International Journal of Mathematical Archive-5(4), 2014, 198-206
Available online through www.ijma.info ISSN 2229 – 5046

Approximation of Unbounded Function by q-Bernstein-Kantorovich Operator in Locally-Global Weighted $L_{P,w}(X)$-Space

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(Received on: 29-03-14; Revised & Accepted on: 13-04-14)

ABSTRACT

The aim of this paper is to study the approximation of unbounded function by q-Bernistein-Kantorovich operator in locally-global weighted-space $L_{P,w}(X)$ (1≤ $P < \infty$) using the weighted Ditzain-Totik modulus of smoothness.

INTRODUCTION

In the two decades interesting generalization of Bernisteinpolymails were proposed by Lupas [1] and by Phillips [2]. Generalizations of the Bernistein polynomials based on the q-integers was attracted alot of interest and was studied widely by a number of authors [3].

Recentely, some new generalizations of well-known positive linear operators, based on q-integers were introduced and studied by several authors [4, 5]. The purpose of this paper is to study approximation of unbounded functions by q-Bernstein-Kantorovich operators.

Before proceding to the study of second order of approximation by q-Bernistein-Kantorovich operators, it is necessary to know some definitions.

Definition: 1 Let X= [0, 1], $L_{P,w}(X)$ be the space of all unbounded continuous functions *f*(1≤ $P < \infty$), which are equipped with the following norm.

$\|f\|_{P,w} = \left[\left|\int_X (fw)_{(x)}\right|^P dx\right]^{\frac{1}{P}}$ where (*fw*) is continuous on [0, 1] and w is a positive weighted function, 0<w(x)<1.

Definition: 2 For $f \in L_{P,w}(X), X = [0,1], 1 \le P < \infty$ let

$\|f\|_{P,\delta,w} = \left[\left(\int_X (Sup\{|f(u)w(u)|: u \in N(x,\delta)\})\right)^P dx\right]^{\frac{1}{P}}$ is the locally global norms of f

where$N(x,\delta) = \{y \in X: |x-y| \le \delta\}, \delta \in R^+, x \in X$

Then we denote

$$L_{P,\delta,w}(X) = \{f: \|f\|_{P,\delta,w} < \infty\}$$

Definition: 3 [8] For q>0 and $n \in N$ let $[n] = [n]_q = q^0 + q^1 + q^2 + \cdots + q^{n-1}$ with [0] = 0 be the q-integer [n]. And the q-fractional [n]! is defined by $[n]! = [n]_q! = [1]_. + [2]_. \ldots\ .[n]$ with [0]!=1 and for integers $0 \le k < n$ then $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ Be the q-binomial coefficient

Definition: 4 [6] For $f \in c[0,1]$ let $\int_0^A f(t) d_q t = A(1-q)\sum_{n=0}^{\infty} f_{(Aq^n)}\,.\,q^n$ (0<q<1) be the q-analouge of integration in the interval [0, A] and $B^*_{n,q}(f,x) = \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 f((\frac{[k]+q^k t}{[n+1]})\, d_q t$ be the modificatedKantrorovich type q-Bernisteinpolynimail,

where $P_{n,k}(q,x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k}$, $(1-x)_q^n = \prod_{s=0}^{n}(1-q^s x)$, $0 \le x \le 1, n \in N$.

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Definition: 5 For $f \in L_{P,w}(X)$, $X = [0,1]$,0<q<1 let

$$\int_0^1 (fw)_{(t)}\, d_q t = (1-q)\sum_{n=0}^{\infty}(fw)_{(q^n)} \,.\, q^n$$

be the q-analogue of integration in the interval [0, 1] and $B^*_{n,q}(f,x) = \sum_{k=0}^{n} P_{n,k}\,(q,x)\int_0^1 (fw)\,(\frac{[k]+q^k t}{[n+1]})d_q t$ be the modificated Kantorovich type q-Bernstein polynomial,
where $P_{n,k}(q,x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k}$, $(1-x)_q^{n-k} = \prod_{s=0}^{n-k-1}(1-q^s x)$, $n \in N$

Remark: 1 For $q \to 1$ then
$B^*_{n,q}(f,x) = B^*_n(f,x) = \sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k}\int_0^1 (f\,w)(\frac{K+t}{n+1})\,dt$, which is say the classical Kantorovich operator n=1, 2 …

