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âg Exterior and âg Frontier in Topological Spaces

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ABSTRACT

In this paper, we introduce âg exterior, âg frontier and study some of its properties.

Key words: *âg open, âg closed, âg int A, âg cl A*

AMS subject classification: *54C10, 54C08, 54C05.*

1. INTRODUCTION

Levine [6] introduced generalized closed sets in topology as a generalisation of closed sets. Many authors like Arya *et. al* [2], Balachandran *et. al* [3], Bhattacharya *et al* [4], Arokiarani[1], Gnanambal [5], Malgan [7] and Nagaveni [8] have worked on generalized closed sets, Palaniappan *et. al* [9] introduced âg closed sets and worked on them. In this paper âg exterior and âg frontier are introduced and their properties are investigated.

Throughout this paper X denote the topological space (X, τ) on which no separation axioms are assumed unless otherwise stated.

2. PRELIMINARIES

Definition: 2.1 A subset A of a topological space X is called
1) A pre open set if A ⊂ int cl A and a preclosed set if cl int A ⊂ A.
2) A regular open set if A = int cl A and a regular closed set if A = cl int A.
3) A semi open set if A ⊂ cl int A and a semi closed set if int cl A ⊂ A.
4) A α – open set if A ⊂ int cl int A and a α – closed set if cl int cl A ⊂ A.

The intersection of all α – closed subsets of X containing A is called the α closure of A and is denoted by α cl A. α cl A is a α closed set.

Definition: 2.2 A subset A of a topological space X is called â generalized closed set (briefly âg closed set) if int cl int A ⊂ U whenever A ⊂ U and U is open in X.

The intersection of all âg closed subset of X containing A is called âg closure of A and is denoted by âg cl A. In general âg cl A is not âg closed.

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The union of all âg open sets contained in A is denoted by âg int A. In general âg int A is not âg open.

In what follows, we assume that X is a topological space in which arbitrary intersections of âg closed sets of X is âg closed in X. Then âg cl A will be a âg closed set in X and âg int A will be a âg open set in X, for $A \subset X$.

Definition: 2.3 Let X be a topological space and let $x \in X$, A subset N of x is said to be a âg neighbourhood of x if and only if there exists a âg open set G such that $x \in G \subset N$.

Definition: 2.4 Let X be a topological space and $A \subset X$. A point $x \in X$ is called a âg limit point of X if and only if every âg neighbourhood of x contains a point of A other than x. The set of all âg limit points of A is called the âg derived set of A and shall be denoted by $D_{âg}$ (A)

Thus x will be a âg limit point of A if and only if $(N-\{x\}) \cap A \neq \phi$, for every âg neighbourhood N of x.

Definition: 2.5 Let A be a sub set of a topological space X and let $x \in X$. Then x is called an âg adherent point of A if and only if every âg neighbourhood of x contains a point of A. The set of all âg adherent points of A is called the âg adherence of A and shall be denoted by âg Adh A.

Definition: 2.6 A point x is said to be a âg isolated point of a subset A of a topological space X if and only if $x \in A$ but is not a âg limit point of A. That is, there exists some âg neighbourhood N of x such that N contains no point of A other than x. A âg closed set which has no âg isolated point is said to be âg perfect.

Remarks: 2.7 A âg adherent point is either âg limit point or âg isolated point.

3. âg Exterior points and âg Exterior of a set.

Definition: 3.1 Let A be a subset of a topological space X. A point $x \in X$ is said to be âg exterior point of A if it is âg interior point of the complement A' of A. That is, if there exists âg open set G such that $x \in G \subset A'$ or $x \in G$ and $G \cap A = \phi$. The set of all âg exterior points of A is called the âg exterior of A and is denoted by âg ext A.

Thus âg ext A= âg int A'

âg ext A'= âg int A

We also have $A \cap$ âg ext $A = \phi$

That is, no point of A can be in âg exterior of A.

Remark: 3.2 âg ext A is the largest âg open set contained in A'.

Remark: 3.3 âg ext A = $\cup$ { G: G is âg open, $G \subset A'$}

Theorem: 3.4 Let A be a subset of a topological space X. Then $x \in X$ is a âg exterior point of A if and only if x is not an âg adherent point of A, that is, if and oniy if $x \in$ (âg cl A)'.

