International Journal of Mathematical Archive-5(2), 2014, 302-306
Available online through www.ijma.info ISSN 2229 – 5046

E.A IN MENGER FOR COMMON FIXED POINT THEOREM

M RAMANA REDDY
Assistant Professor of Mathematics,
Sree Nidhi Institute of Science and Technology, Hyderabad-501 301, India.

(Received on: 29-01-14; Revised & Accepted on: 11-02-14)

ABSTRACT

In this paper, first we introduce the notion of common fixed point theorems using the E.A. in Menger Spaces with t-norm of Hadzix Type. We provide an application of our main theorem for finite families of mappings.

***Key Words**: Fixed Fixed, Common Fixed point E.A. Menger space.*

1. INTRODUCTION

Sehgal and Bharucha-Reid was introduced the theory of probabilistic metric spaces is an important part of stochastic analysis, and so it is of interest to develop the fixed point theory in such spaces. The first result from the fixed point theory in probabilistic metric spaces. Since then many fixed points theorems for single valued and multivalued mappings in probabilistic metric spaces have been proved in [2–8]. The study of metric spaces was initiated by Gahler [9] and some fixed point theorems in metric spaces were proved in [2, 8, 10–13]. In 1987, Zeng [14] gave the generalization of metric to probabilistic metric as follows.

In 2002, Aamri and Moutawakil [17] generalized the notion of noncompatible mapping to the E.A. property. It was pointed out in [17] that the property E.A. buys containment of ranges without any continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, the E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Recently, some common fixed point theorems in probabilistic metric spaces with fuzzy metric spaces by the E.A. property under weak compatibility have been obtained in [18–20].

2. PRELIMINARIES

A probabilistic 2-metric space is an ordered pair(X, F), where $X$is an arbitrary set and$F$is a mapping from X^3 into the set of distribution functions. The distribution function $F_{x,y,z,}$ (t) will denote the value of $F_{x,y,z}$ at the positive real number t. The function $F_{x,,}$ is assumed to satisfy the following conditions:

(i) $F_{x,y,z}(0) = 0$ for all $x, y, z \in X$;
(ii) $F_{x,y,z}(t) = 1$for all $t > 0$ if and only if atleast two of the three points $x, y, z$are equal;
(iii) for distinct points $x, y \in X$, there exists a point $z \in X$ such that$F_{x,y,z}(t) \neq 1$ for $t > 0$
(iv) $F_{x,y,z}(t) = F_{y,z,x}(t) = F_{z,x,y}(t) = \cdots$ for all $x, y, z \in X$ and $t > 0$;
(v) if $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$, and $F_{w,y,z}(t_3)=1$, then $F_{x,y,z}(t_1+t_2+t_3) = 1$ for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

Shi *et al.* [15] gave the notion ofn^{th} order t-norm as follows.

Definition: [1] A mapping Δ: $\Pi_{i=1}^{n}[0,1] \rightarrow [0,1]$ is called an nth order t-norm if the following conditions are satisfied:

(i) $\Delta(0,0,...,0) = 0$, $\Delta(a,1,1,...,1) = a$for all $a \in [0, 1]$;
(ii) $\Delta(a_1,a_2, a_3,...,a_n) = \Delta(a_2,a_1,a_3,...,a_n) = \Delta(a_2,a_3,a_1,...,a_n) \cdots = \Delta(a_2,a_3,a_4,...,a_n,a_1)$;
(iii) $a_i \geq b_i$, $i = 1, 2, 3..., n$, implies $\Delta(a_1,a_2,a_3,...,a_n) \geq \Delta(b_1,b_2,b_3,...,b_n)$;
(iv) $\Delta(\Delta(a_1,a_2,a_3,...,a_n), b_2,b_3,...,b_n) = \Delta(a_1, \Delta(a_2, a_3,...,a_n, b_2), b_3,...b_n)$
$= \Delta(a_1, a_2, \Delta(a_3,a_4,....,a_n,b_2,b_3), b_4,...,b_n)$
$= \cdots = \Delta(a_1,a_2,a_3,...,a_{n-1}, \Delta(a_n,b_2,b_3,...,b_n))$.

