International Journal of Mathematical Archive-5(4), 2014, 102-108
Available online through www.ijma.info ISSN 2229 – 5046

# $\hat{\alpha}$g Exterior and $\hat{\alpha}$g Frontier in Topological Spaces

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*(Received on: 19-03-14; Revised & Accepted on: 01-04-14)*

***ABSTRACT***

*In this paper, we introduce $\hat{\alpha}$g exterior, $\hat{\alpha}$g frontier and study some of its properties.*

***Key words:*** *$\hat{\alpha}$g open, $\hat{\alpha}$g closed, $\hat{\alpha}$g int A, $\hat{\alpha}$g cl A*

***AMS subject classification:*** *54C10, 54C08, 54C05.*

## 1. INTRODUCTION

Levine [6] introduced generalized closed sets in topology as a generalisation of closed sets. Many authors like Arya *et. al* [2], Balachandran *et. al* [3], Bhattacharya *et al* [4], Arokiarani[1], Gnanambal [5], Malgan [7] and Nagaveni [8] have worked on generalized closed sets, Palaniappan *et. al* [9] introduced $\hat{\alpha}$g closed sets and worked on them. In this paper $\hat{\alpha}$g exterior and $\hat{\alpha}$g frontier are introduced and their properties are investigated.

Throughout this paper X denote the topological space (X, $\tau$) on which no separation axioms are assumed unless otherwise stated.

## 2. PRELIMINARIES

**Definition: 2.1** A subset A of a topological space X is called
1) A pre open set if $A \subset$ int cl A and a preclosed set if cl int A $\subset$ A.
2) A regular open set if A = int cl A and a regular closed set if A = cl int A.
3) A semi open set if A $\subset$ cl int A and a semi closed set if int cl A $\subset$ A.
4) A $\alpha$ – open set if A $\subset$ int cl int A and a $\alpha$ – closed set if cl int cl A $\subset$ A.

The intersection of all $\alpha$ – closed subsets of X containing A is called the $\alpha$ closure of A and is denoted by $\alpha$ cl A. $\alpha$ cl A is a $\alpha$ closed set.

**Definition: 2.2** A subset A of a topological space X is called $\hat{\alpha}$ generalized closed set (briefly $\hat{\alpha}$g closed set) if int cl int A $\subset$ U whenever A $\subset$ U and U is open in X.

The intersection of all $\hat{\alpha}$g closed subset of X containing A is called $\hat{\alpha}$g closure of A and is denoted by $\hat{\alpha}$g cl A. In general $\hat{\alpha}$g cl A is not $\hat{\alpha}$g closed.

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The union of all α̂g open sets contained in A is denoted by α̂g int A. In general α̂g int A is not α̂g open.

In what follows, we assume that X is a topological space in which arbitrary intersections of α̂g closed sets of X is α̂g closed in X. Then α̂g cl A will be a α̂g closed set in X and α̂g int A will be a α̂g open set in X, for $A\subset X$.

**Definition: 2.3** Let X be a topological space and let $x\in X$, A subset N of x is said to be a α̂g neighbourhood of x if and only if there exists a α̂g open set G such that $x\in G\subset N$.

**Definition: 2.4** Let X be a topological space and $A\subset X$. A point $x\in X$ is called a α̂g limit point of X if and only if every α̂g neighbourhood of x contains a point of A other than x. The set of all α̂g limit points of A is called the α̂g derived set of A and shall be denoted by $D_{\hat{\alpha}g}$ (A)

Thus x will be a α̂g limit point of A if and only if $(N-\{x\}) \cap A \neq \phi$, for every α̂g neighbourhood N of x.

**Definition: 2.5** Let A be a sub set of a topological space X and let $x \in X$. Then x is called an α̂g adherent point of A if and only if every α̂g neighbourhood of x contains a point of A. The set of all α̂g adherent points of A is called the α̂g adherence of A and shall be denoted by α̂g Adh A.

**Definition: 2.6** A point x is said to be a α̂g isolated point of a subset A of a topological space X if and only if $x\in A$ but is not a α̂g limit point of A. That is, there exists some α̂g neighbourhood N of x such that N contains no point of A other than x. A α̂g closed set which has no α̂g isolated point is said to be α̂g perfect.

