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MEASUREMENT OF PRODUCTIVE AND TECHNICAL OUTPUT EFFICIENCY

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ABSTRACT

This theory is aimed at presenting how output productive efficiency measures are obtained using linear programming approach? The method is primarily an axiomatic approach. Estimation of output pure technical, scale, overall technical, allocative and overall productive efficiencies is explained by suitable numerical examples.

**Schmidt, P. and Lovell, C.A.K., 'Technical and allocative inefficiency relative to Stochastic Production and Cost Frontier's, journal of econometrics, vol.No.91979, pp.343-366.*

INTRODUCTION

In a path breaking article Farrell* introduced the concept of technical efficiency which is defined in terms of input reduction. The production process he assumed obeyed constant returns to scale, as the underlying production function is linear homogenous. His technical efficiency estimates are obtained using unit output isoquant as reference Technology.

Farrell's method was later extended by Koop**, whose reference technology was provided by a Frontier Production Function that admits any type of returns to scale, increasing, constant or decreasing. By Koop's method can be implemented to measure pure technical scale, overall technical, allocative and overall technical efficiencies provided that the frontier production may be explicitly estimated.

Timmer*** proposed linear programming approach to estimate an explicitly specified Frontier Production Function. The Frontier production function is a full Frontier in the sense that the output of each competing producer fall either below or on the production frontier.

*Farrell, M.J., The measurement of productive efficiency, Journal of Royal Statistical Society series-A, 1957, pp. 253-290.

**Koop, R.J., The measurement of productive efficiency, A reconsideration, The Quarterly Journal of Economics, (1981), pp.477-500

***Timmer, C.P., using a probabilistic frontier production function to measure technical efficiency, 'Journal of Political Economy'. vol. No.5, 1971, pp.383-394.

Farrell's method gives input oriented productive efficiencies. But his approach can be extended to outline output oriented productive efficiency measures such as output pure technical scale and overall Technical, allocative and overall productive efficiency measured.

For efficiency measurement, the chief tool is production function either parametric or non-parametric. A parametric frontier production may be fitted for data by either statistical estimation methods or methods of mathematical programming.

Schmidt*(1976), Schmidt and Lovell (1979) proposed statistical estimation procedures to estimate stochastic frontier production functions. These fitted frontier productions are implemented to measure technical and allocative efficiencies.

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PRODUCTIVE EFFICIENCY-OUTPUT APPROACH

A Producer of a firm may combine multiple inputs to produce more than one outputs. Suppose the production process is such that variation of inputs is not possible. At given level of inputs, it is enquired if further output augmentation is possible. So the producer is output technical inefficient, otherwise efficient.

Consider two output and one input production process, for which the underlying production function is,

$$X = \phi(u_1, u_2)$$

If production is efficient equality holds as such we call as frontier production function.

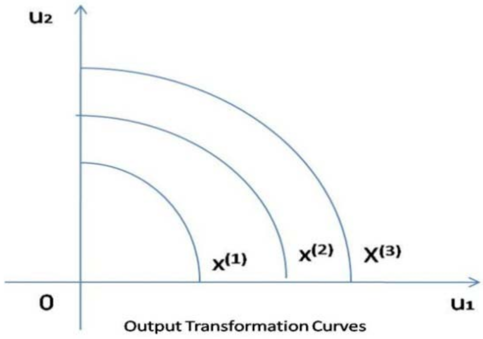
X: Input

u_1, u_2: Outputs

We shall assume that outputs are interdependent in the sense that one output augmentation leads to the reduction in other output.

For a given level of input, the locus of all output combinations producible by input x_0 is known to be output isoquant or the product transformation curve.

The product transformation curves are concave to the origin, do not intersect with each other, farther an output isoquant from output origin, greater the input represents.