For $n = 2$, we have a binary t-norm, which is commonly known a space t-norm.

Basic examples of t-norm are the Lukasiewicz t-norm Δ_L, $\Delta_L(a, b) = \max(a+b-1, 0)$, t-norm Δ_P,$\Delta_P(a, b) = ab$, and t-norm Δ_M,$\Delta_M(a, b) = \min\{a, b\}$

Corresponding author: M Ramana Reddy. E-mail: mrreddyou@gmail.com*

Definition: [2] A special class of t-norms (called a Hadzic-type$t$-norm) is introduced as follows.

Let Δ be at-norm and let Δ_n: [0, 1] → [0, 1] ($n \in N$) be defined in the following way:

$\Delta_1(x) = \Delta(x, x)$, $\Delta_{n+1}(x) = \Delta(\Delta n\ (x), x)$ ($n \in N$, $x \in [0, 1]$). (1)

We say that the t-norm Δ is of H type if Δ is continuous and the family $\{\Delta_n (x), n \in N\}$ is equicontinuous at x=1.

The family $\{\Delta_n(x), n \in N\}$is equicontinuous at x=1 if for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$such that the following implication holds:

$$x > 1 - (\lambda) \text{ implies } \Delta(x) > 1 - \lambda \ \ \forall n \in N. \tag{2}$$

A trivial example of t-norm of H type is $\Delta = \Delta_M$.

Remark: [3] Every t-norm $\Delta_M$is of Hadzic type but the converse need not be true; There is a nice characterization of continuous t-norm.

(i) If there exists a strictly increasing sequence$\{b_n\}_{n\in N} \in [0,1]$ such that $\lim_{n\to\infty} b_n = 1$ and $\Delta(b_n,b_n) = b_n$ for all $n \in N$, then Δ is of Hadzic type.

(ii) If Δ is continuous and Δ is of Hadzic type, then there exists a sequence $\{b_n\}_{\in N}$ as in (i).

Definition: [4] If Δ is a t-norm and $(x_1, x_2, x_3,...,) \in [0,1]^n$ ($n \in N$), then $\Delta^n{}_{i=1}x_i$ is defined recurrely by 1, if $n = 1$ and $\Delta^n{}_{i=1}x_i = \Delta(\Delta^{n-1}{}_{i=1}x_i,\ x_n)$ for all $n \geq 2$. If $\{b_n\}_{\in N}$ is a sequence of numbers from [0,1], then $\Delta^\infty{}_{i=1}x_i$ is defined as $\lim_{n\to\infty}\Delta^n{}_{i=1}x_i$ (this limit always exists and $\Delta^\infty{}_{i=1}x_i$ as $\Delta^\infty{}_{i=1}x_{n+i}$.

Definition: [5] Let X be any nonempty set and D the set of all left-continuous distribution functions. A triplet (X, F, Δ) is said to be a 2-Menger space if the probabilistic 2-metric space (X, F) satisfies the following condition:

Definition: 6 A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be

(i) converge with limit x if $\lim_{n\to\infty} F_{xn,x,a}(t) = 1$ for all $t > 0$ and for every $a \in X$,

(ii) Cauchy sequence in X, if given $\epsilon > 0$, $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $F_{xn,,}(\epsilon) > 1 - \lambda\ \forall m, n > N_{\epsilon,\lambda,}$ for every $a \in X$, (3)

(iii) complete if every Cauchy sequence in X is convergent in X.

In 1996, Jungck's [16] introduced the notion of weakly compatible as follows.

Definition: [7] Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Definition: [8] (see [17]). Let f and g be two self-maps of a metric (X, d). The maps f and g are said to satisfy the E.A. property if there exists a sequence$\{x_n\}$in$X$such that

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = u \text{ for some } u \in X. \tag{4}$$

Now in a similar mode, we can state the E.A. property in 2-Menger space as follows.

