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ALMOST AND WEAKLY S_S-CONTINUOUS FUNCTIONS

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ABSTRACT

In this paper, the relationships among almost semi continuity, weakly semi continuity, almost S_S –continuity and weakly S_S –continuity are discussed. The relationship between almost S_S –continuity and weakly S_S –continuity are established. Moreover, the topological important characterizations, restriction, extension, composition and cross product are investigated. We obtained some preservation theorems by using almost S_S –continuity and weakly S_S –continuity on special types of spaces namely Hausdorff and connected spaces. Finally, the relationship with graphs is studied.

Keywords: *S_S –open, S_S –continuous, almost continuous, almost semi-continuous, weak semi-continuous, almost S_S –continuous, weak S_S –continuous.*

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1. INTRODUCTION

In 1961[11], Levine defined weak continuity and gave several characterizations about this function and in 1963 [12], he defined another type of continuity called semi continuity. Arya and Bahmini [2] introduced weaker forms of semi continuity which are almost semi continuity and weakly semi continuity. Recently, the authors in [10], defined a new class of open sets called S_S –open and discussed the relation between S_S –open and other types of sets. Also they established the notion of S_S –continuity and several properties have been found. For a subset A of a space X, $cl(A)$ and $int(A)$ represent the closure and interior respectively. A subset A of X is called semi-open [11] (α-open [15], pre-open [13], regular open [19]) set if $A \subseteq cl\ int(A)$, (resp., $A \subseteq int\ cl\ int(A)$, $A \subseteq int\ cl(A)$, $A = int\ cl(A)$). The complement of semi-open (α-open, pre-open, regular open) set is called semi-closed (resp., α- closed, pre- closed, regular closed) set. A subset A of topological space (X, τ) is called θ-open (resp., δ-open) set [21] if for each $x \in A$, there is an open (resp., open) set U such that $x \in U \subseteq cl(U) \subseteq A$ (resp., $x \in U \subseteq int\ cl(U) \subseteq A$). The intersection of all semi-closed sets containing A is called the semi-closure [4] of A and it is denoted by $sclA$. The semi-interior of a set A is the union of all semi-open sets contained in A and is denoted by $sintA$. A subset A of a topological space X is said to be θ -semi-open [8] (resp., semi- θ -open [5]) set if for each $x \in A$, there is semi-open set U such that $x \in U \subseteq cl(U) \subseteq A$ (resp., $x \in U \subseteq scl(U) \subseteq A$). For more properties of semi- θ-open sets (see [22]) also. A subset A of a topological space X is said to be regular-semi-open [3] if there exists a regular-open set U such that $U \subseteq A \subseteq clU$ equivalently A is regular-semi-open [20] if and only if $A = sintscl\ A$. A set A is called semi-regular [6], if it is both semi-open and semi-closed and more applications of this set can be found in [7] also. The family of all semi open (resp., regular open, pre-open, regular-semi-open, θ -open, θ -semi-open, semi- θ -open, semi-regular) sets of X is denoted by $SO(X)$ (resp., $RO(X)$, $PO(X)$, $RSO(X)$, $\theta O(X)$, $\theta SO(X)$, $S\theta O(X)$, $SR(X)$. The main purpose of this paper is to study the notion of almost S_S –continuity and weakly S_S –continuity. Furthermore, basic properties of these functions and several results on almost S_S –continuity and weakly S_S –continuity are investigated and retaliations between weakly S_S –continuity and graph of functions are established.

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2. PRELIMINARIES

Proposition: 2.1 Let (Y, τ_Y) be subspace of the space (X, τ), then the following properties are true:

1) If A is semi open in X and $A \subseteq Y$, then A is semi open in Y.[11]
2) If A is semi open in Y and Y is semi open in X, then A is semi open in X.[1]

Definition: 2.2 ([5]) A topological space (X,τ) is called semi regular, if for each semi-closed F and each $x \notin F$ there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $x \in V$.

Theorem: 2.3 ([5]) A space X is semi-regular if and only if for every point $x \in X$ and every semi open G containing x, there exists a semi open Hsuch that $x \in H \subseteq scl(H) \subseteq G$.