Proof: For $q \to 1$ we have

$[n]_q = [n] = q^0 + q^1 + \cdots + q^{n-1} = 1 + 1_{+n-time} + \cdots 1 = n,$

Thus
[n]=n and we get [n]! =[1] · [2] · … · $[n] = 1 \cdot 2 \cdot \ldots \cdot n = n!$,

Thus
$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ and $P_{n,k}(q,x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k} = \binom{n}{k} x^k(1-x)^{n-k}$

Then

$$B^*_{n,k}(f,x) = \sum_{k=0}^{n} P_{n,k}\,(q,x)\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right)d_q t$$

$$= \sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k}\int_0^1 (fw)\left(\frac{K+t}{n+1}\right)dt = B^*_n(f,x)$$

Thus
$B^*_{n,q}(f,x) = B^*_n(f,x)$

Definition: 6 For $f \in L_{P,w}(X), X = [0,1]$, let $\Delta_h^2(f,x) = \sum_{i=0}^{2}\binom{2}{i}(-1)_{2.i}\,f(x-h+ih), x \pm h \in X$ be the difference of second order of *f*

Definition: 7 For $f \in L_{p,\delta,w}(X)$ let $\omega_2(f,\delta)_{P,w} = Sup_{0<P\leq\delta}\|\Delta_h^2(f,x)\|_{P,w}, \delta \geq 0$ is called the second usual weighted modulus of smoothness of *f* and $\omega_2(f,\delta)_{P,\delta,w} = Sup\|\Delta_h^2(fw,x)\|_{P,\delta,w}$ is the locally second usual weighted modulus of smoothness of *f*.

Definition: 8 For $f \in L_{P,\delta,w}(X), X = [0,1], \delta \geq 0$ then $K_2(f,\delta^2)_{P,\delta,w} = inf_{g\in L_{P,w}(X)}\{\|f-g\|_{P,\delta,w} + \delta^2\|g''\|_{P,\delta,w}\}$ is the locally K-functional of second order of *f*, where $g \in L_{P,w}(X)$ such that

$(gw) \in C^2[0,1] = \{(gw),(gw)^{'},(gw)^{''} \in C[0,1]\}$

Remark: 2 [11] For $f \in L_P(X) = \left\{f : \|f\|_P = \left[\int_X |f(x)|^P dx\right]^{\frac{1}{P}} < \infty\right\}$ then $\|f\|_P \leq \|f\|_{P,\delta}$

Remark: 3 For $f \in L_{P,\delta,w}(X), X = [0,1], \delta \geq 0$ then $K_2(f,\delta^2)_{P,\delta,w} \leq C\omega_2(f,\delta)_{P,\delta,w}$ where C is a constant.

Proof: For $f, g \in C^2[0,1]$ we get

$K_2(f,\delta^2)_P = inf_{g\in C^2[0,1]}\{\|f-g\|_P + \delta^2\|g^{''}\|_P\} \leq C\omega_2(f,\delta)_P$ by [7] For $f,g \in L_{P,\delta,w}(X)$ and by definition(1) we have $(fw),(gw) \in C^2[0,1]$ thus

$K_2(fw,\delta^2)_P = inf_{(gw)\in C^2[0,1]}\{\|fw-gw\|_P + \delta^2\|(gw)^{''}\|_P\} \leq C\omega_2(fw,\delta)_P$

Then by remark (2) we have

$$K_2(fw,\delta^2)_{P,\delta} = inf_{(gw)\in C^2[0,1]}\{\|fw-gw\|_{P,\delta} + \delta^2\|(gw)^{''}\|_{P,\delta}\} \leq \omega_2(fw,\delta)_{P,\delta}$$

We get $K_2(f,\delta^2)_{P,\delta,w} = inf_{g\in L_{P,\delta,w}(x)}\{\|f-g\|_{P,\delta,w} + \delta^2\|g''\|_{P,\delta,w}\} \le C\omega_2(f,\delta)_{P,\delta,w}$

then $K_2(f,\delta^2)_{P,\delta,w} \le C\omega_2(f,\delta)_{P,\delta,w}$

Remark: 4 For $f \in L_{P,w}(X). X = [0,1], 1 \le P < \infty$ then

$$\|f\|_{P,w} \le \|f\|_{P,\delta,w}$$

Proof: By remark (2) and since $(fw) \in C^2[0,1]$ by definition (1) then $\|f\|_P \le \|f\|_{P,\delta}$ we get