**Remarks: 2.7** A α̂g adherent point is either α̂g limit point or α̂g isolated point.

**3. α̂g Exterior points and α̂g Exterior of a set.**

**Definition: 3.1** Let A be a subset of a topological space X. A point $x\in X$ is said to be α̂g exterior point of A if it is α̂g interior point of the complement A' of A. That is, if there exists α̂g open set G such that $x\in G\subset A'$ or $x\in G$ and $G\cap A=\phi$. The set of all α̂g exterior points of A is called the α̂g exterior of A and is denoted by α̂g ext A.

Thus α̂g ext A= α̂g int A'

α̂g ext A'= α̂g int A

We also have A∩ α̂g ext A=ϕ

That is, no point of A can be in α̂g exterior of A.

**Remark: 3.2** α̂g ext A is the largest α̂g open set contained in A'.

**Remark: 3.3** α̂g ext A = ∪ { G: G is α̂g open, $G\subset A'$}

**Theorem: 3.4** Let A be a subset of a topological space X. Then $x\in X$ is a α̂g exterior point of A if and only if x is not an α̂g adherent point of A, that is, if and oniy if $x \in$ (α̂g cl A)'.

**Proof:** Let x be a α̂g exterior point of A . So, x is a α̂g interior point of A'. Hence A' is a α̂g neighbourhood of x containing no point of A. It follows x is not a α̂g adherent point of A, that is $x\in$ (α̂g cl A)', by theorem 3.5 [9]

Conversely, let x be not a α̂g adherent point of A. Then, there exists a α̂g neighbourhood N of x which contains no point of A. This implies $N\subset A'$. It follows A' is a α̂g neighbourhood of x.

Hence x is a α̂g interior point of A'. That is, x is a α̂g exterior point of A.

**Corollary: 3.5** It follows from the above theorem,

α̂g ext A= (α̂g cl A)'

From this we conclude that

α̂g int A= α̂g ext A'= (α̂g cl A')'.

## 4. Properties of âg exterior

**Theorem 4.1**: Let X be a topological space and let A, B be subsets of X. Then
i) âg ext X = $\phi$, âg ext $\phi$=X,
ii) âg ext A$\subset$A',
iii) âg ext A= âg ext [(âg ext A)'].
iv) A$\subset$B $\Rightarrow$âg ext B$\subset$ âg ext A.
v) âg int A$\subset$ âg ext [âg ext A].
vi) âg ext(A$\cup$ B)= âg extA $\cap$âg extB.

**Proof:**
i) obvious
ii) obvious
iii) âg ext [(âg ext A)'] = âg ext [(âg int A')']
= âg int [[(âg int A')']']
= âg int [âg int A'] = âg int A' = âg ext A

iv) A$\subset$B$\Rightarrow$B'$\subset$A'$\Rightarrow$ âg int B'$\subset$ âg int A'$\Rightarrow$ âg ext B $\subset$ âg ext A.

v) âg ext A $\subset$A' by (ii)

By (iv) âg ext A'$\subset$ âg ext [âg ext A]

That is âg int A $\subset$ âg ext [âg ext A]

vi) âg ext (A$\cup$B) = âg int (A$\cup$B)'
= âg int (A'$\cap$B')
= âg int A' $\cap$âg int B'
= âg ext A $\cap$ âg ext B

## 5. âg Frontier points and âg Frontier of a set

**Definition: 5.1** A point x of a topological space X is said to be a âg frontier point or âg boundary point of a subset A of X, if and only if it is neither a âg interior point nor a âg exterior point of A. The set of all âg frontier points of A is called the âg frontier of A and shall be denoted by âg Fr (A).

**Theorem: 5.2** Let X be a topological space. Then a point x$\in$X is a âg frontier point of A if and only if every âg neighbourhood of x intersects both A and A'.

**Proof:** x$\in$ âg Fr(A) $\Leftrightarrow$ x$\notin$ âg int A and x $\notin$ âg ext A = âg int A'
$\Leftrightarrow$ neither A nor A' is a âg neighbourhood of x.
$\Leftrightarrow$ no âg neighbourhood of x can be contained in A or in A'
$\Leftrightarrow$ every âg neighbourhood of x intersects both A and A'.