Definition: 2.4 ([14]) A space X is said to be extremally disconnected if the closure of every open set in X is open or equivalently if the interior of every closed set in X is closed.

Definition: 2.5 A function $f: X \to Y$ is called:

1) totally continuous **[14]**, if the inverse image of each open set in Y is clopen subset of X.
2) almost continuous **[16]**, if the inverse image of each regular open subset of Y is open subset of X.
3) almost semi continuous (resp., weakly semi continuous) **[9],** if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a semi open (resp., semi open) set U in X containing x such that $f(U) \subseteq int\big(cl(V)\big)$ (resp., $f(U) \subseteq cl(V)$).

Definition: 2.6 [18] A topological space (X,τ) is called Ultra –Hausdorff. If for each pair of distinct points x and y in X, there are disjoint clopen sets G and H such that $x \in G$ and $y \in H$.

The following definitions and results can be found in [10].

Definition: 2.7 A semi open subset A of a space X is called S_S –open if for each $x \in A$, there is a semi closed set F such that $x \in F \subseteq A$.

The family of all S_S –open subsets of topological space (X,τ) is denoted by $S_SO(X)$.

Proposition: 2.8 If a space X is semi T_1, then $SO(X) = S_SO(X)$.

Proposition: 2.9 A subset A of a space (X, τ) is S_S –open if and only if for each $x \in A$, there exist S_S –open set B such that $x \in B \subseteq A$.

Proposition: 2.10 Let (X,τ) be topological space. If X is semi regular, then $\tau \subseteq S_SO(X)$.

Proposition: 2.11 Let X and Y be two topological spaces and $X \times Y$ be product topological space. If A is S_S –open subset of X and B is S_S –open subset of Y, then $A \times B$ is S_S –open subset of $X \times Y$.

Proposition: 2.12 Let (Y, τ_Y) be subspace of (X, τ) and $A \subseteq Y$. If A is S_S –open set of Y and Y is semi-regular set in X, then A is S_S –open set in X.

Definition: 2.13 A function $f: X \to Y$ is called S_S –continuous, if the inverse image of every open set in $Y$is an S_S –open set in X.

3. ALMOST S_S-CONTINUOUS FUNCTION

Definition: 3.1 A function $f: X \to Y$ is called almost S_S-continuous if for each $x \in X$ and each open set H in Y containing $f(x)$ there is S_S-open set G containing x such that $f(G) \subseteq int\ (cl(H))$.

The proof of the following results are clear from the definitions:

Proposition: 3.2 The following properties are true:

1) Every S_S-continuous function is an almost S_S-continuous function.
2) Every almost S_S- continuous function is almost semi continuous.

Proof: Follows from their definitions.

The converse of Proposition 3.2 is not true in general as it is shown in the following examples:

Example: 3.3 Let $X=\{a,b,c\}$ with $\tau=\{\phi, X,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Define the function $f:(X,\tau)\longrightarrow(X,\tau)$ by: $f(a)=c, f(b)=b, f(c)=a$. Clearly $S_SO(X,\tau)=\{\phi, X, \{b\},\{a,c\}\}$. Then f is almost S_S-continuous. But it is not S_S-continuous since $f^{-1}(\{a\})=\{c\}\notin S_SO(X,\tau)$.

Example: 3.4 Let $X=\{a,b,c\}$ with $\tau=\{\phi, X,\{a\},\{b\},\{a,b\},\{a,c\}\}$ and $\sigma=\{\phi, X,\{a\},\{a,b\}\}$. Define the function $f:(X,\sigma)\longrightarrow(X,\tau)$ by :$f(a)=b, f(b)=f(c)=a$. Then this function is almost semi continuous which is not almost S_S- continuous because $f^{-1}(\{b\})=\{a\}\notin S_SO(X,\sigma)$.

Proposition: 3.5 A $f:X\rightarrow Y$ is almost S_S-continuous if and only if the inverse image of every regular open subset of Y is an S_S-open subset of X.