$$\|f\|_{P,w} \le \|f\|_{P,\delta,w}$$

Definition: 9 For $f \in L_{P,w}(X), X = [0,1], \delta \ge 0$ then $\omega_2^{\phi}(f,\delta)_{P,w} = Sup_{0<P\le\delta}\left\|\Delta_{h\phi}^2(fw,x)\right\|_{P,w}$ be the Ditzian-Totik modulus of smoothness of second order of *f*,

where $\phi(x) = (x(1-x))^{1/2}$ and $\omega_2^{\phi}(f,\delta)_{P,\delta,w} = Sup_{0<P\le\delta}\left\|\Delta_{h\phi}^2(fw,x)\right\|_{P,\delta,w}$ is the locally Ditzian-Totik modulus of smoothness of second order of *f*.

Definition: 10 Let $f \in L_{P,w}(X), X = [0,1], \delta \ge 0$ then $\omega_{2,r}^{\phi}(f,\delta)_{P,w} = Sup_{0<P\le\delta}\left\|\phi^r\Delta_{h\phi}^2(f,x)\right\|_{P,w}$ is called the r-th-weighted Ditzian-Totik modulus of second order of smoothness of *f*, where r is a non negative integer. And $\omega_2^{\phi}(f,\delta)_{P,\delta,w} = Sup_{0<P\le\delta}\left\|\phi^r\Delta_{h\phi}^2(fw,x)\right\|_{P,\delta,w}$ is the r-th locally weighted Ditzian-Totik modulus of smoothness of second order of *f*.

AUXILLARY RESULLTS

To prove the main results, we shall need the following lemmas.

Lemma: 1 [8] For all $n \in N, x \in [0,1], 0 < q \le 1$ then

$$B_{n,q}^*(t^m,x) = \sum_{j=0}^{m}\binom{m}{j}\frac{[n]^j}{[n+1]^m[m-j+1]}\sum_{i=0}^{m-j}\binom{m-j}{i}(q^n-1)^i B_{n,q}(t^{j+i},x)$$

where $B_{n,q}(t^{j+i},x) = \sum_{k=0}^{n}\frac{[k]^{j+i}}{[n]^{j+i}}P_{n,k}(q,x)$

Lemma: 2 Let $e_i(t) = t^i, i = 0,1,2$ for every $t \in [0,1], n \in N, 0 < q \le 1$ we have:

a) $B_{n,q}^*(e_0,x) = 1$

b) $B_{n,q}^*(e_1,x) = \frac{2q[n]}{[2][n+1]}x + \frac{1}{[2][n+1]}$

c) $B_{n,q}^*(e_2,x) = \frac{q(q+2)}{[3]}\cdot\frac{q[n][n+1]}{[n+1]^2}x^2 + \frac{4q+7q^2+q^3}{[2][3]}\cdot\frac{[n]}{[n+1]^2}x + \frac{1}{[3][n+1]^2}$

Proof:

A) By using lemma (1) we have

$B_{n,q}^*(t^m,x) = \sum_{j=0}^{m}\binom{m}{j}\frac{[n]^j}{[n+1]^m[m-j+1]}\sum_{i=0}^{m-j}\binom{m-j}{i}(q^n-1)^i B_{n,q}(t^{j+i},x)$ then for m = 0 we get

$$B_{n,q}^*(t^0,x) = B_{n,q}^*(e_0,x) = \sum_{j=0}^{0}\binom{0}{j}\frac{[n]^j}{[n+1]^0[0-j+1]}\sum_{i=0}^{0-j}\binom{0-j}{i}(q^n-1)^i B_{n,q}(t^{j+i},x)$$

$$= \binom{0}{0}\frac{[n]^0}{[n+1]^0[1]}\sum_{i=0}^{0}\binom{0}{i}(q^n-1)^i B_{n,q}(t^{0+i},x)$$

$$= \binom{0}{0}(q^n-1)^0 B_{n,q}(t^{0+0},x) = \sum_{k=0}^{n}\frac{[k]^{0+0}}{[n]}P_{n,k}(q,x)$$

$$= \sum_{k=0}^{n}P_{n,k}(q,x) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k(1-x)_q^{n-k}$$

$$= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}(1-q^s x) = 1$$

Then $B_{n,q}^*(e_0,x) = 1$

Proof:

B) By using lemma (1) for m=1 we have

$$B^*_{n,q}(t^1,x) = B^*_{n,q}(e_1,x) = \sum_{j=0}^{1}\binom{1}{j}\frac{[n]^j}{[n+1]\ \ [1-j+1]}\sum_{i=0}^{1-j}\binom{1-j}{i}(q^n-1)^i B_{n,q}(t^{j+i},x)$$

$$= \left[\binom{1}{0}\frac{[n]^0}{[n+1]\ \ [2]}\sum_{i=0}^{1}\binom{1}{i}(q^n-1)^i B_{n,q}(t^{0+i},x)\right]$$

$$= \left[\binom{1}{1}\frac{[n]}{[n+1]\ \ [1]}\sum_{i=0}^{0}\binom{0}{i}(q^n-1)^i B_{n,q}(t^{1+i},x)\right]$$

$$= \frac{1}{[n+1]\ \ [2]}\left[\binom{1}{0}(q^n-1)^0 B_{n,q}(t^{0+0},x) + \binom{1}{1}(q^n-1)B_{n,q}(t^{0+1},x)\right] + \frac{n}{[n+1]}\binom{0}{0}(q^n-1)^0 B_{n,q}(t^{1+0},x)$$

$$= \frac{1}{[n+1]\ \ [2]}\left[B_{n,q}(t^{0+0},x) + (q^n-1)B_{n,q}(t^{0+1},x)\right] + \frac{[n]}{[n+1]}B_{n,q}(t^{1+0},x)$$

$$= \frac{1}{[n+1]\ \ [2]}\left[\sum_{k=0}^{n}P_{n,k}(q,x) + (q^n-1)\quad \sum_{k=0}^{n}\frac{[k]}{[n]}P_{n,k}(q,x)\right] + \frac{[n]}{[n+1]}\sum_{k=0}^{n}\frac{[k]}{[n]}P_{n,k}(q,x)$$

$$= \frac{1}{[2]\ \ [n+1]} + \left[\frac{q^n-1}{[2][n+1]} + \frac{[n]}{[n+1]}\right]x = \frac{2q[n]}{[2][n+1]}x + \frac{1}{[2][n+1]}$$

Proof:

C) Also by using lemma (1) for m=2 we get

$$B^*_{n,q}(t^2,x) = B^*_{n,q}(e_2,x) = \sum_{j=0}^{2}\binom{2}{j}\frac{[n]^j}{[n+1]^2[2-j+1]}\sum_{i=0}^{2-j}\binom{2-j}{i}(q^n-1)^i B_{n,q}(t^{j+i},x)$$

$$= \binom{2}{0}\frac{[n]^0}{[n+1]^2[2-0+1]}\sum_{i=0}^{2-0}\binom{2-0}{i}(q^n-1)^i B_{n,q}(t^{0+i},x) + \binom{2}{1}\frac{[n]^1}{[n+1]^2[2-1+1]}\sum_{i=0}^{2-1}\binom{2-1}{i}(q^n-1)^i B_{n,q}(t^{1+i},x) + \binom{2}{2}\frac{[n]^2}{[n+1]^2[2-2+1]}\sum_{i=0}^{2-2}\binom{2-2}{i}(q^n-1)^i B_{n,q}(t^{2+i},x)$$

$$= \frac{1}{[n+1]^2[3]}\left[\binom{2}{0}(q^n-1)^0 B_{n,q}(t^{0+0},x) + \binom{2}{1}(q^n-1)^1 B_{n,q}(t^{0+1},x)\right] + \binom{2}{2}(q^n-1)^2 B_{n,q}(t^{0+2},x) + \frac{2[n]}{[n+1]^2[2]}\left[\binom{1}{0}(q^n-1)^0 B_{n,q}(t^{1+0},x) + \binom{1}{1}(q^n-1)^1 B_{n,q}(t^{1+1},x)\right] + \frac{[n]^2}{[n+1]^2}\left[\binom{0}{0}(q^n-1)^0 B_{n,q}(t^{2+0},x)\right]$$

$$= \frac{1}{[n+1]^2[3]}\left[B_{n,q}(t^{0+0},x) + 2(q^n-1)\quad B_{n,q}(t^{0+1},x) + (q^n-1)^2 B_{n,q}(t^{0+2},x)\right] + \frac{2[n]}{[n+1]^2[2]}\left[B_{n,q}(t^{1+0},x)\quad + (q^n-1)B_{n,q}(t^{1+1},x)\right] + \frac{[n]^2}{[n+1]^2}B_{n,q}(t^{2+0},x)$$