Output Transformation Curves

The rate of output transformation t** is,

$$ROT = -\frac{du_2}{du_1} = \frac{\partial\phi}{\partial u_1} \Big/ \frac{\partial\phi}{\partial u_2}$$

$$ROT = -\frac{du_2}{du_1}$$

$$ROT = \frac{\partial\phi}{\partial u_1} \Big/ \frac{\partial\phi}{\partial u_2}$$

** If input x_0 is constant, the output is isoquant is,

$$X_0 = \phi(u_1, u_2)$$

Total differential,

$$dx_0 = -\frac{d\phi}{du_1}\ du_1 + \frac{d\phi}{du_2}\ du_2$$

Since output is constant, $dx_0 = 0$, consequently

$$ROT = -\frac{du_2}{du_1} = \frac{\partial\phi}{\partial u_1} \Big/ \frac{\partial\phi}{\partial u_2}$$

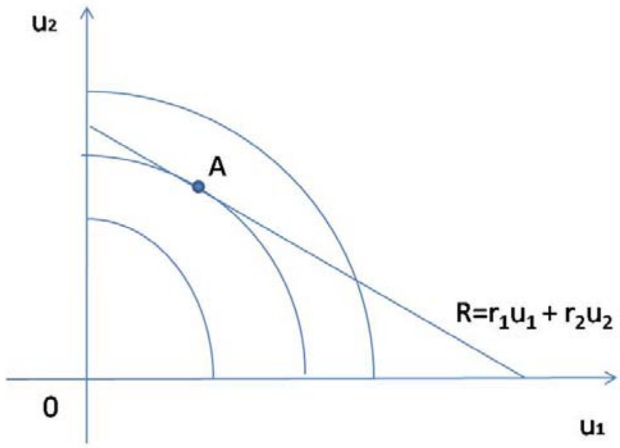
TOTAL REVENUE FUNCTION:

If r1 and r2 are prices per unit of first and second outputs respectively, the total revenue is,

$$R = r_1 u_1 + r_2 u_2$$

The producer is said to attain equilibrium, if the ratio of marginal inputs is equal to the ratio of outputs prices*.

$$\frac{\partial\phi}{\partial x_1} \Big/ \frac{\partial\phi}{\partial x_2} = \frac{r_1}{r_2}$$



Consider the Lagrangian Function

$$R = r_1u_1 + r_2u_2 + \lambda\big(x_0 - \phi(u_1, u_2)\big)$$

The necessary conditions for constrained maximum revenue are,

$$\frac{\partial\phi}{\partial u_1} = r_1 - \lambda\frac{\partial\phi}{\partial u_1} = 0$$

$$\frac{\partial\phi}{\partial u_2} = r_2 - \lambda\frac{\partial\phi}{\partial u_2} = 0$$

Where λ is the Lagrangian multiplier

Upon simplification

$$\frac{\partial\phi}{\partial u_1} \Big/ \frac{\partial\phi}{\partial u_2} = \frac{r_1}{r_2}$$

The producer who operates at A is in equilibrium. He produces u1 and u2 units of first and second outputs by employing x units of input.

$$\left[\frac{\partial\phi}{\partial x_1}\right]_A \Big/ \left[\frac{\partial\phi}{\partial x_2}\right]_A = \frac{r_1}{r_2}$$

The second order condition for attainability of maximum revenue is the determinant if Hessian matrix should be positive definite

$$\begin{vmatrix} -\lambda\frac{\partial^2 x}{\partial {u_1}^2} & -\lambda\frac{\partial^2 x}{\partial u_1 \partial u_2} & -\frac{\partial x}{\partial u_1} \\ -\lambda\frac{\partial^2 x}{\partial u_1 \partial u_2} & -\lambda\frac{\partial^2 x}{\partial {u_2}^2} & -\frac{\partial x}{\partial u_2} \\ -\frac{\partial x}{\partial u_1} & -\frac{\partial x}{\partial u_2} & 0 \end{vmatrix}$$

OUTPUT SETS

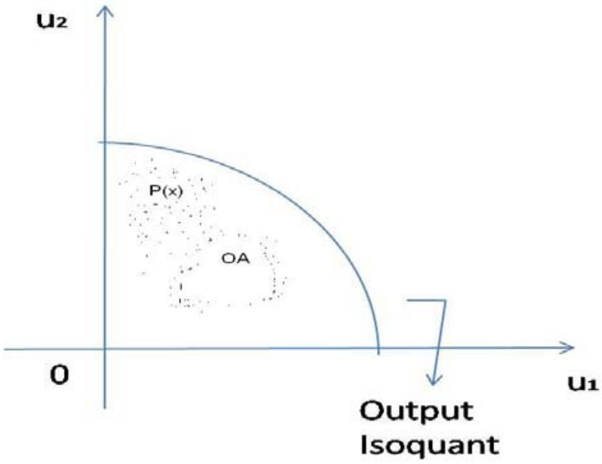
The collection of all output vectors which can be producible by a given input vector forms an output set