Proof: let f be an almost S_S-continuous function, and let H be a regular open subset of Y. Let $x\in f^{-1}(H)$, so $f(x)\in H$. Since f is almost S_S-continuous function, then there is an S_S-open set U in X containing x such that $f(U)\subseteq int\ cl(H)=H$. Therefore, $x\in U\subseteq f^{-1}(H)$. Hence, by Proposition 2.9, $f^{-1}(H)$ is S_S-open.

Conversely. Let H be an open subset of Y containing $f(x)$, then $int\ cl(H)$ is regular open subset of Y containing $f(x)$. So by hypothesis, $f^{-1}(int\ cl(H))$ is S_S-open subset of x containing. Hence, by Proposition 2.9, there is an S_S-open subset U such that $x\in U\subseteq f^{-1}(int\ cl(H))$. Therefore, $f(U)\subseteq int\ cl(H)$ and hence f is almost S_S-continuous.

Corollary: 3.6 A $f:X\rightarrow Y$ is almost S_S-continuous if and only if the inverse image of every regular closed subset of Y is an S_S-closed subset of X.

Proof: Follows from Proposition 3.5.

Proposition: 3.7 Let $f:X\rightarrow Y$ be a function and X is semi T_1-space then f is almost S_S-continuous if and only if f is almost semi continuous

Proof: Follows from Proposition 2.8.

Corollary: 3.8 Let $f:X\rightarrow Y$ be an almost continuous function and X be a semi regular space, then f is almost S_S-continuous

Proof: Follows from proposition 2.10.

Proposition: 3.9 For a function $f:X\longrightarrow Y$, the following statements are equivalent:

1) f is almost S_S-continuous.
2) For each $x\in X$ and each δ-open set H in Y containing $f(x)$, there is an S_S-open set U in X containing x such that $f(U)\subseteq H$.
3) The inverse image of each δ-open set in Y is S_S-open in X.

Proof:
(1)⟹(2): Let H be δ-open set in Y containing $f(x)$, so there is an open set G in Y such that $f(x)\in G\subseteq int(cl(G)\subseteq H$. Since f is almost S_S-continuous, then there is an S_S-open subset U of containing x such that $f(U)\subseteq int\ cl(G)\subseteq H$.

(2) ⟹(3): Let H be a δ-open set in Y containing $f(x)$, then by (2), there is S_S-open subset U of containing x such that $f(U)\subseteq H$. Implies that $x\in U\subseteq f^{-1}(f(U))\subseteq f^{-1}(H)$. Therefore by proposition 2.9, $f^{-1}(H)$ is S_S-open set.

(3) ⟹(1): Let H be an open set in Y containing $f(x)$, then $int(cl(H))$ δ-open set in Y containing $f(x)$. Therefore by (3), $f^{-1}(int(cl(H)))$ is an S_S-open set in X containing x and by Proposition 2.9, there exists an S_S-open set U in X containing x such that $x\in U\subseteq f^{-1}(int(cl(H)))$. Therefore $f(U)\subseteq int\ cl(H)$. Hence f is almost S_S-continuous.

Proposition: 3.10 A function $f:X\rightarrow Y$ is almost S_S-continuous if and only if $f^{-1}(V)\subset S_S-int(f^{-1}(int(cl(V)))$ for every $V\in PO(Y)$.

Proof: Sufficiency. Suppose that f is almost S_S-continuous and let V be any pre-open set in Y, then $int(cl(V))$ is regular open set in Y. Since f is almost S_S-continuous, then $f^{-1}(int(cl(V))$ is S_S-open subset of X and since $V\subseteq int(cl(V))$, thus $f^{-1}(V)\subseteq f^{-1}(int(cl(V)))$. Hence $f^{-1}(V)\subseteq S_S-int(f^{-1}(int(cl(V)))$.

Necessity. Let V be a regular open set in Y, then V is pre-open set. Therefore, by hypothesis, we have

$f^{-1}(V)\subseteq S_S-int(f^{-1}(int(cl(V)))=S_S-int(f^{-1}(V))$.

