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QUALITATIVE BEHAVIOR OF THE DYNAMIC EQUATION USING FIXED POINT THEOREM

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ABSTRACT

In this paper we will establish the qualitative behavior of the dynamic equation of the form $\left(x(t)-h(t)x(t-\tau)\right)^{\Delta}-b(t)\mathrm{f}(\mathrm{x}(t-\sigma))=0,\ t\in T$ *on the time Scale T using Fixed Point Theorem Example is inserted to illustrate the result.*

Keywords: *Dynamic equation, Time Scale, Qualitative Behavior, First order, Fixed Point Theorem*

2010MSC: *74G55, 34N05,*

1. INTRODUCTION

The theory of time scales, which provides new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. Dynamic equations can not only unify the theories of differential equation and difference equations, but also extend these classical cases to cases "in between", e.g. to so-called q-difference equations The theory of dynamic equations on time scales is an adequate mathematical apparatus for the stimulation of processes and phenomena observed to biotechnology, chemical technology economic, neural networks, physics, social science etc. [1-3]. Motivated by this observation, in this paper we are concerned with first order nonlinear dynamic equation using Fixed Point Theorem

$$\left(x(t)-h(t)x(t-\tau)\right)^{\Delta}-b(t)\mathrm{f}(\mathrm{x}(t-\sigma))=0,\ t\in T \tag{1}$$

Where T is a time scale. Throughout this paper we assume the following conditions without further mention:

(H_1): τ,σ are fixed nonnegative constants such that the delay functions $\tau(t)=t-\tau<t$ and $\sigma(t)=t-\sigma<t$ satisfy $\tau(t):T\to T$ and $\sigma(t):T\to T$ for all $t\in T$;

(H_2): $h(t)$ is real valued rd-continuous positive functions defined on T ;

(H_3): $b(t)$ is a positive and rd-continuous function on T such that $0\le b(t)<1$.

By a solution of equation (1), we mean a nontrivial real- valued function which has the properties $\left(x(t)-h(t)x(t-\tau)\right)\in C'_{rd}\left[t_y,\infty\right)$ and $\left(x(t)-h(t)x(t-\tau)\right)^{\Delta}\in C'_{rd}\left[t_y,\infty\right)$, $t_y\ge t_0$ and satisfying equation (1.1) for all $t\ge t_y$. A solution $x(t)$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non oscillatory. Since we are interested in qualitative behavior of solutions, we will assume that the time scale T under considerations is not bounded above and therefore the time scale is assumed in the form $\left[t_0,\infty\right)_T=\left[t_0,\infty\right)\cap T$.

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We note that if $T = \mathbb{R}$ we have $\sigma(t) = t,\ \mu(t) = 0, f^{\Delta}(t) = f'(t).$ then equation (1) becomes $\left(x(t) - h(t)x(t-\tau)\right)' - b(t)f(x(t-\sigma)) = 0,\ t \in \mathbb{R}$ If $T = \mathbb{N}$

We have $\sigma(n) = n+1,\ \mu(n) = 1, y^{\Delta}(n) = \Delta y(n). = \mathrm{y(n+1) - y(n)}$ then equation (1) becomes $\Delta\left(x(t) - h(t)x(t-\tau\right) - b(t)f(x(t-\delta)) = 0,\ n \in \mathbb{N}$

If $T = h\mathbb{N}, h > 0,$ we have $\sigma(t) = t+h,\ \mu(t) = h, y^{\Delta}(\mathrm{t}) = \Delta_h(t). = \dfrac{\mathrm{y}(t+h) - \mathrm{y}(t)}{h}$ then equation (1.1) becomes $\Delta_h\left(x(t) - h(t)x(t-\tau)\right) - b(t)f(x(t-\sigma)) = 0,\ t \in h\mathbb{N}$

If $T = q^{\mathbb{N}} = \left\{t : t = q^n, n \in \mathbb{N}\right\}, q > 1,$ we have $\sigma(t) = q(t),\ \mu(t) = (q-1), y^{\Delta}(\mathrm{t}) = \Delta_q(t). = \dfrac{\mathrm{y(q}t) - \mathrm{y}(t)}{h}$ then equation (1) becomes the second order q-neutral difference equations.

$$\Delta_q\left(x(t) - h(t)x(t-\tau\right) - b(t)f(x(t-\sigma)) = 0,\ t \in q^{\mathbb{N}}$$