$$P(x) = \{u : x \; can \; prove \; u\}$$

If $p(x)$ can satisfy certain structural properties, $L(u)$ and $P(x)$ determine each other completely. The duality between $L(u)$ and $P(x)$ is seen from the definitions given below:

$$L(u) = \{x : u \; \varepsilon \; P(x)\}^*$$

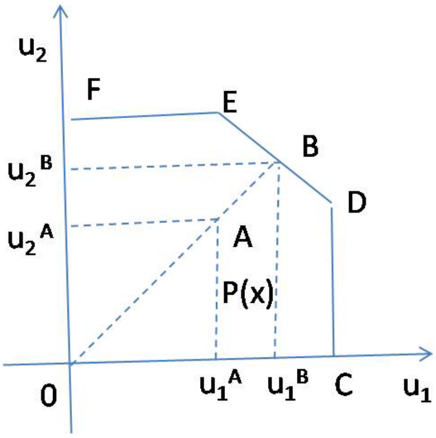
$$P(x) = \{u : x \; \varepsilon \; L(u)\}$$



The dotted region and its boundary in non-negative quadrant of the graph is the output set which is constituted by all output vectors each of which can be produced by the input x. the points which fall on the output isoquant are efficient points and the others are inefficient points. The production unit that operates at A is output technical inefficient unit. To achieve output technical efficiency there is a need for further augmentation of outputs radially along the ray that emanates from origin in the direction of frontier

*Henderson,j.m and quandt R.E(1985)"micro Economic Theory……" A mathematical approach, MC GRAHILL Book company New Delhi.

Suppose the production frontier is piece-wise linear



The linear segments C-D, D-E and E-F constitute the isoquant of $p(x)$.at C, u1 alone is produced and at F u2 alone is produced. The segment ED is constituted by efficient points where the points on CD and EF represent weak efficient points.

The production unit that operates at A is output technical efficient. It is possible to increase the outputs radially along the ray the emanates from origin in the direction of frontier to the point B. Any further movement in upward direction takes the outputs out of $p(x)$, so that these outputs cannot be produced by x.

B is output technical efficiency can be estimated as the solution of the optimization problem,

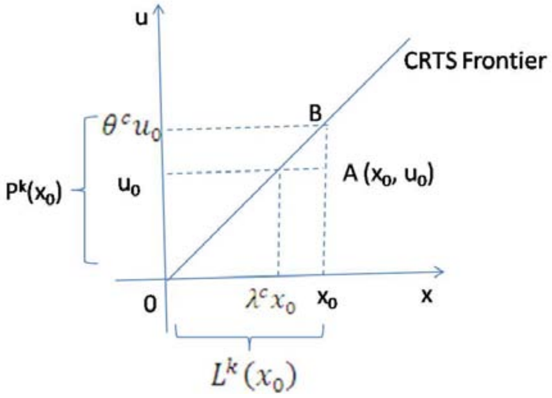
$$Max\ \theta$$

Such that $\theta u\ \epsilon\ P(x)$

Output technically efficiency

$$OTE = Max\ \theta = \frac{OB}{OA}$$

$$OTE \geq 1*$$



The producer who operates A is output technical efficient. The returns to the scale of constant. x_0 Produces u_0 the input level set gathers all points from $\lambda^c x_0$ and on its right hand side. The input set marked on the graph is $L^k(x_0)$ and it is constituted by all input vectors capable of producing of x_0 under constant returns to scale. The output set begins at Origen and gathers all points on vertical axis up to $\theta^c u_0$. The output set $P^k(x_0)$ is consistent with constant returns to scale. By keeping input, x_0 constant the point A is vertically projected on to the CRTS frontier so that the point of contract is B. to achieve output technical efficiency, further output augmentation is necessary. Potential output is $\theta^c u_0$. And increase in the output is $\theta^c u_0 - u_0$.