Hence, $f^{-1}(V)$ is an S_S-open set, so f is almost S_S-continuous.

Lemma: 3.11[17] If V is any subset of Y, then $scl(V) = int(cl(V))$ for every $V \in PO(Y)$.

Corollary: 3.12 A function $f: X \to Y$ is almost S_S-continuous if and only if $f^{-1}(V) \subseteq S_S - int(f^{-1}(scl(V)))$ for every $V \in PO(Y)$.

Proof: Follows from Proposition 3.10 and Lemma 3.11.

Proposition: 3.13. Let $f: X \to Y$ be any function and let $\{H_\alpha : \alpha \in \Lambda\}$ be semi-regular cover of X. If $f|_{H_\alpha}: H_\alpha \to Y$ is almost S_S-continuous, then f is almost S_S-continuous.

Proof: Let W be a regular open subset of Y. then $f^{-1}(W) = X \cap f^{-1}(W) = \cup\{H_\alpha \cap f^{-1}(W) : \alpha \in \Lambda\} = \cup\{(f|_{H_\alpha})^{-1}(W): \alpha \in \Lambda\}$. Since$f|_{H_\alpha}$is almost S_S-continuous, then $(f|_{H_\alpha})^{-1}$(W) is S_S-open subset of H_α but H_α is semi-regular cover of X for each $\alpha \in \Lambda$ thus by Proposition 2.12, $(f|_{H_\alpha})^{-1}$(W) is S_S-open subset of X. Hence f is almost S_S-continuous.

Proposition: 3.14 Let $f: X \to Y$ and $g: Y \to Z$ be any two functions. If f is almost S_S-continuous and g is continuous then $g \circ f$ is almost S_S-continuous.

Proof: Let H be a regular open subset of Z. since g is continuous function, then $g^{-1}(H)$ is an open subset of Y. But f is almost S_S-continuous, then $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is an S_S-open subset of X. Hence $g \circ f$ is almost S_S-continuous.

Proposition: 3.15 If Y is hyperconnected, then every function $f: X \to Y$ is almost S_S-continuous.

Proof: Let Y be hyperconnected space and let H be an open subset of Y containing $f(x)$. Then $cl(H) = Y$ this implies that $int\, cl(H) = Y$. Thus for every S_S-open set U in X containing x, $f(U) \subset Y = int\, cl(H)$. Hence f is almost S_S-continuous.

Proposition: 3.16 If $f: X \to Y$ is almost S_S-continuous where Y is T_2-space and let $A = \{(x, y): y = f(x)\}$, then for each $(x, y) \notin A$, there is an S_S-open set $U \subseteq X$ containing x and an open subset V containing y such that $f(U) \cap cl(V) = \phi$.

Proof: Suppose that $(x, y) \notin A$ then $y \neq f(x)$. Since Y is T_2-space, there are two disjoint open subsets V and H such that $f(x) \in H$ and $y \in V$ and so $int\, cl(H) \cap cl(V) = \phi$. Since f is almost S_S-continuous, then there is an S_S-open set U containing x such that $f(U) \subseteq int\, cl(H)$. This implies that $f(U) \cap cl(V) = \phi$.

By (Y, ζ_δ) we mean the topology on the space Y in which the class of δ–open subsets of Y is a base for the topology.

Proposition: 3.17 A function $f: (X, \tau) \longrightarrow (Y, \zeta)$ is almost S_S-continuous if and only if $f: (X, \tau) \longrightarrow (Y, \zeta_\delta)$ is S_S-continuous.

Proof: Suppose that $f: (X, \tau) \longrightarrow (Y, \zeta)$ is almost S_S-continuous. Let $H \in \zeta_\delta$. Then H is δ-open in Y, therefore by Proposition 3.9, $f^{-1}(H)$ is S_S-open subset of X. Hence f is S_S-continuous.