If $T = \mathbb{N}^2 = \left\{t^2 : t \in \mathbb{N}\right\},$ we have $\sigma(t) = \left(\sqrt{t}+1\right)^2,\ \mu(t) = 1 + 2\sqrt{t}, y^{\Delta}(\mathrm{t}) = \Delta_N(t). = \dfrac{\mathrm{y}\left(\left(\sqrt{t}+1\right)^2\right) - \mathrm{y}(t)}{1+2\sqrt{t}}$ then equation (1) becomes $\Delta_N\left(x(t) - h(t)x(t-\tau\right) - b(t)f(x(t-\sigma)) = 0,\ t \in \mathbb{N}^2$

If $T = \left\{t_n : \mathrm{n} \in \mathbb{N}\right\},$ where $\left\{t_n\right\}$ is the set if harmonic numbers defined by the (n[th] **harmonic number** is the sum of the reciprocals of the first *n* natural numbers) $t_0 = 0, t_n = \sum_{k=1}^{n} \frac{1}{k},\ n \in \mathbb{N}_0,$ we have $\sigma(t_n) = t_{n+1},\ \mu(t_n) = \dfrac{1}{n+1}, y^{\Delta}(\mathrm{t}) = \Delta t_n\, \mathrm{y}(t_n) = \left(n+1\right)\Delta\, \mathrm{y}(t_n)$ then equation (1.1) becomes $\Delta t_n\left(x(t_n) - h(t_n)x(t_n-\tau\right) - b(t_n)f(x(t_n-\sigma)) = 0,\ t \in T$

2. MAIN RESULT

To prove our main results, we will use the following Theorem which called **Krasnoselskii's Fixed Point Theorem.**

Theorem: 2.1 ([6, 12]). ***(Krasnoselskii's fixed point theorem)*** Let X be a Banach Spaces. Let Ω be a bounded closed convex subset of X and Let M_1, M_2 be maps of Ω into X such that $M_1x + M_2y \in \Omega$ for every $x, y \in \Omega.$ If M_1 is contractive and M_2 is completely continuous, then the equations $M_1x + M_2x = x$ has a solution in Ω

Now we state and prove our main results:

Theorem: 2.2. With respected to the Dynamic Equation (1) Assume that the functions u and $\mathrm{v} \in C'_{rd}\left(\left[t_0, \infty\right), \left(0, \infty\right]\right)$ constants $c > 0, K_2 > K_1 \geq 0$, $t_1 \geq t_0 + m$ $\quad m = \max\left\{\tau, \sigma\right\}$ such that the following conditions(2)-(4) holds:

$$u(t) \leq v(t),\ t \geq t_0, \tag{2}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0,\ t_0 \leq t \leq t_1 \tag{3}$$

$$\frac{1}{u(t-\tau)}\left(u(t) - K_1 + \int_{t_0}^{t} b(s)f\left(v\left(s-\sigma\right)\right)\Delta s\right) \leq a(t) \leq \frac{1}{v(t-\tau)}\left(v(t) - K_2 + \int_{t_0}^{t} b(s)f\left(u\left(s-\sigma\right)\right)\Delta s\right) \leq c < 1.\ t \geq t_1 \tag{4}$$

Then Eq. (1) has uncountably many positive solutions which are bounded by the functions $u, v.$

Proof: Let $C'_{rd}\left(\left[t_0\infty\right), R\right)$ be the set of all continuous bounded functions with the nom $\left\|x\right\| = \sup_{t \geq t_0}\left|x(t)\right|$. Then $C'_{rd}\left(\left[t_0\infty\right), R\right)$ is a Banach Space. We define a closed, bounded and convex subset Ω of $C'_{rd}\left(\left[t_0\infty\right), R\right)$ as follows:

$$\Omega = \left\{ x = x(t) \in C'_{rd}\left(\left[t_0\infty\right), R\right) : u(t) \le x(t) \le v(t),\ t \ge t_0 \right\}.$$

For $K \in [K_1, K_2]$ we define two maps $M_1, M_2 : \Omega \to C'_{rd}\left(\left[t_0\infty\right), R\right)$ as follows

$$\left(M_1x\right)\left(t_1\right) = \begin{cases} K + h(t)x(t-\tau),\ t \ge t_1 \\ \left(M_1x\right)\left(t_1\right), \quad t_0 \le t \le t_1 \end{cases} \tag{5}$$

$$\left(M_2x\right)\left(t_2\right) = \begin{cases} -\int_{t_0}^{t} b(s)f(s(t-\sigma))\Delta s,\ t \ge t_1 \\ \left(M_2x\right)\left(t_2\right) + v(t) - v(t_1), \quad t_0 \le t \le t_1 \end{cases} \tag{6}$$