To access output technical efficiency the point A is compared with B on the frontier.

$$OTE = \frac{\|u^B\|}{\|u^A\|} = \frac{\|(Max\ \theta)\ u^A\|}{\|u^A\|} = \frac{(Max\ \theta)\|\ u^A\|}{\|u^A\|} = Max\ \theta$$

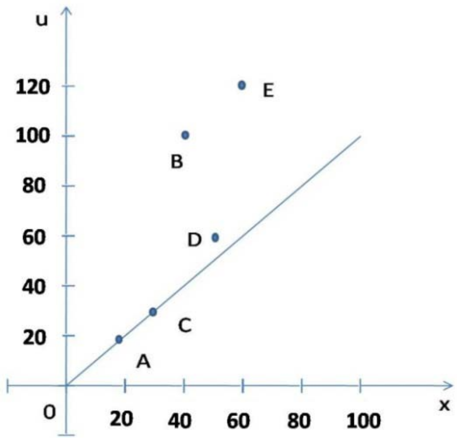
The technical efficiency accessed in this manner is overall output technical efficiency which can be decomposed in to the product of output pure technical and sale efficiencies.

NUMERICAL EXAMPLE

Given below are inputs and output of the six production units:

Unit	X	U
A	20	20
B	40	100
C	30	30
D	50	60
E	60	120

Assuming constant returns to the scale frontier is true frontier compute input and output over all technical efficiencies



The production unit B determines the CRTS frontier.

It is both overall input and output technical efficient.

The equation of the CRTS frontier is $u = 2.5\ x$

$\boldsymbol{A}(\mathbf{20},\mathbf{20})$

$$20 = 2.5\,x^*$$
$$x^* = \frac{20}{2.5} = 8$$
$$x_0 = 20$$
$$Input\ Technical\ Efficiency = \frac{x^*}{x_0} = \frac{8}{20} = 0.4$$
$$u^* = 2.5 * 20 = 50$$
$$Output\ Technical\ Efficiency = \frac{u^*}{u_0} = \frac{50}{20} = 2.5$$

$\boldsymbol{B}(\mathbf{40},\mathbf{100})$

$$100 = 2.5\,x^*$$
$$x^* = \frac{100}{2.5} = 40$$
$$x_0 = 40$$
$$Input\ Technical\ Efficiency = \frac{x^*}{x_0} = \frac{40}{40} = 1$$
$$u^* = 2.5 * 40 = 100$$
$$Output\ Technical\ Efficiency = \frac{u^*}{u_0} = \frac{100}{100} = 1$$

$\boldsymbol{C}(\mathbf{30},\mathbf{30})$

$$30 = 2.5\,x^*$$
$$x^* = \frac{30}{2.5} = 12$$
$$x_0 = 30$$
$$Input\ Technical\ Efficiency = \frac{x^*}{x_0} = \frac{12}{30} = 0.4$$
$$u^* = 2.5 * 30 = 75$$
$$Output\ Technical\ Efficiency = \frac{u^*}{u_0} = \frac{75}{30} = 2.5$$

$\boldsymbol{D}(\mathbf{50},\mathbf{60})$

$$60 = 2.5\,x^*$$
$$x^* = \frac{60}{2.5} = 24$$
$$x_0 = 25$$
$$Input\ Technical\ Efficiency = \frac{x^*}{x_0} = \frac{24}{50} = 0.48$$
$$u^* = 2.5 * 60 = 150$$
$$Output\ Technical\ Efficiency = \frac{u^*}{u_0} = \frac{150}{60} = 2.5$$

$\boldsymbol{E}(\mathbf{60},\mathbf{120})$

$$120 = 2.5\,x^*$$
$$x^* = \frac{120}{2.5} = 48$$
$$x_0 = 60$$
$$Input\ Technical\ Efficiency = \frac{x^*}{x_0} = \frac{48}{60} = 0.80$$
$$u^* = 2.5 * 60 = 150$$
$$Output\ Technical\ Efficiency = \frac{u^*}{u_0} = \frac{150}{120} = 1.25$$

CONCLUSIONS

1. Production unit A is input and output technical efficient. 60 percent of inputs are freely disposed. 60 percent of potential output is lost due to output technical inefficiency.
$$\left(1 - \frac{1}{2.5}\right) * 100 = 60\%$$
2. Production unit B is both input and output technical efficient.

3. Production unit C is technical inefficient. 60 percent of inputs and potential output are lost due to technical inefficiency.
4. 52 percent of inputs are lost due to technical inefficiency, 60 percent of potential output is lost due to output technical inefficiency.
5. The output technical inefficiency is 0.80. 20 percent of inputs are lost due to input technical inefficiency. Same amount of potential output is lost due to output technical inefficiency.

OUTPUT TECHNICAL EFFICIENCY

PIECE WISE LINEAR TECHNOLOGY - OUTPUT EFFICIENCY

The chief tool to measure the output efficiency is production function. A piece wise linear production function is linear approximation of a continuous once differentiable production frontier. Piece wise linear technology is axiomatic, based on the following axioms.