Conversely, suppose that $f: (X, \tau) \longrightarrow (Y, \zeta_\delta)$ is S_S-continuous, and let H be an open subset of Y containing $f(x)$, then $int\, cl(H)$ is δ-open in Y containing $f(x)$. Thus $f^{-1}(int(cl(H))$ is S_S-open subset of X containing x and since $f(f^{-1}(int(cl(H))) \subseteq int\, cl(H)$. Take $U = f^{-1}(int\, cl(H))$, thus $f(U) \subseteq int\, cl(H)$. Hence f is almost S_S-continuous.

Proposition: 3.18 Let $f: X \to Y$ be a function if for each $x \in X$, there is a semi-regular set U containing x such that $f|_U: : U \to Y$ is almost S_S-continuous, then $f: X \to Y$ is almost S_S-continuous.

Proof: Let $x \in X$ and H be an open subset of Ycontaining $f(x)$, then by hypothesis, there is semi clopen set U containing x such that $f|_U$ is almost S_S-continuous. This implies that there is an S_S-open W in U such that $(f|_U)(W) \subseteq int\, cl(H)$. Since U is semi clopen then by proposition 2.12, W is an S_S-open subset in X. Therefore, $f(W) \subseteq int\, cl(H)$. Hence f is almost S_S-continuous.

4. WEAKLY S_S-CONTINUOUS FUNCTION

Definition: 4.1 A function $f: X \to Y$ is called weakly S_S-continuous. If for each $x \in X$ and each open set H in Y containing $f(x)$, there is an S_S-open set G containing x such that $f(G) \subseteq cl(H)$.

Proposition: 4.2 The following are true:
1) Every weakly S_S-continuous function is weakly semi continuous.
2) Every almost S_S-continuous function is weakly S_S-continuous.

Proof: Follows directly from their definitions

The converse of the above proposition is not true as it is shown in the following examples:

Example: 4.3 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. The function $f: (X, \tau) \to (X, \sigma)$ defined by $f(a) = a$, $f(b) = b$, $f(c) = c$ then f is weakly S_S-continuous.but it is not almost S_S-continuous.

Example: 4.4 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. The function $f: (X, \tau) \to (X, \sigma)$ defined by $f(a) = a$, $f(b) = b \textit{ and } f(c) = c$ is weakly semi continuous but it is not weakly S_S-continuous.

Theorem: 4.5 let Y be regular space. Then $f: X \to Y$ is weakly S_S-continuous if and only if f is S_S-continuous.

Proof: Suppose that f is weakly S_S-continuous. Let $x \in X$ and V be an open set in Y containing $f(x)$. Since Y is regular space, then there is an open set W in Y such that $f(x) \in W \subseteq cl(W) \subseteq V$. Since f is weakly S_S-continuous, then there is S_S-open set U such that $f(U) \subseteq cl(W) \subseteq V$. Hence f is S_S-continuous.

The converse part is obvious.

Lemma: 4.6 [16] If $f: X \to Y$ is continuous, then for any subset U of X, $f(cl(U)) \subseteq cl(f(U))$.

Proposition: 4.7 If $f: X \to Y$ is continuous, then f is weakly S_S-continuous.

Proof: For each $x \in X$, let H be an open subset of Y containing $f(x)$. Since f is continuous, then $f^{-1}(H)$ is an open subset of X containing x implies that $cl\ (f^{-1}(\text{H}))$ is regular closed. So, $cl\ (f^{-1}(H))$ is S_S-open.

Since $f(cl\ (f^{-1}(\text{H}))) \subseteq cl(\text{H})$ and take U= $cl\ (f^{-1}(\text{H}))$. Thus by Lemma 4.6. f is weakly S_S-continuous.

Proposition: 4.8 if a function $f: X \to Y$ is weakly S_S-continuous, then for each $x \in X$ and each θ-open H in Y containing $f(x)$, there is an S_S-open U in X containing x such that $f(U) \subseteq H$.

Proof: Let f be weakly S_S-continuous and let $x \in X$ and H be θ-open set in Y containing $f(x)$. Then there exist an open set G in Y such that $f(x) \in G \subseteq cl(G) \subset H$. Since f is weakly S_S-continuous, then there is an S_S-open U in X containing x such that $f(U) \subseteq cl(G) \subseteq H$. This implies that $f(U) \subseteq H$.