We first show that for any $x, y \in \Omega.$, $M_1x + M_2y \in \Omega$. In fact for every $x, y \in \Omega.$ and $t \ge t_1$ with using eq. (4), we obtain

$$\begin{aligned} \left(M_1x\right)(t) + \left(M_2y\right)(t) &= K + h(t)x(t-\tau) - \int_{t_0}^{t} b(s)f(x(s-\sigma))\Delta s \\ &\le K + h(t)v(t-\tau) - \int_{t_0}^{t} b(s)f(u(s-\sigma))\Delta s \\ &\le K + v(t) - K_2 \le v(t) \end{aligned}$$

For $t \in [t_0, t_1]$ we have

$$\begin{aligned} \left(M_1x\right)\left(t_1\right) + \left(M_2y\right)\left(t_2\right) &= \left(M_1x\right)\left(t_1\right) + \left(M_2y\right)\left(t_2\right) + v(t) - v(t_1) \\ &\le v(t_1) + v(t) - v(t_1) = v(t) \end{aligned}$$

For $t \ge t_1$, we get

$$\begin{aligned} \left(M_1x\right)(t) + \left(M_2y\right)(t) &\ge K + h(t)u(t-\tau) - \int_{t_0}^{t} b(s)f(v(s-\sigma))\Delta s \\ &\ge K + u(t) - K_1 \ge u(t) \end{aligned}$$

For $t \in [t_0, t_1]$ with regard to (3), we get

$$v(t) - v(t_1) + u(t_1) \ge u(t),\ t_0 \le t \le t_1$$

Then, for $t \in [t_0, t_1]$ and any $x, y \in \Omega.$, we obtain

$$\begin{aligned} \left(M_1x\right)(t) + \left(M_2y\right)(t) &= \left(M_1x\right)\left(t_1\right) + \left(M_2y\right)\left(t_2\right) + v(t) - v(t_1) \\ &\ge u(t_1) + v(t) - v(t_1) \ge u(t) \end{aligned}$$

Thus, we have proved that $M_1x + M_2y \in \Omega$ for any $x, y \in \Omega.$.

Next we shall show that M_1 is a contraction mapping on Ω. Indeed for any $x, y \in \Omega.$, and $t \le t_1,$ we have

$$\left|\left(M_1x\right)(t) - \left(M_2y\right)(t)\right| = \left|h(t)\right|\left|x(t-\tau) - y(t-\tau)\right| \le c\left\|x - y\right\|$$

This implies that

$$\left\|M_1x - M_2y\right\| \le c\left\|x - y\right\| \tag{7}$$

Since $t \in [t_0, t_1]$ We conclude that M_1 is a contraction mapping on Ω

Now we show that M_2 is completely continuous. First, we will show that M_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \le t_1$, we have

$$\left|(M_2x_k)(t) - (M_2x)(t)\right| \le \left|\int_{t_0}^{t} b(s)\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]\Delta s\right|$$

$$\le \int_{t_0}^{t} b(s)\left\|\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]\right\|\Delta s$$

and

$$\cdot \int_{t_0}^{t} b(s)f(v(s-\sigma))\Delta s < \infty \tag{8}$$

Since $\left\|\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]\right\| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{k\to\infty}\left\|(M_2x_k)(t) - (M_2x)(t)\right\| = 0$$

This means that M_2 is continuous.

We now show that $M_2\Omega$ is relatively compact. by (8), for $x \in \Omega$ and any $\varepsilon > 0$, there exist $t^* \ge t_1$ large enough so that

$$\int_{t^*}^{\infty} b(s)f(x(s-\sigma))\Delta s < \frac{\varepsilon}{2} \cdot$$

Then, form $x \in \Omega,\quad T_2 > T_1 \ge t^*$, we have

$$\left|(M_2x)(T_2) - (M_2x)(T_1)\right| \le \int_{T_2}^{\infty} b(s)f(x(s-\sigma))\Delta s + \int_{T_1}^{\infty} b(s)f(x(s-\sigma))\Delta s < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \cdot$$