$$= \frac{1}{[3][n+1]^2}\left[\frac{[n]^2}{[n+1]^2} + \frac{2[n](q^n-1)}{[2][n+1]^2} + \frac{(q^n-1)^2}{[3][n+1]^2}\right]\left(1-\frac{1}{[n]}\right)x^2 + \left[\frac{[n]^2}{[n][n+1]^2} + \frac{2[n](q^n-1)}{[2][n][n+1]^2} + \frac{(q^n-1)^2}{[3][n][n+1]^2} + \frac{2[n]}{[2][n+1]^2} + \frac{2(q^n-1)}{[3][n+1]^2}\right]x$$

$$= \frac{q^2+2q}{[3]}\cdot\frac{q[n]\ \ [n-1]}{[n+1]^2}x^2 + \frac{(4q+7q^2+q^3)[n]}{[2][3][n+1]^2}x + \frac{1}{[3][n+1]^2}$$

Then

$$B^*_{n,q}(e_2,x) = \frac{q^2+2q}{[3]}\cdot\frac{q[n][n-1]}{[n+1]^2}x^2 + \frac{(4q+7q^2+q^3)[n]}{[2][3][n+1]^2}x + \frac{1}{[3][n+1]^2}$$

Lemma: 4 For $f \in L_{P,\delta,w}(X)\cdot X = [0,1], 0 < q < 1$ we have $\left\|B^*_{n,q}(f)\right\|_{P,w} \le C\|f\|_{P,\delta,w}$ where C is a constant.

Proof: $\left\|B^*_{n,q}(f)\right\|_{P,\delta,w} = \left\{\int_X \left|B^*_{n,q}(f,x)\right|^P dx\right\}^{1/P}$

$$= \left\{\int_X \left|\sum_{k=0}^{n}P_{n,q}(q,x)\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right)d_q t\right|^P dx\right\}^{1/P}$$

And by using Jenson inquality we have

$$\left\|B_{n,q}^{*}(f)\right\|_{P,w} \leq \left\{\int_0^1 \left|(fw)\left(\frac{[k]+q^k t}{[n+1]}\right)\right|^P d_q t\right\}^{1/P} \cdot \left\{\int_X \left|\sum_{k=0}^{n} P_{n,k}(q,x)\right| dx\right\}$$

$$\leq \|f\|_{P,w} \cdot \sum_{k=0}^{n} \int_X \left|P_{n,k}(q,x)\right| dx \leq C\|f\|_{P,w}$$

Then
$\left\|B_{n,q}^{*}(f)\right\|_{P,w} \leq C\|f\|_{P,w}$ and by remark (4) we get

$\left\|B_{n,q}^{*}(f)\right\|_{P,\delta,w} \leq C\|f\|_{P,\delta,w}$

Lemma: 5 For $f, g \in L_{P,w}(X), X = [0,1], 0 < q \leq 1$ we have

a) $B_{n,q}^{*}(f+g,x) = B_{n,q}^{*}(f,x) + B_{n,q}^{*}(g,x)$

b) $B_{n,q}^{*}(\propto f,x) = \propto B_{n,q}^{*}(f,x)$ where $\propto$ is a constant

Proof:

(A) Since $B_{n,q}^{*}(f,x) = \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q\, t$ then

$$B_{n,q}^{*}(f+g,x) = \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 ((f+g)w)\left(\frac{[k]+q^t t}{[n+1]}\right) d_q t$$

$$= \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 \left[(fw)\left(\frac{[k]+q^k t}{[n+1]}\right) + (gw)\left(\frac{[k]q^k t}{[n+1]}\right)\right] d_q t$$

$$= \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t + \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 (gw)\left(\frac{[k]q^k t}{[n+1]}\right) d_q t$$

$$= B_{n,q}^{*}(f,x) + B_{n,q}^{*}(g,x)$$

Proof:

(B) $B_{n,q}^{*}(\propto f,x) = \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 (\propto f)w\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t$

$$= \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 \propto (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t$$

$$= \propto \sum_{k=0}^{n} P_{n,k}(q,x)\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t = \alpha B_{n,q}^{*}(f,x)$$