Definition: [9] A pair of self-mappings(f,g) of 2-Menger spaces(X, F, Δ) is said to old the E.A. property if there exists a sequence$\{x_n\}$ in X such that

$$\lim_{n\to\infty} F_{fx_n,gx_n,p}(t) = 1 \ \ \forall t > 0,\ p \in X. \tag{5}$$

Example: [10] Let $X = [0,\infty)$ be the usual metric space. Define $f, g: X \to X$ by $fx = x/4$ and $gx = 3x/4$ for all $x \in X$. Consider the sequence $\{x_n\} = 1/n$. Since $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = 0$, then f and g satisfy the E.A. property.

Although E.A property is generalization of the concept of noncompatible maps, yet it requires either completeness of the whole space or any of the range spaces or continuity of maps. But on the contrary, the new notion of the CLR property (common limit range property) recently given by Sintunavarat and Kumam [21]does not impose such conditions. The importance of the CLR property ensures that one does not require the closeness of range subspaces.

Definition: [11] (see [21]) Two maps f and g on 2-Menger spaces X satisfy the common limit in the range of g (CLRg) property if $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = gx$ for some $x \in X$.

Example: [12.] Let $X = [0, \infty)$ be the usual metric space. Define $f, g: X \to X$ by $fx = x+1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = 1 + (1/n)$. Since $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = 2 = g_1$, f and g satisfy the CLRg property.

Now we state a lemma which is useful in our study.

Lemma: 13 (see [22]) Let (X, F, Δ) be a 2-Menger space. If there exists $q \in (0,1)$ such that $F(x, y, z, qt) \geq F(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq x$, $z \neq y$, and $t > 0$, then $x = y$.

3. MAIN RESULT

Theorem: 3.1 Le (X, F, Δ) be a complete 2-Menger space with continuous t-norm Δ of Hype Type. Let T and A be self mappings on x. Then T and A have a unique common fixed point in X if and only if there exist two self-mapping P, Q of x satisfying the following.

(3.1) $P(X) \subset T(X)$ and $Q(X) \subseteq A(X)$

(3.2) The pairs {P.A} and {Q.T} are weakly compatible

(3.3) There exists $K \in (0,1)$ such that for every x, y, $s \in X$ and t > 0.
$F[Pu, Qv, s, kt]^2 \geq \min\{F(Au, Tv, s, (x))^2. F(Au, Pu, s, (x). F (Tv, Qv, s,(x)). F(Au, Tv, s, (x)). F(Au, Qv, s (2x)\}$.

(3.4) One of the subset A(x).T(x).P(x).Q(x) is a closed subset of x. Indeed A.T.P.Q have a unique common fixed point in X.

Proof: For any point x_0 in X, there exists a point $x_1 \in X$, such that $Px_0 = STx_1$. For this point x_1, we can choose a point x_2 in X, such that $Qx_1 = ABx_2$ and so on, in this manner we can define a sequence $\{y_n\}$ in X such that

$y_{2n} = Px_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = Ax_{2n+2}$, for n = 0, 1, 2,……

Now we shall prove $Fy_{2n}, y_{2n+1,}, s,(kx) \geq Fy_{2n-1}, y_{2n},s, (x)$ for all x > 0, where $k \in (0,1)$.

Suppose that $Fy_{2n}, y_{2n+1},s,(kx) < Fy_{2n-1}, y_{2n},s(x)$. Then by using (4.3) and $Fy_{2n}, y_{2n+1},s, (kx) \leq Fy_{2n}, y_{2n+1},s,(x)$, we have