Corollary: 4.9 If a function $f: X \to Y$ is weakly S_S-continuous, then the inverse image of each θ-open set in Y is an S_S-open set in X.

Proof: Let H be a θ-open set in Y and let $f(x) \in H$. Since f is weakly S_S-continuous, then by Proposition 4.8. there is an S_S-open U in X containing x such that. This implies that $f(x) \in f(U) \subseteq H$ for all $x \in X$, since $f(U) \subseteq H$ then $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(H)$. So $U \subseteq f^{-1}(H)$, since U is S_S-open then by Proposition 2.9, $f^{-1}(H)$ is S_S-open.

By (Y, ξ_θ) we mean the topology on the space Y in which the class of θ–open subsets of Y is a base for the topology.

Theorem: 4.10 If a function $f: (X, \tau) \to (Y, \xi)$ is weakly S_S-continuous then $f: (X, \tau) \to (Y, \xi_\theta)$ is S_S- continuous

Proof: Let $H \in \xi_\theta$, then H is θ-open in Y. Since f is weakly S_S-continuous, then by Corollary 4.9, $f^{-1}(H)$ is S_S-open in X. Thus f is S_S-continuous.

Proposition: 4.11 Let $f: X \to Y$ be any function. If the inverse image of every regular closed subset of Y is an S_S-open subset of X, then f is weakly S_S-continuous.

Proof: Let H be an open subset of Y containing $f(x)$, then $cl\ int(H)$ is regular closed in Y containing $f(x)$. Thus by hypothesis, $f^{-1}(cl\ int(H))$ is S_S-open subset of X containing x since $f(f^{-1}(cl\ int(H))) \subseteq cl\ int(H) = cl\ (H)$. Hence f is weakly S_S-continuous.

Proposition: 4.12 Let $f: X \to Y$ be any function and X be extremally disconnected, then the following statements are equivalent:

1) f is weakly S_S-continuous.
2) The inverse image of each clopen subset of Y is S_S-open subset of X

Proof:
(1)⇒ **(2)**: Suppose that $f: X \to Y$ be weakly S_S-continuous and let A be clopen subset of X. Let $x \in f^{-1}(A)$,then$f(x) \in A$. Since f is weakly S_S-continuous, then there is an S_S-open set U in X containing x such that $f(U) \subseteq cl(A)= A$ implies that $x \in U \subseteq f^{-1}(A)$. Hence $f^{-1}(A)$ is S_S-open set in X.

(2) ⇒(1): Suppose the inverse image of clopen subset in Y is S_S-open. Let H be an open subset of Y containing $f(x)$. Since X is extremally disconnected then $cl(H)$ is clopen set in Y. So by hypothesis, $f^{-1}(cl(H))$ is S_S-open set in X.

Since
$f(f^{-1}(cl(H))) \subseteq cl(H)$. Take $U = f^{-1}(cl(H))$ then $f(U) \subseteq cl(H)$. Hence f is weakly S_S-continuous.

Proposition: 4.13 Let $f: X \to Y$ be a function and Y is extremally disconnected, then f is almost S_S-continuous if and only if it is weakly S_S-continuous.

Proof:
Necessity. Suppose that f is almost S_S-continuous, then it is clear that f is weakly S_S-continuous.

Sufficiency. Let H be an open subset of Y containing $f(x)$. Since f is weakly S_S-continuous, then there is an S_S-open set U in X containing x such that $f(U) \subseteq cl(H)$

But Y is extremally disconnected, then $cl(H) = int\, cl(H)$ implies that $f(U) \subseteq cl(H)= int\, cl(H)$. Hence f is almost S_S-continuous.

Corollary: 4.14 Let $f: X \to Y$ is weakly S_S-continuous and Y is extremally disconnected. If U is semi-regular in X then $f|_U: U \to Y$ is weakly S_S-continuous.

Proof: Follows from proposition 4.13 and proposition 3.18.