Then$B_{n,q}^{*}(\propto f,x) = \propto B_{n,q}^{*}(f,x)$

Thus $B_{n,q}^{*}$ be linear operator

Remark: 5 For every $n \in N, 0 < q < 1$ then
$[n]_q = [n] \leq n$

Proof: by induction for n we have
$[n] = q^0 + q^1 + \cdots + q^{n-1}$ then

For n=1, $[1] = q^{1-1} = q^0 = 1$ thus [1]= 1

For n=2, $[2] = q^0 + q^{2-1} = q^0 + q^1 = 1 + q^1 < 2$ (0<q<1) then [2]<2

Suppose it is true for n thus
$[n] = q^0 + q^1 + \cdots + q^{n-1} < n$ then

$[n+1] = q^0 + q^1 + \cdots + q^{(n+1)-2} + q^{(n+1)-1} = q^0 + q^1 + \cdots + q^{n-1} + q^n$

Then $[n+1] = q^0 + q^1 + \cdots + q^{n-1} + q^n < n + q^n < n+1$

Thus $[n+1] < n+1$

Then $[n] \leq n$ for every n ∈ N

Lemma: 6 Let $f \in L_{P,\delta,w}(X)$ and g_n be a polynomial such that $g_n \in \prod_n \cap L_{P,w}(X)$ where $\prod_n$ set of all algebraic polynomials, then

$$\|f - g_n\|_{P,\delta,w} \leq Cn^{\frac{1}{P}} \quad \omega_{2,r}^{\phi}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$$

Proof: by using remark (5) we have for all $n \in N$, $[n] \leq n$then $\frac{1}{n} \leq \frac{1}{[n]}$ we get

$$\omega_{2,r}^{\phi}\left(f, \frac{1}{n}\right)_{P,\delta,w} \leq \quad \omega_{2,r}^{\phi}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$$

Since $\|f - g_n\|_{P,\delta,w} \leq Cn^{\frac{1}{P}} \quad \omega_{2,r}^{\phi}\left(f, \frac{1}{n}\right)_{P,\delta,w}$ by [9]

We get $\|f - g_n\|_{P,\delta,w} \leq Cn^{\frac{1}{P}} \quad \omega_{2,r}^{\phi}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$

Lemma: 7 For $f, g \in L_{P,w}(X), X = [0,1] \cdot n \in N, 0 < q < 1$ we have

a. $f \geq 0$ then $B_{n,q}^*(f,x) \geq 0$

b. $f \leq g$ then $B_{n,q}^*(f,x) \leq B_{n,q}^*(g,x)$

Proof: A) $f \geq 0$ then $f\left(\frac{[k]+q^k t}{[n+1]}\right) \geq 0$ for $k \in N$ we have

$\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t \geq 0$ thus

$\sum_{k=0}^{n} P_{n,k}(q,x) \int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t \geq 0$ we get

$B_{n,q}^*(f,x) \geq 0$

Proof: B) Since $f \leq g$ thus $f\left(\frac{[k]+q^k t}{[n+1]}\right) \leq g\left(\frac{[k]+q^k t}{[n+1]}\right)$ then

$\int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t \leq \int_0^1 (gw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t$ we get

$$= \sum_{k=0}^{n} P_{n,k}(q,x) \int_0^1 (fw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t$$

$$\leq \sum_{k=0}^{n} P_{n,k}(q,x) \int_0^1 (gw)\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t$$

We have $B_{n,q}^*(f,x) \leq B_{n,q}^*(g,x)$

Now we need the following theorem:

Theorem: 1 [Korevkin Theorem] [10] Let L_n be a linear positive monotone operator such that

1. $L_n(1, x) = 1$
2. $L_n(t, x) = x + \propto(x)$
3. $L_n(t^2, x) = x^2 + B(x)$

Then for any $f \in C[a,b]$

$$\left\|L_n(f,.) - f(.)\right\|_P \leq 3W < \left(f, \sqrt{B(x) - 2x \propto (x)}\right)_P$$

Lemma: 8 Let L_n be a linear positive monotone operator, which satisfies the above conditions then for any $f \in L_{P,\delta,w}(X), X = [0,1]$ we have

$$\|L_n(f,.) - f(.)\|_{P,\delta,w} \leq 3\omega_2\left(f, \sqrt{B(x) - 2x \propto (x)}\right)_{P,\delta,w}$$