**Theorem: 5.3** âg Fr(A) = âg Fr(A')

**Proof:** x $\in$âg Fr(A) $\Leftrightarrow$every âg neighbourhood of x
intersect both A and A' $\Leftrightarrow$every âg neighbourhood of x
intersect both (A')' and A'$\Leftrightarrow$ x $\in$âg Fr(A').

**Theorem: 5.4** Let A be any subset of a topological space X. Then âg int A, âg ext A and âg Fr (A) are disjoint and x= âg int A$\cup$ âg ext A $\cup$âg Fr(A).Further âg Fr(A) is a âg closed set.

**Proof:** âg ext A = âg int A'. Also âg int A$\subset$A, and âg int A'$\subset$ A'. Since A$\cap$A'= $\phi$, âg int A $\cap$ âg ext A = $\phi$.

Again by the definition of âg frontier,
x$\in$ âg Fr(A) $\Leftrightarrow$ x $\notin$âg int A and x$\notin$ âg ext A.
$\Leftrightarrow$ x $\notin$( âg int A) $\cup$ (âg ext A).
$\Leftrightarrow$ x $\in$ ( âg int A $\cup$ âg ext A)'.

Hence âgFr(A)=(âg int A∪ âg ext A)'

It follows that âg Fr(A) ∩ âg int A = ϕ and âg Fr(A) ∩ âg ext A = ϕ

Also X= âg int A∪ âg ext A ∪ âg Fr(A).

Since âg int A and âg ext A are âg open sets, âg Fr(A) is a âg closed set.

**Definition: 5.5** Let X be a topological space and let A, B be subsets of X. Then
i) A is said to be âg dense in B if and only if B ⊂âg cl A.
ii) A is said to be âg dense in X or every where âg dense if and only if âg cl A = X.
iii) A is said to be nowhere âg dense or non âg dense if and only if âg int (âg cl A) = ϕ
iv) A is said to be âg dense in itself if and only if A⊂$D_{âg}$(A).

**Remark: 5.6** A is everywhere âg dense if and only if every point of X is âg adherent point of A.

**Theorem: 5.7** Let X be a topological space and let A⊂X. Then A is âg perfect if and only if A is âg dense in itself and âg closed.

**Proof:** A is âg perfect ⇔ A is âg closed and A has no âg islolated points ⇔ A is âg closed and every point of A is a âg limit point of A⇔ A is âg closed and A⊂$D_{âg}$(A) ⇔ A is âg closed and A is âg dense initself.

**Remark: 5.8** A is âg perfect if and only if A = $D_{âg}$(A)

**6. Relations between âg closure, âg interior and âg frontier**

**Theorem: 6.1** Let X be a topological space and let A⊂X. Then
i) âg int A = (âg cl A')'
ii) âg cl A' = (âg int A)'
iii) âg cl A = (âg int A')'

**Proof:**
i) It follows from corollary 3.5 âg ext A = (âg cl A)'

âg ext A = âg int A'

Hence âg int A = âg ext A' = (âg cl A')'

ii) By i), âg int A = (âg cl A')'

Taking complements,
(âg int A)' = âg cl A'

iii) By ii) âg cl A' = (âg int A)'

Replace A by A'

âg cl A = (âg int A')'

**Theorem: 6.2** Let X be a topological space and let A⊂ X. Then âg cl A = âg int A ∪ âg Fr(A)

**Proof**: (âg cl A)' = âg ext A

Taking complements,

âg cl A = (âg ext A)' = âg int A ∪ âg Fr(A)

**Remark: 6.3** âg Fr(A) ⊂ âg cl A

**Theorem: 6.4** Let X be a topological space and let $A \subset X$. Then âg cl A = A $\cup$ âg Fr(A).

**Proof:** $A \subset$ âg cl A and âg Fr(A) $\subset$ âg cl A

Hence $A \cup$ âg Fr(A) $\subset$ âg cl A

Also âg Fr(A) = [âg int $A \cup$ âg ext A]'

= (âg int A)'$\cap$ (âg ext A)'

Again, since âg int $A \subset A$ and âg cl A = âg int $A \cup$ âg Fr(A), it follows that âg cl $A \subset A \cup$ âg Fr(A).

Hence âg cl A = $A \cup$ âg Fr(A).