Proposition: 4.15 For a function $f: X \to Y$ the following statements are equivalent:

(1) f is weakly S_S-continuous.
(2) For each open subset $V$of Y, $f^{-1}(V) \subseteq S_S - int(f^{-1}(cl(V)))$
(3) For each open subset $V$of Y, $S_S - cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$

Proof:
(1)⇒(2): let V be an open set in Y and let $x \in f^{-1}(V)$, then $f(x) \in V$. Since f is weakly S_S-continuous, then there exists an S_S-open set in X conatining x such that $f(U) \subseteq cl(V)$, so $x \in U \subset f^{-1}(cl(V))$. Therefore, $x \in S_S - int(f^{-1}(cl(V)))$

(2)⇒(3): Let V be an open set in Y and let $W = Y \setminus cl(V)$, then by (2) $f^{-1}(W) \subseteq S_S - int(f^{-1}(cl(W))) \Rightarrow f^{-1}(Y \setminus cl(V)) \subseteq S_S - int(f^{-1}(cl(Y \setminus cl(V)))) \Rightarrow X \setminus f^{-1}(cl(V)) \subseteq S_S - int(f^{-1}(Y \setminus int(cl(V))) \Rightarrow X \setminus f^{-1}(cl(V)) \subseteq S_S - int(X \setminus f^{-1}(int(cl(V))) \Rightarrow X \setminus f^{-1}(cl(V)) \subseteq X \setminus S_S - int(f^{-1}(int(cl(V))) \subseteq X \setminus S_S - cl(f^{-1}(V)) \Rightarrow S_S - cl(f^{-1}(V)) \subseteq S_S - int(f^{-1}(int(cl(V)))$.

(3) ⇒(1): Let V be an open subset of Y containing $f(x)$, then $x \in f^{-1}(V)$. Let $W = Y \setminus cl(V)$, then by(3), we obtain that
$S_S - cl(f^{-1}(W)) \subseteq f^{-1}(cl(W)) \Rightarrow S_S - cl(f^{-1}(Y \setminus cl(V))) \subseteq f^{-1}(cl(Y \setminus cl(V))) \Rightarrow S_S - cl(X \setminus f^{-1}(cl(V))) \subseteq f^{-1}(Y \setminus int(cl(V))) \Rightarrow X \setminus S_S - int(f^{-1}(cl(V))) \subseteq X \setminus f^{-1}(int(cl(V))) \subseteq X \setminus f^{-1}(V)$ implies that $f^{-1}(V) \subseteq S_S - int(f^{-1}(cl(V))) \subseteq f^{-1}(cl(V))$. Take $H = S_S - int(f^{-1}(cl(V)))$, so H is an S_S-open set containing x and since $f(f^{-1}(V)) \subseteq f(H) \subseteq f(f^{-1}(cl(V))) \subseteq cl(V)$. Therefore $f(H) \subseteq cl(V)$. Hence f is weakly S_S-continuous.

Proposition: 4.16 Let $f: X \to Y$ be any function. If the function $g: X \to X \times Y$ defined by $g(x) = (x, f(x))$ is weakly S_S-continuous at each $x \in X$, then f is also weakly S_S-continuous.

Proof: Let $x \in X$ and $f(x) \in U$ where U is an open set containing $f(x)$ in Y, then $g(x) = (x, f(x)) \in X \times U$ and $X \times U$ ia an open set in $X \times Y$. Since g is weakly S_S-continuous, so there is an S_S-open set W in X containing x such that $g(W) \subseteq cl(X \times U) = X \times cl(U)$. It follows that $f(W) \subseteq cl(U)$. Hence f is weakly S_S-continuous.

Proposition: 4.17 If $f: X \to Y$ is weakly S_S-continuous and Y is ultra Hausdorff then $G(f)$ is S_S-closed subset of $X \times Y$.

