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PART I: NEW APPROACH TO THE CONCEPTS OF ROOTS AND VERTEX POINTS OF POLYNOMIALS

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ABSTRACT

When I learned high school the way how to determine the vertex point of second degree polynomial, I asked my teacher why not possible to determine the vertex points of polynomials for degree $n \geq 3$, he addressed me the concepts of derivatives in calculus. But through time, after I have learned the concepts of polynomials so that I addressed the problem. The aim of this paper is, in general to address elementary polynomials optimization for $n \geq 3$, without considered the concepts and application of calculus, so we can handle none linear polynomials optimization through constructing vertices differentiation polynomials and the concepts of polynomials.

Key words and phrases: *Vertices differentiation polynomials; vertex points and vertex values of polynomials; degenerate polynomials and degenerate points.*

INTRODUCTION

Among mathematician and physicists, before the emergence of calculus, there was difficult to determine the vertex points and vertex values of polynomials for their degree $n \geq 3$. So, in this paper, I attempt to develop a general method to address the ways to determine the vertex points, vertex values, existence and uniqueness of roots and monotonic intervals of nth degree polynomials for $n \geq 3$, from their vertex differentiation polynomials without being considered the concepts of derivatives.

Definition: 1 Vertex point is a point in the domain D such that a polynomial getting either its least values or large values at the turning points of polynomial. This point is called the vertex point of polynomials. Therefore, by definition the vertex point of polynomials exists at the corresponding turning points of polynomials.

Definition: 2 Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots\dots\dots + a_2x^2 + a_1x + a_0$, (1).
where $a_n \neq 0, a_{n-1}, \dots, a_3, a_2, a_1$ and a_0 real coefficients. Let $b_{1,i}$ be vertex points of $P(x)$, then $(x - b_{1,i})^2$ quadratic divisors of $P(x)$ with constant reminder values $P(b_{1,i})$, for $i = 1,2,\dots,n-1$.

Lemma: 1 Let $P(x)$ a polynomial of degree greater than or equal to 2 and $a_n \neq 0$;

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots\dots\dots + a_2x^2 + a_1x + a_0, (1)$$

Let $b_{1,i}$ be vertex points of $P(x)$, for $i = 1,2,\dots,n-1$. Then there exist quotient factors $G_i(x)$ and constant remainder values $P(b_{1,i})$, for $i = 1,2,\dots,n-1$ of $P(x)$ such that;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,j}\right), \text{for } i,j = 1,2,\dots,n-1, \ (2)$$

Where $G_i(x) = a_n \prod_{j=2}^{n-1}\left(x + b_{j,i}\right)$, for $i = 1,2,\dots,n-1$ and $j = 2,3,\dots,n-1$ quotient factors and $P(b_{1,i})$ remainder values at the vertex points $b_{1,i}$, for $i = 1,2,\dots n-1$.

Proof: By induction on n, we can generalize for $n \geq 2$. Let n=2, $a_2 \neq 0$ and $P(x) = a_2x^2 + a_1x + a_0$, (1). Let b_1 be a vertex point by definition (2), $(x - b_1)^2$ a quadratic divisor. Then by division algorithm, there exist constant reminder values $K_1 = P(b_1)$ and quotient factor a_2 such that; $P(x) = a_2(x - b_1)^2 + P(b_1)$, for $b_1 = \frac{a_1}{2a_1}$ (2).

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Similarly, let n=3; $P(x) = a_3x^3+a_2x^2 + a_1x + a_0$, (3). Let $b_{1,i}$ for $i = 1, 2$ vertex points of P(x), by definition (2), $(x - b_{1,i})^2$, for $i = 1, 2$ quadratic divisors. Then by division algorithm, there exist constant reminder values $K_i = P(b2,i)$ for i=1, 2 and quotient factors Gix such that Px =Gix x−b1,i2+P(b1,i), for i=1, 2, (4). Moreover, by using division algorithm, we can state the existence of constant remainder values P $(b_{1,i})$ and quotient factors $G_i(x)$, for $i = 1,2,..,n-1$, for any n^{th} degree polynomial P(x);

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,j}), \text{for } i, j = 1,2,..,n-1, \ (2).$$

Where G_i (x)= $a_n \prod_{j=2}^{n-1}(x + b_{j,i})$, for $i = 1,2,..,n-1$ and $j = 2,3,..,n-1$ are quotient factors and $P(b_{1,i})$ remainder values at the vertex points $b_{1,i}$, for $i = 1,2,..n-1$.