Proof: for $f \in C[0,1]$, by using theorm (1) we get

$$\|L_n(f,.) - f(.)\|_P \le 3\omega \left(f, \sqrt{B(x) - 2x \propto (x)}\right)_P \le 3W_2\left(f, \sqrt{B(x) - 2x \propto (x)}\right)_P$$

Thus

$\|L_n(f,.) - f(.)\|_P \le 3\omega_2\left(f, \sqrt{B(x) - 2x \propto (x)}\right)_P$

For $f \in L_{P,w}(X)$ and from definition (1) we get$(fw) \in C[0,1]$ and $\|f\|_{P,w} = \left\{\left|(fw)_{(x)}\right|^P dx\right\}^{\frac{1}{P}} < \infty$

Then

$\|L_n(fw,.) - (fw)(.)\ \ \|_P \le 3\omega_2\left(fw, \sqrt{B(x) - 2x \propto (x)}\right)_P$ and by remark (2) We have

$\|L_n(fw) - (fw)\|_{P,\delta} \le 3\omega_2\left(fw, \sqrt{B(x) - 2x \propto (x)}\right)_{P,\delta}$

Thus

$\|L_n(f,.) - f(.)\|_{P,\delta,w} \le 3\omega_2\left(f, \sqrt{B(x) - 2x \propto (x)}\right)_{P,\delta,w}$

MAIN RESULTS

Theorem: 2 For $f \in L_{P,\delta,w}(X), X = [0,1], n \in N, 0 < q < 1$ we have

$$\left\|B^*_{n,q}(f,x) - f(x)\right\|_{P,\delta,w} \le C \quad \omega^{\phi}_{2,r}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$$

Proof: Let g_n be any polynomial we have

$$\begin{aligned}\left\|B^*_{n,q}(f,x) - f(x)\right\|_{P,\delta,w} &= \left\|B^*_{n,q}(f) - f - B^*_{n,q}(g_n) + B^*_{n,q}(g_n) - g_n + g_n\right\|_{P,\delta,w}\\ &\le \left\|B^*_{n,q}(f) - B^*_{n,q}(g_n)\right\|_{P,\delta,w} + \left\|B^*_{n,q}(g_n) - g_n\right\|_{P,\delta,w} + \|f - g_n\|_{P,\delta,w}\\ &\le \left\|B^*_{n,q}(f - g_n)\right\|_{P,\delta,w} + \left\|B^*_{n,q}(g_n) - g_n\right\|_{P,\delta,w} + \|f - g_n\|_{P,\delta,w}\end{aligned}$$

By linearly of $B^*_{n,q}$

Then by using lemma (4) we have

$\left\|B^*_{n,q}(f) - f)\right\|_{P,\delta,w} \le C_1\|f - g_n\|_{P,\delta,w} + \left\|B^*_{n,q}(g_n) - g_n\right\|_{P,\delta,w} + \|f - g_n\|_{P,\delta,w}$

and since $\lim_{n\to\infty}\left\|B^*_{n,q}(g_n) - g_n\right\|_{P,\delta,w} = 0$ we have

$\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} \le C_2\|f - g_n\|_{P,\delta,w}$ and by using lemma (6) we have

$\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} \le C_2\|f - g_n\|_{P,\delta,w} \le C_3\omega^{\phi}_{2,r}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$

Theorem: 3 For $f \in L_{P,\delta,w}(X), X = [0,1], 1 \le P < \infty, 0 < q \le 1, n \in N$ we have

$$\lim_{n\to\infty}\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} = 0$$

Proof: by using theorem (2) we have

$$\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} \le C\omega^{\phi}_{2,r}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$$

Then $\lim_{n\to\infty}\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} \le \lim_{n\to\infty} C\ \omega^{\phi}_{2,r}\left(f, \frac{1}{[n]}\right)_{P,\delta,w}$

and since $\lim_{n\to\infty} \frac{1}{[n]} = 0$ we have

$\lim_{n\to\infty}\left\|B^*_{n,q}(f) - f\right\|_{P,\delta,w} \le \lim_{n\to\infty} C\ \omega^{\phi}_{2,r}\left(f, \frac{1}{[n]}\right)_{P,\delta,w} = C\omega^{\phi}_{2,r}(f,0)_{P,\delta,w} = 0$