Proof: Let x be a âg exterior point of A . So, x is a âg interior point of A'. Hence A' is a âg neighbourhood of x containing no point of A. It follows x is not a âg adherent point of A, that is $x \in$ (âg cl A)', by theorem 3.5 [9]

Conversely, let x be not a âg adherent point of A. Then, there exists a âg neighbourhood N of x which contains no point of A. This implies $N \subset A'$. It follows A' is a âg neighbourhood of x.

Hence x is a âg interior point of A'. That is, x is a âg exterior point of A.

Corollary: 3.5 It follows from the above theorem,

âg ext A= (âg cl A)'

From this we conclude that

âg int A= âg ext A'= (âg cl A')'.

4. Properties of âg exterior

Theorem 4.1: Let X be a topological space and let A, B be subsets of X. Then
i) âg ext X = ϕ, âg ext ϕ=X,
ii) âg ext A⊂A',
iii) âg ext A= âg ext [(âg ext A)'].
iv) A⊂B ⇒âg ext B⊂ âg ext A.
v) âg int A⊂ âg ext [âg ext A].
vi) âg ext(A∪ B)= âg extA ∩âg extB.

Proof:
i) obvious
ii) obvious
iii) âg ext [(âg ext A)'] = âg ext [(âg int A')']
= âg int [[(âg int A')']']
= âg int [âg int A'] = âg int A' = âg ext A

iv) A⊂B⇒B'⊂A'⇒ âg int B'⊂ âg int A'⇒ âg ext B ⊂ âg ext A.

v) âg ext A ⊂A' by (ii)

By (iv) âg ext A'⊂ âg ext [âg ext A]

That is âg int A ⊂ âg ext [âg ext A]

vi) âg ext (A∪B) = âg int (A∪B)'
= âg int (A'∩B')
= âg int A' ∩âg int B'
= âg ext A ∩ âg ext B

5. âg Frontier points and âg Frontier of a set

Definition: 5.1 A point x of a topological space X is said to be a âg frontier point or âg boundary point of a subset A of X, if and only if it is neither a âg interior point nor a âg exterior point of A. The set of all âg frontier points of A is called the âg frontier of A and shall be denoted by âg Fr (A).

Theorem: 5.2 Let X be a topological space. Then a point x∈X is a âg frontier point of A if and only if every âg neighbourhood of x intersects both A and A'.

Proof: x∈ âg Fr(A) ⇔ x∉ âg int A and x ∉ âg ext A = âg int A'
⇔ neither A nor A' is a âg neighbourhood of x.
⇔ no âg neighbourhood of x can be contained in A or in A'
⇔ every âg neighbourhood of x intersects both A and A'.

Theorem: 5.3 âg Fr(A) = âg Fr(A')

Proof: x ∈âg Fr(A) ⇔every âg neighbourhood of x
intersect both A and A' ⇔every âg neighbourhood of x
intersect both (A')' and A'⇔ x ∈âg Fr(A').

Theorem: 5.4 Let A be any subset of a topological space X. Then âg int A, âg ext A and âg Fr (A) are disjoint and x= âg int A∪ âg ext A ∪âg Fr(A).Further âg Fr(A) is a âg closed set.

Proof: âg ext A = âg int A'. Also âg int A⊂A, and âg int A'⊂ A'. Since A∩A'= ϕ, âg int A ∩ âg ext A = ϕ.

Again by the definition of âg frontier,
x∈ âg Fr(A) ⇔ x ∉âg int A and x∉ âg ext A.
⇔ x ∉(âg int A) ∪ (âg ext A).
⇔ x ∈ (âg int A ∪ âg ext A)'.

Hence âgFr(A)=(âg int A$\cup$ âg ext A)'

It follows that âg Fr(A) $\cap$ âg int A = ϕ and âg Fr(A) $\cap$ âg ext A = ϕ

Also X= âg int A$\cup$ âg ext A $\cup$ âg Fr(A).

Since âg int A and âg ext A are âg open sets, âg Fr(A) is a âg closed set.