Theorem: 1 Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2+a_1x+a_o$, (1). Where $a_n \neq 0, a_{n-1}, \dots, a_3, a_2, a_1$ and a_0 are coefficients of P(x). Suppose P(x) can be written as;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i}), \text{for } i = 1,2,.,n-1, \ (2)$$

Where $G_i(x) = a_n \prod_{j=1}^{n-2}(x + b_{j+1,i})$ degenerate polynomials, $b_{1,i}$ vertex points and $P(b_{1,i})$ vertex values of P(x), for $i = 1,2, \dots \dots, n-1$. Then $b_{1,i}$, for $i = 1,2,..,n-1$ satisfy the vertex equation;

$$na_nb_{1,i}^{n-1} + (n-1)a_{n-1}b_{1,i}^{n-2} + (n-2)a_{n-2}b_{1,i}^{n-3} + \cdots 2_3a_2b_{1,i} + a_1 = 0, (3)$$

Proof (1): By induction on n, if n =2, it is trivial that $b_{1,i}$ for i=1 and $G_1(x) = a_2$ is constant such that;

$P(x) = a_2(x - b_{1,1})^2 + P(b_{1,1})$, (1). Then $b_{1,1}$ satisfies $2a_2b_{1,1} + a_1 = 0$, (2). From (1) and (2);

$$P(x) = a_2\left(x + \frac{a_1}{2a_2}\right)^2 + P\left(-\frac{a_1}{2a_2}\right), (3)$$

Let for n=3 and $a_3 \neq 0$, such that $P(x) = a_3x^3+a_2x^2 + a_1x + a_0$, (4). From (4), we have two polynomials such that;

$$P(x) = a_3(x + b_{2,i})(x - b_{1,i})^2 + P(b_{1,i}), \text{for } i = 1, 2, (5)$$

where $b_{1,1}, b_{1,2}$ and $b_{2,1}, b_{2,2}$ satisfy;

$$b_{2,i} = \frac{a_2}{a_3} + 2b_{1,i}, \text{for } i = 1,2, (6)$$

$$b_{1,i}^2 - 2b_{1,i}b_{2,i} = \frac{a_1}{a_3}, \text{for } i = 1,2, (7)$$

From (6) and (7), we can reduced in to an equation containing only variables $b_{1,i}$;

$3a_3b_{1,i}^2 + 2a_2b_{1,i} + a_1 = 0, for\ i = 1,2$, (8). The vertex equation

Next, consider for n=4; $P(x) = a_4x^4 + a_3x^3+a_2x^2 + a_1x + a_0$, (9)

From (9), we have we have three polynomials such that;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i}), for\ i = 1,2,3, (10)$$

Where $G_i(x) = a_4(x + b_{3,i})(x + b_{2,i})$,the points $b_{1,i}$and $b_{j,i}$, for i=1,2,3 , j=2,3 satisfy;

$$b_{3,i} + b_{2,i} = \frac{a_3}{a_4} + 2b_{1,i}, for\ i = 1,2,3, (11)$$

$$b_{1,i}^2 - 2b_{1,i}(b_{3,i} + b_{2,i}) + b_{2,i}b_{3,i} = \frac{a_2}{a_4}, for\ i = 1,2,3, (12)$$

$$b_{1,i}^2(b_{3,i} + b_{2,i}) - 2b_{1,i}b_{2,i}b_{3,i} = \frac{a_1}{a_4}, for\ i = 1,2,3, (13)$$