Proof: Let $(x, y) \in X \times Y - G(f)$, then $f(x) \neq y$. Since Y is ultra Hausdorff space, so there exist two disjoint clopen sets V and H such that $y \in V$ and $f(x) \in H$. Since f is weakly S_S-continuous, then there exists an S_S-open set U in X containing x such that $f(U) \subseteq cl(H)$. Since $cl(H) \cap V = \phi$, thus $f(U)) \cap V = \phi$ and that implies $(x, y) \in U \times V \subset X \times Y - G(f)$ and by proposition 3.14, $U \times V$ is an S_S-open set. Therefore, $X \times Y - G(f)$ is S_S-open set in $X \times Y$ and hence $G(f)$ is S_S-closed subset of $X \times Y$.

Theorem: 4.18 If a function $f: X \to Y$ is weakly S_S-continuous and Y is Hausdorff, then for each $(x, y) \notin G(f)$, there is an S_S-open set U in X and $V \subseteq Y$ such that $x \in U, y \in V$ and $f(U) \cap int(cl(V)) = \phi$.

Proof: Suppose that $(x, y) \notin G(f)$, then $y \neq f(x)$. Since Y is Hausdorff, then there exist two disjoint open sets V and H such that $y \in V$ and $f(x) \in H$. Therefore, $int(cl(V)) \cap cl(H) = \phi$. Since f is weakly S_S-continuous, there exist an S_S-open U subset of X such that $f(U) \subset cl(H)$. Hence $f(U) \cap int(cl(V)) = \phi$.

Proposition: 4.19 if f is weakly S_S-continuous function and g is totally continuous from a space X into ultra Hausdorff Y, then the set $A = \{x \in X: f(x) = g(x)\}$ is S_S-closed in X.

Proof: If $x \notin A$, then we have $f(x) \neq g(x)$. Since Y is ultra Hausdorff, then there are two disjoint clopen sets U_1 and U_2 such that $f(x) \in U_1$ and $g(x) \in U_2$. Since f is weakly S_S-continuous, then there is an S_S-open set H_1 in X containing x such that $f(H_1) \subseteq U_1$. Since g is totally continuous, then there is clopen set H_2 containing x such that $(g(H_2) \subseteq U_2)$. Therefore, by Proposition 2.9, we obtain that $H_1 \cap H_2$ is an S_S-open set in X containing x and $(H_1 \cap H_2) \cap A = \phi$. Consequently A is S_S-closed.

Proposition: 4.20 If $f: X \longrightarrow Y$ is weakly S_S-continuous and Y is urysohn space, then the set$\{(x_1,x_2): f(x_1) = f(x_2)\}$ is S_S-closed in $X \times X$.

Proof: Let $B = \{(x_1, x_2): f(x_1) = f(x_2)\}$. If $(x_1, x_2) \notin B$, then $f(x_1) \neq f(x_2)$ in the space Y. Since Y is urysohn space, then there exist two open sets V and W such that $f(x_1) \in V$ and $f(x_2) \in W$ and $cl(V) \cap cl(W) = \phi$. Since f is weakly S_S-continuous, then there exist S_S-open sets U_1 and U_2 in X such that $x_1 \in U_1$ and $f(U_1) \subseteq cl(V)$ and $x_2 \in U_2$and $f(U_2) \subseteq cl(W)$. Thus $(f(U_1) \cap f(U_2) = \phi)$. Hence $(U_1 \times U_2) \cap B = \phi$ and by Proposition 2.14, $U_1 \times U_2$ is an S_S-open set in $X \times X$. Hence $(x_1, x_2) \notin S_{s\_}clB$ and therefore, B is S_S-closed in $X \times X$.

Proposition: 4.21 if $f: X \longrightarrow Y$ is open and weakly S_S-continuous, then $f(S_S\text{-}cl(A)) \subset cl(f(A))$ for each A an open subset of X.

Proof: Let A an open subset of X containing x. Since f is open, then $f(A)$ is an open subset of Y and since f is weakly S_S-continuous, then by Proposition 4.15, $S_S\text{-}cl(f^{-1}(f(A))) \subseteq f^{-1}(cl(f(A)))$ implies that $S_S\text{-}cl(A) \subseteq f^{-1}(cl(f(A)))$. Hence $f(S_S\text{-}cl(A)) \subseteq cl(f(A))$.

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