X. Hence X is fuzzy $\mathfrak{I}_S$–closed.

5. FUZZY $\mathfrak{I}_S$–CLOSEDNESS: FUZZY SUBSETS AND FUNCTIONS

In this section our intention is to characterize fuzzy $\mathfrak{I}_S$–closedness for arbitrary fuzzy sets in an fts. X. The invariance of the property under suitable maps will also be dicussed in this section.

We recall the definition of fuzzy S-closed sets (FSC sets, shortly).

Definition 5.1: [16] A fuzzy set A in an fts. (X,τ) is said to be fuzzy S-closed set(FSC sets, shortly) if for every fuzzy cover $\{U_\alpha : \alpha \in \Lambda \}$ of A by fuzzy semiopen sets in X there exists a finite subset Λ_0 of Λ such that $A \leq \vee_{\alpha\in\Lambda_0} ClU_\alpha$.

We now generalize the concept of fuzzy S-closedness for an arbitrary fuzzy set in our setting in the following way.

Definition 5.2: A fuzzy set A in an fts. (X,τ) with an fuzzy *ideal* $\mathfrak{I}$, is said to be fuzzy $\mathfrak{I}_S$–closed relative to X if for every fuzzy cover $\{ U_\alpha : \alpha \in \Lambda \}$ of A by fuzzy semiopen sets in X, there exists a finite subset Λ_0 of Λ such that $(1 - \vee_{\alpha\in\Lambda_0} ClU_\alpha) \in \mathfrak{I}$.

Theorem 5.1: Let A and B are two fuzzy sets in an fts. (X,τ) with an fuzzy *ideal* $\mathfrak{I}$. If A is fuzzy $\mathfrak{I}_S$–closed relative to X and B is fuzzy regular open set in X then $(A \wedge B)$ is fuzzy $\mathfrak{I}_S$–closed relative to X.

Proof: Let $\{ U_\alpha : \alpha \in \Lambda \}$ be a fuzzy cover of $(A \wedge B)$ by fuzzy semiopen sets in X. Then $\{U_\alpha : \alpha \in \Lambda\} \vee \{1 - B\}$ is a fuzzy cover of A by fuzzy semiopen sets in$X$ (since $1 - B$ is fuzzy regular closed and hence fuzzy semiopen). So there exists a finite subset Λ_0 of Λ such that $1 - \vee_{\alpha\in\Lambda_0} Cl(U_\alpha) \vee (1 - B) \in \mathfrak{I}$ (since A is fuzzy $\mathfrak{I}_S$–closed relative to X).

Now $\{1 - (\vee_{\alpha\in\Lambda_0} ClU_\alpha)\} \leq \{1 - \vee_{\alpha\in\Lambda_0} Cl(U_\alpha) \vee (1 - B)\} \in \mathfrak{I} \implies \{1 - \vee_{\alpha\in\Lambda_0} Cl(U_\alpha)\} \in \mathfrak{I}$. Thus $(A \wedge B)$ is fuzzy $\mathfrak{I}_S$–closed relative to X.

Corollary 5.1: If (X,τ) be an fts. with an fuzzy i*deal* $\mathfrak{I}$ and A be is fuzzy regular open set in X, then A is fuzzy $\mathfrak{I}_S$–closed relative to X.

Theorem 5.2: Let A and B are two fuzzy sets in an fts. (X,τ) with an fuzzy *ideal* $\mathfrak{I}$. If A and B are fuzzy $\mathfrak{I}_S$–closed relative to X then so is $(A \vee B)$.

Proof: Let $\{ U_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of $(A \vee B)$ by fuzzy semiopen sets in$X$. Then $\{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of A as well as B by fuzzy semiopen sets in X. Then there exists finite subsets Λ_0 and Λ_0' of Λ such that

$\{1 - \vee_{\alpha\in\Lambda_0} ClU_\alpha\} \in \mathfrak{I}$ and $\{1 - \vee_{\alpha\in\Lambda_0'} ClU_\alpha\} \in \mathfrak{I}$. Thus $\{1 - \vee_{\alpha\in\Lambda_0} ClU_\alpha\}\vee\{1 - \vee_{\alpha\in\Lambda_0'} ClU_\alpha\} \in \mathfrak{I}$ (Finite additivity). That is, $1 - \vee_{\alpha\in\Lambda_0\cup\Lambda_0'} ClU_\alpha \in \mathfrak{I}$. Hence $(A \vee B)$ is fuzzy $\mathfrak{I}_S$–closed relative to X.

Theorem 5.3: [12] Let $f: X \to Y$ be an arbitrary crisp function. Let $A_\alpha \in I^X$ and $B_\alpha \in I^Y$, $\alpha \in \Lambda$, Then (i) $f(A) = 0_Y$ *iff*
(i) $A = 0_X$;
(ii). if $A_1 \le A_2$, then $f(A_1) \le f(A_2)$;
(iii).$f(\vee_{\alpha\in\Lambda} A_\alpha) = \vee_{\alpha\in\Lambda} f(A_\alpha)$;
(iv).$f(\wedge_{\alpha\in\Lambda} A_\alpha) \le \wedge_{\alpha\in\Lambda} f(A_\alpha)$;
(v). if $B_1 \le B_2$, then $f^{-1}(B_1) \le f^{-1}(B_2)$;

We now mention the following result follows from the *Theorem* 4.3.