By using equation from (11) - (13), construct an equation containing only point $b_{1,i}$, for i=1, 2, 3;
$4a_4b_{1,i}^3+3a_3b_{1,i}^2 + 2a_2b_{1,i} + a_1 = 0, for\ i = 1,2,3$, (14) vertex equations of P(x)

Finally, consider for n = 5; $P(x) = a_5x^5 + a_4x^4 + a_3x^3+a_2x^2 + a_1x + a_0$, (15)

So, we have four polynomials such that;

$$P(x) = G_i(x)(x - b_{1,i})^2 + P(b_{1,i}), for\ i = 1,2,..,4, (16)$$

Where $G_i(x) = a_5(x + b_{4,i})(x + b_{3,i})(x + b_{2,i})$, then equation containing only $b_{1,i}$, for i=1,2,3,4; $5a_5b_{1,i}^4+4a_4b_{1,i}^3+3a_3b_{1,i}^2 + 2a_2b_{1,i} + a_1 = 0, for\ i = 1,2,3,4$, (17), vertex equation

By induction on n, we can generalize for all $n \geq 2$. Then from those formulation for vertex equations for $n = 2, n = 3, n = 4$ and $n = 5$. Thus, for any n[th] degree polynomial P (x), we can generalize for vertex equation by induction on n. Therefore, vertex equation for P(x) defined by;

$$na_n {b_{1,i}}^{n-1} + (n-1)a_{n-1}{b_{1,i}}^{n-2} \dots\dots + 2a_2 b_{1,i} + a_1 = 0, for\ i = 1,2,3, \dots\dots n-1 \ (18)$$

Theorem: 2 Let $b_{1,k}$ vertex points, for $k = 1,2,3, \dots, n-1$ and $\left(x - b_{1,k}\right)^2$ quadratic factors of P(x), for $k = 1,2, \dots, n-1$. Then there exists a polynomial equation of degree $(n-2)$ such that, $C_{n-2}{b_{k,i}}^{n-2} + C_{n-3}{b_{k,i}}^{n-3} + \cdots \dots + C_2{b_{k,i}}^2 + C_1 b_{k,i} + C_0 = 0$, (1), $For\ i = 1,2, \dots, n-1\ and\ k = 2,3, \dots, n-1$

Where coefficients $C_j, for\ j = 0,1,2, \dots, n-2$ of (3) described in terms of vertex points $b_{1,k}, for\ k = 1,2,3, \dots, n-1$ and coefficients of P(x), such that for each k, coefficients C_j given by;

$$C_j = \sum_{i=j+1}^{n-1} (-1)^{j+1} (i-j) a_{i+1} {b_{1,k}}^{i-(j+1)}, (2)$$

Proof: By induction on n, generalized (2). Let's consider P(x) of degree n=3, such that $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, (3). Where $a_3 \neq 0$, a_2 $a_1, and\ a_0$ are real coefficients of (1).From (3), we have two polynomials such that;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P(b_{1,i}), for\ i = 1, 2, (4)$$

Let $G_i(x) = a_3\left(x + b_{2,i}\right)$, then $b_{1,1}, b_{1,2}$ and $b_{2,1}, b_{2,2}$ satisfy the relation;

$$b_{2,i} = \frac{a_2}{a_3} + 2b_{1,i}, for\ i = 1,2, (5)$$

$${b_{1,i}}^2 - 2b_{1,i}b_{2,i} = \frac{a_1}{a_3}, for\ i = 1,2, (6)$$

From (5) and (6), there exist two polynomial equation of degree 1 containing $b_{j,i}$, for i=1, 2, j=2;

$$a_3 b_{2,i} - \left[\, a_2 + 2a_3 b_{1,i}\right] = 0, for\ i = 1,2, (7), \text{Where } C_1 = a_3, \text{ and } C_0 = -\left[\, a_2 + 2a_3 b_{1,i}\right]$$