CONVEXITY

Let $T = \{(x, u); x\ produces\ u\}$ be a production possibility set.

$(x_i, u_i)\ \varepsilon\ T, \qquad i = 1,2,3, \dots, k$

$$=> \left(\sum_{i=1}^{k} \lambda_i x_i\,,\ \sum_{i=1}^{k} \lambda_i u_i\right)\ \varepsilon\ T$$

Where $\lambda_1 \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$

INEFFICIENCY

$(\bar{x}, \bar{u})\ \varepsilon\ T\ => (x, u)\ \varepsilon\ T$

Where $x \geq \bar{x}, \quad u \leq \bar{u}$

$(\bar{x}, \bar{u})\ \varepsilon\ T\ =>$ There exists λ_1, such that

$(\bar{x}, \bar{u}) = (\sum \lambda_i x_i, \sum \lambda_i u_i),\ \sum \lambda_i = 1,\ \lambda_i \geq 0$

$\sum \lambda_i x_i \leq x$

$\sum \lambda_i u_i \geq u$

$\lambda_i \geq 0$

$\sum \lambda_i = 1$

MINIMUM EXTRACTION

T is the intersection of all T_α

$T = \cap_\alpha T_\alpha$

$T = \{(x, u): \sum \lambda_i x_i \leq x\ \ , \sum \lambda_i u_i \geq u\ ,\ \lambda_i \geq 0, \sum \lambda_i = 1\}$

The production possibility set T admits variable returns to scale.

RAY EXPANSION

$(\bar{x}, \bar{u})\ \varepsilon\ T => \lambda(\bar{x}, \bar{u})\ \varepsilon\ T,\ \lambda > 0$

$(\lambda\bar{x}, \lambda\bar{u}) = (\sum \lambda\lambda_i x_i, \sum \lambda\lambda_i u_i) \epsilon\ T$

$\sum \lambda\lambda_i x_i \leq x$

$\sum \lambda\lambda_i u_i \geq u$

$\sum \delta_i x_i \leq x$

$\sum \delta_i u_i \geq u$

$\delta_i \geq 0$

$T = \{(x,u): \sum \delta_i x_i \leq x \ , \sum \delta_i u_i \geq u \ , \ \delta_i \geq 0\}$

The production possibility set T admits constant returns to scale.

OUTPUT PURE TECHNICAL EFFICIENCY

OPTE = Max θ

Such that

$\sum \lambda_i x_i \leq x_0$

$\sum \lambda_i u_i \geq u_0$

$\lambda_i \geq 0$

$\sum \lambda_i = 1$

Where x_i , u_i are the input and output vectors of i^{th} production unit, x_0 , u_0 are the input and output vectors of the production unit whose efficiency is under evaluation.

OUTPUT OVERALL TECHNICAL EFFICIENCY

OOTE = Max θ

Such that

$\sum \lambda_i x_i \leq x_0$

$\sum \lambda_i u_i \geq \theta u_0$

$\lambda_i \geq 0$

OUTPUT SCALE EFFICIENCY

It is a derived measure

$OSE = \frac{OOTE}{OPE}$

OVERALL OUTPUT PRODUCTIVE EFFICIENCY

To estimate overall output productive efficiency, potential revenue has to be estimated. Potential revenue can be obtained by solving the following linear programming problem.

$R(x_0, r_0) = Max\ r_0 u$

Such that

$\sum \lambda_i x_i \leq x_0$

$\sum \lambda_i u_i \geq \theta u_0$

$\lambda_i \geq 0$

OUTPUT OVERALL EFFICIENCY

$$OOE = \frac{R(x_0, r_0)}{r_0 u_0}$$

Where $r_0 u_0$ is observed revenue and $R(x_0, r_0)$ is potential revenue.