$$\begin{aligned}
[F(y_{2n}, y_{2n+1}, s,(kx))]^2 &= [F(Px_{2n}, Qx_{2n+1}s,(kx)]^2, \\
&\geq \min\{[F(y_{2n-1}, y_{2n}s,(x)]^2, F(y_{2n-1}, y_{2n},s,(x)).F(y_{2n}, y_{2n+1},s,(x)),F(y_{2n-1}, y_{2n},s,(x)).F(y_{2n-1}, y_{2n+1},s,(2x)\} \\
&\geq \min\{[Fy2_{n-1}, y_{2n,} s,(x)]^2, Fy_{2n-1}, y_{2n,}s,(x).Fy_{2n}, y_{2n+1}s,(x), [Fy2n-1, y_{2n},s,(x)]^2 Fy_{2n}, y_{2n+1},s,(x))\} \\
&\geq \min\{[Fy_{2n}, y_{2n+1},s,(kx)]^2, [Fy_{2n}, y_{2n+1}s,(kx)], [Fy_{2n}, y_{2n+1},s,kx)] [Fy_{2n}, y_{2n+1},s,kx)], \\
&\qquad [Fy_{2n}, y_{2n+1},s,kx)]\} \\
&= \{[Fy_{2n}, y_{2n+1},s,(kx)]^2\}
\end{aligned}$$

which is a contradiction. Thus we have

$Fy_{2n}, y_{2n+1,} s, (kx) \geq Fy_{2n-1}, y_{2n},s,(x)$

Similarly we can have $Fy_{2n+1}, y_{2n+2}, s, (kx) \geq Fy_{2n}, y_{2n+1}(x)$.

Therefore, for every $n \in N$, $Fy_n, y_{n+1}, s, (kx) \geq Fy_{n-1}, y_n, s(x)$.

$\{y_n\}$ is a Cauchy sequence in X. let $\varepsilon \in (0,1) be\ given\ since\ the\ t-norm\Delta$ is of H type there exists $\lambda \in (0,1)$ such that for all m,n $\in N$ with m>n $\Delta^{2^{m-n}}(1-\lambda) > 1-\epsilon$.

Since $\lim_{n\to\infty} F(y_{0,}y_1 \frac{t}{s^n})$ =1there exists $n_{0,} \in N$such that is complete, $\{y_n\}$ converges to a point z in X, and the subsequences $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_{2n}\}$ also converges to z with t-norm Δ is H type.

Now suppose that P is continuous, since P and A are weak compatible it follows from $(AB)Px_{2n} \to Pz$ and $PPx_{2n} \to Pz$ as $n \to \infty$.

Now putting $u = Px_{2n}$ and $v = x_{2n+1}$ in the equation (4.5), we have

$$\begin{aligned}
[FPPx_{2n}, Qx_{2n+1}, s, kx)]^2 \geq \min\Delta\{&[F(A)Px_{2n}, S,Tx_{2n+1}(x)]^2, N\ F(A)P\, x_{2n}, PPx_{2n}, s, x). FTx_{2n+1}, Qx_{2n+1}, s, x), \\
&F(A)Px_{2n}, Tx_{2n+1},s,(x)\}.
\end{aligned}$$

Taking the limit $n \to \infty$, we have

$[FPz, z, s, kx)]^2 \geq \min\Delta\{[FPz, z, s, x)]^2, FPz, Pz, s, x)]^2. Fz, z, s, (x), FP\, z, z, s, (x)\}$

which is a contradiction. Thus we have Pz = z. Since $P(X) \subset ST(X)$, there exists a point $u \in X$ such that z = Pz = Tp.

Again putting $u = Px_{2n}$ and $v = p$ in (4.3), we have

$$[FPPx_{2n}, Qp, s, (kx)]^2 \geq \min\Delta[F(A)Px_{2n}, Tp, s, (x)]^2, F(A)Px_{2n}, PPx_{2n}, s, x). FTp, Qp, s, (x), F(A)Px_{2n}, Tp, s, (x). F(A)Px_{2n}, F(A)Px_{2n}, Qp, s, (2x),$$

Taking the limit $n \to \infty$, we have

$$[Fz, Qp, s, (kx)]^2 \geq \Delta[Fz, Qp, s, (x)]^2$$

which is a contradiction, therefore $z = Qp$. Since Q and T are weak compatible of type (α) and $Tp = Qp = z$, $(T)Qp = Q(T)p$ and hence $STz = (T)Qp = Q(T)p = Qz$.