Then $\lim_{n\to\infty}\left\|B_{n,q}^{*}(f)-f\right\|_{P,\delta,w}=0$

Theorem: 4 For $f\in L_{P,w}(X), X=[0,1], 0<q<1, 0<\delta<1$ then

$$\left\|B_{n,q}^{*}(f)-f\right\|_{P,\delta,w}\le 3\ \ \omega_2\ (f,\delta)_{P,\delta,w}$$

Proof: by using lemma (5) and lemma (7) we get $B_{n,q}^{*}(f,x)$ be a linear positive monotone operator and by using lemma (3) we get

$B_{n,q}^{*}(1,x)=1$ and

$$B_{n,q}^{*}(t,x)=\frac{2q[n]x}{[2][n+1]}+\frac{1}{[2][n+1]}=x-x+\frac{2q[n]x}{[2][n+1]}+\frac{1}{[2][n+1]}=x+\propto(x)$$

Where $\propto(x)=-x+\frac{2q[n]x}{[2][n+1]}+\frac{1}{[2][n+1]}$

And since $\lim_{n\to\infty}\frac{2q[n]}{[2][n+1]}=1$ we get

$$\propto(x)=-x+x+\frac{1}{[2][n+1]}=\frac{1}{[2][n+1]}$$

Also, by using lemma (3) we get

$$B_{n,q}^{*}(t^2,x)=\frac{q(q+2)}{[3]}\cdot\frac{q[n][n+1]}{[n+1]^2}x^2+\frac{4q+7q^2+q^3}{[2][3]}\cdot\frac{[n]}{[n+1]^2}x+\frac{1}{[3][n+1]^2}$$

$$=x^2-x^2+\frac{(q^3+2q^2)[n][n+1]}{[3][n+1]^2}x^2+\frac{(4q+7q^2+q^3)[n]}{[2][3][n+1]^2}x+\frac{1}{[3][n+1]^2}$$

$$=x^2+B(x)$$

where

$$B(x)=-x^2+\frac{(q^3+2q^2)[n][n-1]}{[3][n+1]^2}x^2+\frac{(4q+7q^2+q^3)[n]}{[2][3][n+1]^2}x+\frac{1}{[3][n+1]^2}$$

And since

$\lim\limits_{n\to\infty}\frac{(q^3+2q^2)[n][n-1]}{[3][n+1]^2}=1, \lim\limits_{n\to\infty}\frac{(4q+7q^2+q^3)[n]}{[2][3][n+1]^2}=0$ we get

$B(x)=-x^2+x^2-(0)x+\frac{1}{[3][n+1]^2}=\frac{1}{[3][n+1]^2}$ thus

$$B(x)=\frac{1}{[3][n+1]^2}$$

Then by using lemma (8) we get

$$\left\|B_{n,q}^{*}(f,.)-f(.)\right\|_{P,\delta,w}\le 3\omega_2\left(f,\sqrt{B(x)-2x\propto(x)}\right)_{P,\delta,w}$$

$$=3\omega_2\left(f,\sqrt{\frac{1}{[3][n+1]^2}-\frac{2x}{[2][n+1]}}\right)_{P,\delta,w}$$

and since $\sqrt{\frac{1}{[3][n+1]^2}-\frac{2x}{[2][n+1]}}\le\sqrt{\frac{1}{[3][n+1]^2}}=\frac{1}{\sqrt{[3]}[n+1]}$

Then $\left\|B_{n,q}^{*}(f,.)-f(.)\right\|_{P,\delta,w}\le 3\omega_2\left(f,\sqrt{\frac{1}{[3][n+1]^2}-\frac{2x}{[2][n+1]}}\right)_{P,\delta,w}\le 3\omega_2\left(f,\frac{1}{\sqrt{[3]}[n+1]}\right)_{P,\delta,w}$

Let $\delta = \frac{1}{\sqrt{[3][n+1]}}$ we have

$$\left\|B^*_{n,q}(f,.) - f(.)\right\|_{P,\delta,w} \leq 3\omega_2\left(f, \frac{1}{\sqrt{[3][n+1]}}\right)_{P,\delta,w} = 3\omega_2(f,\delta)_{P,\delta,w}$$

Thus $\left\|B^*_{n,q}(f,.) - f(.)\right\|_{P,\delta,w} \leq 3\omega_2(f,\delta)_{P,\delta,w}$

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Source of support: Nil, Conflict of interest: None Declared