Result 5.1: Let (X,τ) be an fts. with an fuzzy *ideal* $\mathfrak{I}$ and let$f: X \to Y$ be a function, then $f(\mathfrak{I}) = \{f(I):\ I \in \mathfrak{I}\}$ is an fuzzy *ideal* on Y.

Definition 5.3: [17] A function $f: X \to Y$ is said to be fuzzy semicontinuous if $f^{-1}(B)$ is fuzzy semiopen in X for every fuzzy open set B in Y.

Definition 5.4: [17] A function $f: X \to Y$ is said to be fuzzy irresolute if $f^{-1}(B)$ is fuzzy semiopen in X for every fuzzy semiopen set B in Y.

Theorem 5.3: Let $f: (X, \tau, \mathfrak{I}) \to (Y, \tau)$ be a fuzzy continuous irresolute function and a fuzzy set A be fuzzy $\mathfrak{I}_S$ –closed relative to X. Then $f(A)$ is fuzzy $(f(\mathfrak{I}))_S$ −closed relative to Y.

Proof: Let $\{ V_\alpha : \alpha \in \Lambda \}$ be a fuzzy cover of $f(A)$ by fuzzy semiopen sets in Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of A by fuzzy semiopen sets in X (since f is fuzzy irresolute). Hence $\{1 - \vee_{i=1}^{n} Clf^{-1}(V_{\alpha_i})\} \in \mathfrak{I}$ for some positive integer n. As f is fuzzy continuous $Clf^{-1}(A) \le f^{-1}(ClA)$ for every fuzzy set A in Y. Thus $\{1 - \vee_{i=1}^{n} Cl(V_{\alpha_i})\}$ $\le \{1 - \vee_{i=1}^{n} ff^{-1}(Cl(V_{\alpha_i}))\} \le \{1 - \vee_{i=1}^{n} f^{-1}(Cl(V_{\alpha_i}))\} \le \{1 - \vee_{i=1}^{n} Cl(f^{-1}(V_{\alpha_i}))\}$. So $\{1 - \vee_{i=1}^{n} Cl(V_{\alpha_i})\} \in f(\mathfrak{I})$ and consequently $f(A)$ is fuzzy $(f(\mathfrak{I}))_S$ −closed relative to Y.

Lemma 5.1: [20] If an fts. (X,τ) is FED, then $SclA = Cl_\theta A$ for each $A \in$ SO(X).

Corollary 5.2: [20] If an fts. (X,τ) is FED, then $SclA = ClA = Cl_\theta A$ for each $A \in$ SO(X).

Theorem 5.3: Let $f: (X, \tau, \mathfrak{I}) \to (Y, \tau_1)$ be a fuzzy irresolute function and X be FED. If a fuzzy set A in X be fuzzy $\mathfrak{I}_S$ –closed relative to X, then $f(A)$ is fuzzy $(f(\mathfrak{I}))_S$ −closed relative to Y.

Proof: Let $\{ V_\alpha: \alpha \in \Lambda\}$ be a fuzzy cover of $f(A)$ by fuzzy semiopen sets in Y. Then $\{f^{-1}(V_\alpha): \alpha \in \Lambda\}$ is a fuzzy cover of A by fuzzy semiopen sets in X(since f is irresolute). Since A is fuzzy $\mathfrak{I}_S$ –closed relative to X, there exists a finite subset Λ_0 of Λ such that $\{1 - \vee_{\alpha\in\Lambda_0} Cl(f^{-1}(V_\alpha))\} \in \mathfrak{I}$. As X is FED,
$\{1 - \vee_{\alpha\in\Lambda_0} Scl(f^{-1}(V_\alpha))\} \in \mathfrak{I}$.

Now $\{1 - \vee_{\alpha\in\Lambda_0} Scl(f^{-1}(V_\alpha))\} \ge \{1 - \vee_{\alpha\in\Lambda_0} f(Scl(f^{-1}(V_\alpha)))\} \ge \{1 - \vee_{\alpha\in\Lambda_0} Scl(V_\alpha)\} \ge \{1 - \vee_{\alpha\in\Lambda_0} Cl(V_\alpha)\}$ (As f is fuzzy irresolute). Therefore, $\{1 - \vee_{\alpha\in\Lambda_0} Cl(V_\alpha)\} \in f(\mathfrak{I})$. Hence $f(A)$ is fuzzy $(f(\mathfrak{I}))_S$ −closed.

Corollary 5.3: If $f: X \to Y$ is an fuzzy irresolute surjection from an $\mathfrak{I}_S$ –closed fts. X to an fts. Y and X is FED, then Y is fuzzy $f(\mathfrak{I})_S$ –closed.

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