Definition: 5.5 Let X be a topological space and let A, B be subsets of X. Then
i) A is said to be âg dense in B if and only if B $\subset$âg cl A.
ii) A is said to be âg dense in X or every where âg dense if and only if âg cl A = X.
iii) A is said to be nowhere âg dense or non âg dense if and only if âg int (âg cl A) = ϕ
iv) A is said to be âg dense in itself if and only if A$\subset D_{âg}$(A).

Remark: 5.6 A is everywhere âg dense if and only if every point of X is âg adherent point of A.

Theorem: 5.7 Let X be a topological space and let A$\subset$X. Then A is âg perfect if and only if A is âg dense in itself and âg closed.

Proof: A is âg perfect $\Leftrightarrow$ A is âg closed and A has no âg islolated points $\Leftrightarrow$ A is âg closed and every point of A is a âg limit point of A$\Leftrightarrow$ A is âg closed and A$\subset D_{âg}$(A) $\Leftrightarrow$ A is âg closed and A is âg dense initself.

Remark: 5.8 A is âg perfect if and only if A = $D_{âg}$(A)

6. Relations between *âg* closure, *âg* interior and *âg* frontier

Theorem: 6.1 Let X be a topological space and let A$\subset$X. Then
i) âg int A = (âg cl A')'
ii) âg cl A' = (âg int A)'
iii) âg cl A = (âg int A')'

Proof:
i) It follows from corollary 3.5 âg ext A = (âg cl A)'

âg ext A = âg int A'

Hence âg int A = âg ext A' = (âg cl A')'

ii) By i), âg int A = (âg cl A')'

Taking complements,
(âg int A)' = âg cl A'

iii) By ii) âg cl A' = (âg int A)'

Replace A by A'

âg cl A = (âg int A')'

Theorem: 6.2 Let X be a topological space and let A$\subset$ X. Then âg cl A = âg int A $\cup$ âg Fr(A)

Proof: (âg cl A)' = âg ext A

Taking complements,

âg cl A = (âg ext A)' = âg int A $\cup$ âg Fr(A)

Remark: 6.3 âg Fr(A) $\subset$ âg cl A

Theorem: 6.4 Let X be a topological space and let A⊂ X. Then âg cl A = A ∪ âg Fr(A).

Proof: A⊂ âg cl A and âg Fr(A) ⊂ âg cl A

Hence A∪ âg Fr(A) ⊂ âg cl A

Also âg Fr(A) = [âg int A∪ âg ext A]'

= (âg int A)'∩ (âg ext A)'

Again, since âg int A⊂ A and âg cl A = âg int A∪ âg Fr(A), it follows that âg cl A⊂ A∪ âg Fr(A).

Hence âg cl A = A∪ âg Fr(A).

Remark: 6.5 Every âg closed subset is the disjoint union of its âg interior and âg frontier.

Theorem: 6.6 Let X be a topological space and let A, B be subsets of X. Then
i) âg Fr(A) = âg cl A∩ âg cl A' = âg cl A - âg int A.
ii) âg int A = A- âg Fr(A).
iii) (âg Fr (A))' = âg int A∪ âg int A'.
iv) âg Fr(âg int A) ⊂ âg Fr(A)
v) âg Fr (âg cl A) ⊂âg Fr(A).
vi) âg Fr(A∪B) ⊂âg Fr(A) ∪âg Fr(B).
vii)âg Fr(A∩ B) ⊂âg Fr(A) ∪âg Fr(B).

Proof:
i) âg Fr(A) = (âg int A ∪ âg ext A)' = (âg int A)'∩ (âg ext A)'
= (âg cl A')" (âg cl A)" by theorem 3.5
= âg cl A' ∩ âg cl A

Now âg cl A∩ âg cl A' = âg cl A – (âg cl A')' = âg cl A - âg int A.

Hence âg Fr(A) = âg cl A - âg int A.

ii) A- âg Fr(A) = A- (âg cl A - âg int A) = âg int A.

iii) (âg Fr(A))' = (âg cl A ∩ âg cl A')' = (âg cl A)' ∪ (âg cl A')'.

By cor 3.5 (âg cl A')' = âg int A.