Again by putting $u = x_{2n}$ and $v = z$ in (4.3), we have

$$[FPx_{2n}, Qz,s,(kx)]^2 \geq \min\Delta\{[FAx_{2n}, Tz,s,(x)]^2, FAx_{2n}, Px_{2n},s,(x). FTz, Qz, s,(x), FAx_{2n}, Tz, s, (x).$$

Letting $n \to \infty$, we have

$$[Fz, Qz, s, (kx)]^2 \geq [Fz, Qz, s, (x)]^2$$

which is a contradiction, therefore we have $Qz = z$. Thus $Qz = STz = z$. Similarly since P and AB are weak compatible of type (α) we have $ABz = Pz = z$. Now we prove $Az = z$. Suppose that $Az \neq z$ then by putting $u = Az$ and $v = z$ in (3.3), we have

$$[FPAz, Qz, s, (kx)]^2 \geq \min\{[F(A)Az, Tz, s, (x)]^2, F(A)Az, PAz, s, (x). FTz, Qz(x), F(A)Az, Tz, s, (x).$$

which yields

$$[FAz, z(kx)]^2 \geq [FAz, z(x)]^2$$

which is a contradiction, there fore we have $Az = z$. Similarly if we put $u= Bz$ and $y=z$ in (3.4), we have

$$[FPz, Qz(kx)]^2 \geq \min\{[F(A)z, Tz(x)]^2, F(A)z, Pz(x). FTz, Qz(x), F(A)z, Tz(x).$$

which gives

$$[FBz, z(kx)]^2 \geq [FBz, z(x)]^2$$

which is a cotradication, therefore we have $z = z$. So $Az = z = z$. Finally we show that $Tz = z$. By using (3.3), we have

$$[Fz, Qz, s, (kx)]^2 \geq \min\{[Fz, (T)Sz, s, (x)]^2, Fz, z, s, (x). F(T)Sz, Qz, s, (x), Fz, (T)z, s, (x). Fz, Qz, s, (2x),$$

which gives

$$[Fz, z, s, (kx)]^2 \geq [Fz, z, s, (x)]^2$$

which is a contradiction, therefore we have $Tz = z$. So $Az = Tz = z$. Thus combining the results, we have $Pz = Qz = Az = Tz = z$. Thus z is a common fixed point of A, T, P and Q.

For uniqueness let w ($z \neq w$) be another common fixed point of A, T , P and Q, then by (3.3), we have

$$\begin{aligned}[Fz, w, s, (kx)]^2 &= [FPz, Qw, s, (kx)]2 \\ &\geq \min\{[Fz, w, s, (x)]^2, Fz, z, s, (x). Fw, w, s, (x), Fz, w, s, (x). Fz, w, s, (2x), [Fz, w, s, (x)]^2\end{aligned}$$

which is a contradiction, therefore $z = w$. Hence z is a unique common fixed point of A, T, P and Q.

If we put $T = I$ (I is identity mapping on X) in Theorem 3.1., we obtain the following result due to Pathak *et al*. [17].

Corollary: 4.1 Let (X, F, t) be a complete Menger space with $t(x; y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and P, Q, A and T be mappings from X into itself such that

(3.5) $P(X) \subset T(X)$ and $Q(X) \subset A(X)$,

(3.6) the pairs {P, A} and {Q, S} are weak compatible of type (α)

(3.7) P is continuous,

(3.8) $[FPu, Qv, s, (kx)]^2 \geq \min\{FAu, Tv, s, (x)]^2$, FAu, Pu, s, (x). FTv, Qv, s, (x), FAu, Qv, s,(2x), FAu, Tv, s, (x).

for all u, v ∈ X and x ≥ 0, where k ∈ (0, 1). Then P, Q, A and T have a unique common fixed point.

REFERENCES

[1] V. M. Sehgal and A. T. Bharucha-Reid, "Fixed points of contraction mappings on probabilistic metric spaces," Mathematical Systems Theory, vol.6, pp.97–102, 1972.