OUTPUT ALLOCATIVE EFFICIENCY

Output allocative efficiency is a derived measure

$$OAE = \frac{OOE}{OOTE}$$

NUMERICAL EXAMPLE

Given below are the inputs and outputs of five production units

UNIT	X_1	X_2	U_1	U_2
A	10	10	10	10
B	20	25	20	15
C	30	20	25	10
D	35	10	10	12
E	40	25	20	20

OUTPUT PURE TECHNICAL EFFICIENCY

$$\delta_1^D = Max\ \theta$$

Subject to

$$10\lambda_1 + 20\lambda_2 + 30\lambda_3 + 35\lambda_4 + 40\lambda_5 \leq 35$$

$$10\lambda_1 + 25\lambda_2 + 20\lambda_3 + 35\lambda_4 + 25\lambda_5 \leq 35$$

$$10\lambda_1 + 20\lambda_2 + 25\lambda_3 + 10\lambda_4 + 20\lambda_5 \leq 10\theta$$

$$40\lambda_1 + 60\lambda_2 + 70\lambda_3 + 50\lambda_4 + 90\lambda_5 \leq 20\theta$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1$$

$$\lambda_i \geq 0$$

Optimal solution of this linear programming problem gives output pure technical efficiency of the production unit D.

OUTPUT PURE TECHNICAL EFFICIENCY

$$\delta_1^A = 1.5625$$

OUTPUT OVERALL TECHNICAL EFFICIENCY

To find output overall technical efficiency of the production unit D. we solve,

$$\theta^D = Max\ \theta$$

Such that

$$10\lambda_1 + 20\lambda_2 + 30\lambda_3 + 35\lambda_4 + 40\lambda_5 \leq 35$$

$$10\lambda_1 + 25\lambda_2 + 20\lambda_3 + 35\lambda_4 + 25\lambda_5 \leq 35$$

$$10\lambda_1 + 20\lambda_2 + 25\lambda_3 + 10\lambda_4 + 20\lambda_5 \leq 10\theta$$

$$10\lambda_1 + 15\lambda_2 + 10\lambda_3 + 12\lambda_4 + 20\lambda_5 \leq 20\theta$$

$$\lambda_i \geq 0$$

Solving LPP we obtain,

$$\delta^D = 2.9166$$

OUTPUT SCALE EFFICIENCY

Output scale efficiency is derived measure.

$$\delta = \delta_1 . \delta_2$$

Where δ_2 is output scale efficiency measure.

$$\delta_2 = \frac{\delta}{\delta_1}$$

$$\delta_2^D = \frac{\delta^D}{\delta_1^D} = \frac{2.9166}{1.5625} = 1.8666$$

INFERENCE

δ^D, δ_1^D *And* δ_2^D Values revels that the production unit D experiences heavy output losses due to pure technical, scale and hence overall technical efficiencies

OVERALL PRODUCTIVE EFFICIENCY

To compute the overall productive efficiency, primarily, positional revenue has to be estimated, by solving the following linear programming problem of production unit D.

$$\pi = Max\ (0.35\ u_1 + 0.35\ u_2)$$

Such that

$$10\lambda_1 + 20\lambda_2 + 30\lambda_3 + 35\lambda_4 + 40\lambda_5 \leq 35$$

$$10\lambda_1 + 25\lambda_2 + 20\lambda_3 + 35\lambda_4 + 25\lambda_5 \leq 35$$

$$10\lambda_1 + 20\lambda_2 + 25\lambda_3 + 10\lambda_4 + 20\lambda_5 - u_1 \geq 0$$

$$10\lambda_1 + 15\lambda_2 + 10\lambda_3 + 12\lambda_4 + 20\lambda_5 - u_2 \geq 0$$

$$\lambda_i \geq 0$$

Where 0.35 and 0.35 are unit selling prices of first and second outputs.

The optimal solution of this LPP gives potential revenue estimated as, $\pi = 35$,

Which is measured in some monetary units? The observed revenue is

$$R = 0.35 * 10 + 0.35 * 12$$

$$R = 3.5 + 4.2$$

$$R = 7.7$$

OVERALL PRODUCTIVE EFFICIENCY

$$OPE = \frac{35}{7.7} = 4.5454$$

ALLOCATIVE EFFICIENCY

This is another derived measure, defined as,

$$OPE = OTE * AE$$

$$AE = \frac{OPE}{OTE}$$

For the production unit D allocative efficiency is,

$$AE = \frac{4.5454}{2.9166} = 1.5584$$

SUMMARY AND CONCLUSIONS

This study is aimed at presenting inefficiency problem of managerial economics on simple graphical terms first and extending them onto analytical grounds to measure various output productive effectiveness such as output pure technical, scale, overall technical, overall productive and scale efficiencies. Numerical illustrations are given to have clear insight into the problem of inefficiency and its measurement.

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