For $x \in \Omega,\quad t_1 \le T_1 < T_2 \le t^*$, we get

$$\left|(M_2x)(T_2) - (M_2x)(T_1)\right| \le \int_{T_1}^{T_2} b(s)f(x(s-\sigma))\Delta s \le \max_{t_1\le s\le t^*}\left\{b(s)f(x(s-\sigma))\right\}(T_2 - T_1) \cdot$$

Thus there exists $\delta_1 = \dfrac{\varepsilon}{A}$ where $A = \max_{t_1\le s\le t^*}\left\{b(s)f(x(s-\sigma))\right\}$, such that

$\left|(M_2x)(T_2) - (M_2x)(T_1)\right| < \varepsilon$ if $0 < T_2 - T_1 < \delta_1$

Finally, for any $x \in \Omega,\quad t_0 \le T_1 < T_2 \le t_1$, there exists a $\delta_2 > 0$ such that

$$\left|(M_2x)(T_2) - (M_2x)(T_1)\right| = \left|v(T_1) - v(T_2)\right| = \left|\int_{T_1}^{T_2} v^{\Delta}(s)\Delta s\right|$$

$$\le \max_{t_0\le s\le t_1}\left\{\left|v^{\Delta}(s)\right|\right\}(T_2 - T_1) < \varepsilon \text{ if } 0 < T_2 - T_1 < \delta_2 .$$

Then $\{M_2x : x \in \Omega\}$ is uniformly bounded and equicontinous on $[t_0,\infty)$ and hence $M_2\Omega$ is a relatively compact subset of $C'_{rd}([t_0\infty),R)$. By Theorem (2.1) there is an $x_0 \in \Omega$ such that $M_1x_0 + M_2x_0 = x_0$. We concluded that $x_0(t)$ is a positive solution of (1).

Next we show that Eq.(1) has uncountably many bounded positive solutions in Ω. Let the constant $\widetilde{K} \in [K_1,K_2]$ be such that $\widetilde{K} \ne K$. We infer similarly that there exist mappings $\widetilde{M_1},\widetilde{M_2}$ satisfying (5), (6), where K,M_1,M_2 are replaced by $\widetilde{K},\widetilde{M_1},\widetilde{M_2}$ respectively. We assume that $x, y \in \Omega,\ M_1x + M_2x = x,\ \widetilde{M_1}y + \widetilde{M_2}y = y,$ which are the bounded positive solutions of Equation (1) that is

$$x(t) = K + h(t)x(t-\tau) - \int_t^{t_1} b(s)f(x(s-\sigma))\Delta s,\ t \ge t_1 \cdot$$

$$y(t) = \widetilde{K} + h(t)y(t-\tau) - \int_t^{t_1} b(s)f(y(s-\sigma))\Delta s,\ t \ge t_1$$

From condition (8) it follows that there exists as $t_2 > t_1$, satisfying

$$\int_t^{t_1} b(s)\left[f(x(s-\sigma)) + f(y(s-\sigma))\right]\Delta s < \left|K - \widetilde{K}\right| \cdot \tag{9}$$

In order to prove that the set of bounded positive solutions of Eq. (1) is uncountable, it is sufficient to verify that $x \ne y$ for $t \ge t_2$ we get

$$\begin{aligned}\left|x(t) - y(t)\right| &= \left|K + h(t)x(t-\tau) - \int_t^{t_2} b(s)f(x(s-\sigma))\Delta s - \widetilde{K} - h(t)y(t-\tau) + \int_t^{t_2} b(s)f(y(s-\sigma))\Delta s\right| \\ &\ge \left|K - \widetilde{K} + h(t)\left[x(t-\tau) - y(t-\tau)\right] - \int_t^{t_2} b(s)\left[f(x(s-\sigma)) + f(y(s-\sigma))\right]\Delta s\right| \\ &\ge \left|K - \widetilde{K}\right| - h(t)\left\|x-y\right\| - \left|\int_t^{t_2} b(s)\left[f(x(s-\sigma)) + f(y(s-\sigma))\right]\Delta s\right| \\ &\ge \left|K - \widetilde{K}\right| - C\left\|x-y\right\| - \int_t^{t_2} b(s)\left[f(x(s-\sigma)) + f(y(s-\sigma))\right]\Delta s\end{aligned}$$

Thus we have

$$(1+C)\left\|x-y\right\| \ge \left|K - \widetilde{K}\right| - \int_t^{t_2} b(s)\left[f(x(s-\sigma)) + f(y(s-\sigma))\right]\Delta s,\ t \ge t_2$$

From eq.(9) we get that $x \ne y$. Since the interval $\widetilde{K} \in [K_1, K_2]$ contains uncountably man constants, the Eq. (1) has uncountably many positive solutions which are bounded by the functions $u(t), v(t)$. This completes the proof.