**Remark: 6.5** Every âg closed subset is the disjoint union of its âg interior and âg frontier.

**Theorem: 6.6** Let X be a topological space and let A, B be subsets of X. Then
i) âg Fr(A) = âg cl $A \cap$ âg cl A' = âg cl A - âg int A.
ii) âg int A = A- âg Fr(A).
iii) (âg Fr (A))' = âg int $A \cup$ âg int A'.
iv) âg Fr(âg int A) $\subset$ âg Fr(A)
v) âg Fr (âg cl A) $\subset$âg Fr(A).
vi) âg Fr($A \cup B$) $\subset$âg Fr(A) $\cup$âg Fr(B).
vii)âg Fr($A \cap$ B) $\subset$âg Fr(A) $\cup$âg Fr(B).

**Proof:**
i) âg Fr(A) = (âg int A $\cup$ âg ext A)' = (âg int A)'$\cap$ ( âg ext A)'
= (âg cl A')" (âg cl A)" by theorem 3.5
= âg cl A' $\cap$ âg cl A

Now âg cl $A \cap$ âg cl A' = âg cl A – (âg cl A')' = âg cl A - âg int A.

Hence âg Fr(A) = âg cl A - âg int A.

ii) A- âg Fr(A) = A- (âg cl A - âg int A) = âg int A.

iii) (âg Fr(A))' = (âg cl A $\cap$ âg cl A')' = (âg cl A)' $\cup$ (âg cl A')'.

By cor 3.5 (âg cl A')' = âg int A.

So, âg int A' = (âg cl A)'

Hence (âg Fr(A))' = âg int A $\cup$ âg int A'.

iv) âg Fr (âg int A) = âg cl (âg int A) $\cap$ âg cl (âg int A)' by (i)
= âg cl (âg int A) $\cap$ âg cl (âg cl A') $\subset$ âg cl $A \cap$ âg cl A' = âg Fr(A).

v) âg Fr (âg cl A) = âg cl (âg cl A) $\cap$ âg cl (âg cl A)' = âg cl $A \cap$ âg cl (âg cl A)'

Now $A \subset$ âg cl A $\Rightarrow$ (âg cl A)'$\subset$A'$\Rightarrow$ âg cl (âg cl A)'$\subset$ âg cl A'.

Hence âg Fr (âg cl A) $\subset$ âg cl $A \cap$ âg cl A' = âg Fr (A).

vi) âg Fr($A \cup B$) = âg cl ($A \cup B$) $\cap$ âg cl ($A \cup B$)' = (âg cl (A) $\cup$âg cl (B) $\cap$âg cl (A'$\cap$B') $\subset$ (âg cl $A \cup$ âg cl B) $\cap$ (âg cl A' $\cap$âg cl B')
= (âg cl A $\cap$ (âg cl A' $\cap$âg cl B')) $\cup$ (âg cl B $\cap$ (âg cl A'$\cap$ âg cl B'))
= ((âg cl $A \cap$ âg cl A') $\cap$âg cl B') $\cup$ ((âg cl $B \cap$ âg cl B') $\cap$âg cl A')
= (âg Fr(A) $\cap$ âg cl B') $\cup$ (âg Fr(B) $\cap$ âg cl A') $\subset$âg Fr (A) $\cup$âg Fr(B)

vii) $\hat{\alpha}g\ Fr(A\cap B) = \hat{\alpha}g\ cl(A\cap B) \cap \hat{\alpha}g\ cl(A\cap B)' \subset (\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ B) \cap (\hat{\alpha}g\ cl(A' \cup B'))$
$= (\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ B) \cap (\hat{\alpha}g\ cl\ A' \cup \hat{\alpha}g\ cl\ B')$
$= ((\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ B) \cap \hat{\alpha}g\ cl\ A') \cup ((\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ B) \cap (\hat{\alpha}g\ cl\ B')$
$\subset (\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ A') \cup (\hat{\alpha}g\ cl\ B \cap \hat{\alpha}g\ cl\ B') = \hat{\alpha}g\ Fr(A) \cup \hat{\alpha}g\ Fr(B)$.

**Theorem: 6.8** Let X be a topological space and let $A \subset X$. Then
i) if A is open, then $\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A - A$
ii) $\hat{\alpha}g\ Fr(A) = \phi$ if and only if A is âg open as well as âg closed.
iii) A is âg open if and only if $A \cap \hat{\alpha}g\ Fr(A) = \phi$, that is $\hat{\alpha}g\ Fr(A) \subset A'$.
iv) A is âg closed if and only if $\hat{\alpha}g\ Fr(A) \subset A$.