So, âg int A' = (âg cl A)'

Hence (âg Fr(A))' = âg int A ∪ âg int A'.

iv) âg Fr (âg int A) = âg cl (âg int A) ∩ âg cl (âg int A)' by (i)
= âg cl (âg int A) ∩ âg cl (âg cl A') ⊂ âg cl A∩ âg cl A' = âg Fr(A).

v) âg Fr (âg cl A) = âg cl (âg cl A) ∩ âg cl (âg cl A)' = âg cl A∩ âg cl (âg cl A)'

Now A⊂ âg cl A ⇒ (âg cl A)'⊂A'⇒ âg cl (âg cl A)'⊂ âg cl A'.

Hence âg Fr (âg cl A) ⊂ âg cl A∩ âg cl A' = âg Fr (A).

vi) âg Fr(A∪B) = âg cl (A∪B) ∩ âg cl (A∪B)' = (âg cl (A) ∪âg cl (B) ∩âg cl (A'∩B') ⊂ (âg cl A∪ âg cl B) ∩ (âg cl A' ∩âg cl B')
= (âg cl A ∩ (âg cl A' ∩âg cl B')) ∪ (âg cl B ∩ (âg cl A'∩ âg cl B'))
= ((âg cl A∩ âg cl A') ∩âg cl B') ∪ ((âg cl B∩ âg cl B') ∩âg cl A')
= (âg Fr(A) ∩ âg cl B') ∪ (âg Fr(B) ∩ âg cl A') ⊂âg Fr (A) ∪âg Fr(B)

vii)âg Fr (A∩B) = âg cl (A∩B) ∩âg cl (A∩B)'⊂ (âg cl A ∩âg cl B) ∩(âg cl (A'∪B'))
= (âg cl A ∩âg cl B) ∩(âg cl A'∪ âg cl B')
= ((âg cl A ∩âg cl B) ∩ âg cl A') ∪ ((âg cl A ∩âg cl B) ∩(âg cl B')
⊂ (âg cl A∩ âg cl A') ∪(âg cl B∩âg cl B') = âg Fr (A) ∪ âg Fr(B).

Theorem: 6.8 Let X be a topological space and let A⊂X. Then
i) if A is open, then âg Fr (A) = âg cl A- A
ii) âg Fr(A) = ϕ if and only if A is âg open as well as âg closed.
iii) A is âg open if and only if A∩ âg Fr(A) = ϕ, that is âg Fr(A) ⊂A'.
iv) A is âg closed if and only if âg Fr(A) ⊂A.

Proof:
i) By theorem 6.6 âg Fr(A) = âg cl A - âg int A
A is open ⇒A is âg open ⇒âg int A = A
Hence âg Fr(A) = âg cl A – A.

ii) Let âg Fr(A) = ϕ
Then âg Fr(A) = ϕ ⇒ âg cl A - âg int A = ϕ
⇒ âg cl A = âg int A = A
⇒ A is âg open as well as âg closed.

Conversely, let A be âg open as well as âg closed.
âg Fr(A) = âg cl A - âg int A

Since A is âg closed, âg cl A = A

Since A is âg open, âg int A = A

Hence âg Fr(A) = ϕ.

iii) By theorem 6.6 âg Fr(A) = âg cl A∩ âg cl A'

Let A be âg open. Hence A' is âg closed.

âg cl A' = A'

Now, A ∩âg Fr(A) = A ∩ (âg cl A∩ âg cl A') = A∩ (âg cl A ∩A') = (A ∩âg cl A) ∩A' = A∩A' = ϕ.

Conversely, let A∩ âg Fr(A) = ϕ.

This impies A ∩ (âg cl A ∩ âg cl A') =ϕ.
⇒ (A ∩âg cl A) ∩âg cl A' = ϕ.
⇒A∩ âg cl A' = ϕ.
⇒âg cl A' = A' ⇒A' is âg closed ⇒ A is âg open.

iv) Let A be âg closed. Then âg cl A = A.

Hence âg Fr(A) = âg cl A ∩âg cl A' = A∩ âg cl A'

So, âg Fr(A) ⊂ A

Conversely, let âg Fr(A) ⊂A

Then A∪ âg Fr(A) = A

But A ∪âg Fr (A) = âg cl A, by theorem 6.4.

That is A= âg cl A.Hence A is âg closed.

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