Corollary: 2.3. With respected to the Dynamic Equation (1) Assume that the functions u and $v \in C'_{rd}\left([t_0,\infty),(0,\infty]\right)$ constants $c > 0, K_2 > K_1 \ge 0$, $t_1 \ge t_0 + m$ $m = \max\{\tau,\sigma\}$ such that:

$$v^\Delta(t) - u^\Delta(t) \le 0,\ t_0 \le t \le t_1 \tag{10}$$

Then Eq.(1) has uncountably many positive solutions which are bounded by the functions u, v.

Proof: We need to prove that condition (10) implies (3). Let $t \in [t_0, t_1]$ and set $H(t) = v(t) - v(t_1) - u(t) + u(t_1)$

Then, from eq. (10) it follows that

$H(t) = v^\Delta(t) - u^\Delta(t) \le 0, t_0 \le t \le t_1$ Since $H(t_1) = 0$ and $H^\Delta(t) \le 0$ for $t \in [t_0, t_1]$

This implies that

$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \ge 0,\ t_0 \le t \le t_1$

Thus all the conditions of Theorem 2.2 are satisfied.

Examples: Consider the nonlinear dynamic equation

$$\left[x(t) - h(s)x(t-2)\right]^\Delta = b(t)x^3(t-1),\ t \ge t_0\ , \tag{11}$$

where $p(t) = e^{-t}$. We will show that the conditions of corollary 2.3 satisfied. The function $u(t) = 0.5,\ v(t) = 2$ satisfy (2) and also condition (10) for $t \in [t_0, t_1] = [0, 4]$.

For the constants $K_1 = 0.5, K_2 = 1$, condition (4) has the form

$$16e^{-t} \le h(t) \le \frac{1}{2} + \frac{1}{16}e^{-t},\ t \ge t_1 = 4 \tag{12}$$

If the function $h(t)$ satisfies (12), Then Eq. (1) has uncountably many positive solutions which are bounded by the functions u, v.

REFERENCES

1. R. P. Agarwal, M. Bohner and A. Peterson, *Dynamic equations on time scales: A survey*. J. Comp. Appl. Math, Special issue on dynamic equations on time scales, edited by R.P. Agarwal, M.Bohner and D.O'Regan (Preprint in Ulmer Seminare 5), 141 (2002), 1–26.
2. D. K. Anderson and A. Zafer, *Nonlinear oscillation of second order dynamic equations on time scales*, Appl.Math. Let., 22 (2009), 1591-1597.
3. M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhuser, Boston, 2001.
4. S. R. Grace, R. Agarwal, M. Bohner and D. O'Regan, *Oscillation of second order strongly superlinear and strongly sublinear dynamic equations*, Comm. Nonl. Sci Num. Sim., 14 (2009), 3463–3471.
5. M. Huang andW. Feng, *Oscillation for forced second order nonlinear dynamic equations on time scales*, Elect. J. Diff. Eqn., 145 (2005), 1–8.
6. S. H. Saker, *Oscillation of second order nonlinear neutral delay dynamic equations*, J. Comp. Appl. Math., 187(2006), 123–141.
7. P.Mohankumar and A.Ramesh, *Oscillatory Behaviour Of The Solution Of The Third Order Nonlinear Neutral Delay Difference Equation*, International Journal of Engineering Research & Technology (IJERT) ISSN: 2278-0181 Vol. 2 Issue 7,no.1164-1168 July – 2013.
8. B.Selvaraj, P.Mohankumar and A.Ramesh, *On The Oscillatory Behavior of The Solutions to Second Order Nonlinear Difference Equations*, International Journal of Mathematics and Statistics Invention (IJMSI) E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759 Volume 1 Issue 1 ǁ Aug. 2013ǁ PP-19-21.
9. Erbe,LH, Kong, Qk.Zhang,BG: Oscillation Theory for Functional Differential equations, Dekkar New YorK (1995).
10. Zhou.Y: Existence of Nonoscillatory solutions of second order nonlinear differential equations, J. Math. Anal Appl.331 pp91-96(2007).

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