**Proof:**
i) By theorem 6.6 $\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A - \hat{\alpha}g\ int\ A$
A is open $\Rightarrow$ A is âg open $\Rightarrow \hat{\alpha}g\ int\ A = A$
Hence $\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A - A$.

ii) Let $\hat{\alpha}g\ Fr(A) = \phi$
Then $\hat{\alpha}g\ Fr(A) = \phi \Rightarrow \hat{\alpha}g\ cl\ A - \hat{\alpha}g\ int\ A = \phi$
$\Rightarrow \hat{\alpha}g\ cl\ A = \hat{\alpha}g\ int\ A = A$
$\Rightarrow$ A is âg open as well as âg closed.

Conversely, let A be âg open as well as âg closed.
$\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A - \hat{\alpha}g\ int\ A$

Since A is âg closed, $\hat{\alpha}g\ cl\ A = A$

Since A is âg open, $\hat{\alpha}g\ int\ A = A$

Hence $\hat{\alpha}g\ Fr(A) = \phi$.

iii) By theorem 6.6 $\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ A'$

Let A be âg open. Hence A' is âg closed.

$\hat{\alpha}g\ cl\ A' = A'$

Now, $A \cap \hat{\alpha}g\ Fr(A) = A \cap (\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ A') = A \cap (\hat{\alpha}g\ cl\ A \cap A') = (A \cap \hat{\alpha}g\ cl\ A) \cap A' = A \cap A' = \phi$.

Conversely, let $A \cap \hat{\alpha}g\ Fr(A) = \phi$.

This impies $A \cap (\hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ A') = \phi$.
$\Rightarrow (A \cap \hat{\alpha}g\ cl\ A) \cap \hat{\alpha}g\ cl\ A' = \phi$.
$\Rightarrow A \cap \hat{\alpha}g\ cl\ A' = \phi$.
$\Rightarrow \hat{\alpha}g\ cl\ A' = A' \Rightarrow A'$ is âg closed $\Rightarrow$ A is âg open.

iv) Let A be âg closed. Then $\hat{\alpha}g\ cl\ A = A$.

Hence $\hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A \cap \hat{\alpha}g\ cl\ A' = A \cap \hat{\alpha}g\ cl\ A'$

So, $\hat{\alpha}g\ Fr(A) \subset A$

Conversely, let $\hat{\alpha}g\ Fr(A) \subset A$

Then $A \cup \hat{\alpha}g\ Fr(A) = A$

But $A \cup \hat{\alpha}g\ Fr(A) = \hat{\alpha}g\ cl\ A$, by theorem 6.4.

That is $A = \hat{\alpha}g\ cl\ A$. Hence A is âg closed.

**REFERENCES**

1. I. Arokiarani, "Studies on Generalisations of Generalised closed sets and maps in Topological spaces" PhD, Thesis, Bharathiar university, Coimbatore (1997).
2. S.P.Arya and R. Gupta, "On strongly continuous mappings", Kyungpook Math J14 (1974), 131-143.
3. K.Balachandran, P.Sundaram and H.Maki, "On Genrealised continuous maps in Topological spaces", Mem. I.ac.sci-Kochi, Univ Math 12(1991), 5-13.
4. P.Bhattacharya and B.K.Lahiri, "Semi generalized closed sets in Topology", Indian, J. Math 29(1987) 376-382.
5. Y.Gnanambal, "On Generalised pre-regular closed sets in Topological spaces", Indian. J. Pure Appl. Math. 28(1997) 351-360.
6. N.Levine, "Generalised closed sets in Topology", Rend. Circ. Mat. Palermo 19(1970) 89-96.
7. S.R.Malghan, "Generalized closed maps", J. Karnatak Uni. Sci. 27(1982) 82-88
8. N.Nagaveni, "Studies on Generalisations of Homeomorphisms in Topological spaces", Ph.D Thesis. Bharathiyar University(1999)
9. Y.Palaniappan, R.Krishnakumar and V.Senthilkumaran, "On $\hat{\alpha}$ Generalised closed sets in topology", Intl J. Math. Archive. Accepted.

**Source of support: Nil, Conflict of interest: None Declared**