[2] O. Hadˇ zi´ c, "On common fixed point theorems in 2-metric spaces," Univerzitet u Novom Sadu, Zbornik Radova PrirodnoMatematiˇckog Fakulteta, vol.12, pp.7–18, 1982.

[3] O. Hadˇ zi´ c, "Some theorems on the fixed points in probabilistic metric and random normed spaces," Bollettino dell' Unione Matematica Italiana B, vol.1, no.6, pp.381–391, 1982.

[4] O. Hadˇ zi´ c, "Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces,"Fuzzy Sets and Systems, vol. 29, no. 1, pp. 115–125, 1989.

[5] O. Hadzic, "A fixed point theorem for multivalued mappings in 2-Menger spaces," Univerzitet u Novom Sadu, Zbornik Radova Prirodno-Matematiˇckog Fakulteta, vol.24, pp.1–7, 1994.

[6] O. Hadˇ zi´ c, "A fixed point theorem in Menger spaces," Institute Math´ ematique, vol. 40, pp. 107–112, 1979.

[7] O. Hadˇ zi´ candE.Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[8] T. L. Hicks, "Fixed point theory in probabilistic metric spaces," Univerzitet u Novom Sadu, Zbornik Radova PrirodnoMatematiˇckog Fakulteta, vol.13, pp.63–72, 1983.

[9] S. G¨ ahler, "2-metrische R¨ aume und ihre topologische Struktur," Mathematische Nachrichten,vol.26, pp.115–148, 1963.

[10] K. Iseki, "Fixed point theorems in 2-metric spaces," Mathematics Seminar Notes, vol.3, pp.133–136, 1975.

[11] B. G. Pachpatte, "Fixed point theorems for contraction type mappings on a 2-metric space,"Proceedings of the National Academy of Sciences, India A, vol.48, no.2, pp.94–102, 1978.

[12] B. E. Rhoades, "Contraction type mappings on a 2-metric space," Mathematische Nachrichten,vol.91,pp.151–155,1979.

[13] S.-S. Chang and N. J. Huang, "On the generalized 2-metric spaces and probabilistic 2-metric spaces with applications to fixed point theory," Mathematica Japonica, vol.34, no.6, pp. 885–900, 1989.

[14] W. Z. Zeng, "Probabilistic 2-metric spaces," Journal of Mathematical Research and Exposition, vol.7, no.2, pp.241–245, 1987.

[15] Y. Shi, L. Ren, and X. Wang, "The extension of fixed point theorems for set valued mapping," Journal of Applied Mathematics & Computing, vol.13, no.1-2, pp.277–286, 2003.

[16] G. Jungck, "Common fixed points for noncontinuous nonself maps on nonmetric spaces," Far East Journal of Mathematical Sciences, vol.4, no.2, pp.199–215, 1996.

[17] M. Aamri and D. El Moutawakil, "Some new common fixed point theorems under strict contractive conditions,"Journal of Mathematical Analysis and Applications, vol.270, no.1, pp.181– 188, 2002.

[18] M. Imdad and J. Ali, "Jungck's common fixed point theorem and E.A property," Acta Mathematica Sinica, vol.24, no.1, pp.87–94, 2008.

[19] G. Jungck and B. E. Rhoades, "Fixed points for set valued functions without continuity," Indian Journal of Pure and Applied Mathematics, vol.29, no.3, pp.227–238, 1998.

[20] H. K. Pathak, "Common fixed point theorems using property (E.A) in complex-valued metric spaces," Thai Journal of Mathematics. Inpress.

[21] W. Sintunavarat and P. Kumam, "Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces," Journal of Applied Mathematics, vol. 2011, Article ID 637958, 14 pages, 2011.

[22] J. Han, "A common fixed point theorem on Fuzzy 2-metric spaces," Journal of the Chung Cheong Mathematical Society, vol. 23, no. 4, 2010.

Source of support: Nil, Conflict of interest: None Declared