Equations (7) $for\ i = 1,2\ and\ j = 2$ are called polynomial equations of degree 1and $b_{2,i}, for\ i = 1,2$ are called degenerate points of P(x). Next, let n=4; $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, (8). Where $a_4 \neq 0$, a_3 , a_2 $a_1, and\ a_0$ are real constant coefficients. Here, we have three polynomials of the form;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P(b_{1,i}), for\ i = 1, 2,3, (9)$$

Let $G_i(x) = a_4\left(x + b_{3,i}\right)\left(x + b_{2,i}\right)$, then $b_{1,i}$ and $b_{j,i}$, for i=1,2,3 , j=2,3 satisfy;

$$b_{3,i} + b_{2,i} = \frac{a_3}{a_4} + 2b_{1,i}, for\ i = 1,2,3, (10)$$

$${b_{1,i}}^2 - 2b_{1,i}(b_{3,i} + b_{2,i}) + b_{2,i}b_{3,i} = \frac{a_2}{a_4}, for\ i = 1,2,3, (11)$$

$${b_{1,i}}^2(b_{3,i} + b_{2,i}) - 2b_{1,i}b_{2,i}b_{3,i} = \frac{a_1}{a_4}, for\ i = 1,2,3, (12)$$

By using (10) - (12), there exist polynomial equations of degree 2 containing points $b_{j,i}$, for i=1,2,3, j=2,3,;

$$-a_4{b_{j,i}}^2 + \left[2b_{1,i} + a_3\right]b_{j,i} - \left[3a_4{b_{1,i}}^2 + 2a_3b_{1,i} + a_2\right] = 0, (13)$$

Then, $for\ \ i = 1,2,3$ and j=2, 3, (13) are called six polynomial equation of degree 2, where $C_2 = -a_4$, $C_1 = 2b_{1,i} + a_3$ and $C_0 = -\left[3a_4{b_{1,i}}^2 + 2a_3b_{1,i} + a_2\right]$,

Finally, let n = 5; $P(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, (14).Where $a_5 \neq 0, a_4, a_3$, a_2 $a_1, and\ a_0$ are real constant coefficients. From (14), we have four polynomials of the form;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P(b_{1,i}), for\ i = 1,2,..,4 \ (15)$$

Let $G_i(x) = a_5\left(x + b_{4,i}\right)\left(x + b_{3,i}\right)\left(x + b_{2,i}\right)$, then there exist polynomial equations of degree (n-2) = 3 containing $b_{j,i}$, for i=1,2….,4, j=2,3,4;

$$a_5{b_{j,i}}^3 - [2a_5 + a_4]{b_{j,i}}^2 + \left[3a_5{b_{1,i}}^2 + 2a_4b_{1,i} + a_3\right]b_{j,i} - \left[4a_5{b_{1,i}}^3 + 3a_4{b_{1,i}}^2 + 2a_3b_{1,i} + a_2\right] = 0, (16)$$

From (16), we can reduce in to the 3rd degree polynomial equations;

$$C_3{b_{ji}}^3 + C_2{b_{j,i}}^2 + C_1b_{j,i} + C_0 = 0, for\ j = 2,3,4\ and\ i = 1,2,..4, (17)$$

Where C_j in terms of $b_{1,i}, for\ i = 1,2,3,4$ given by; $C_3 = a_5$,$C_2 = -[2a_5 + a_4]$, $C_1 = 3a_5{b_{1,i}}^2 + 2a_4b_{1,i} + a_3$ and $C_0 = -\left[4a_5{b_{1,i}}^3 + 3a_4{b_{1,i}}^2 + 2a_3b_{1,i} + a_2\right]$. Therefore, by induction on n, there exist polynomial equations of degree $(n-2)$ such that;

$$C_{n-2}{b_{j,i}}^{n-2} + C_{n-3}{b_{j,i}}^{n-3} + C_{n-4}{b_{j,i}}^{n-4} + \cdots C_1b_{j,i} + C_0 = 0, (18)$$

The coefficients C_j, $for\ j = 2,3, \dots n-1\ and\ i = 1,2,..n-1$ can be generalize to a formula for $n \geq 3$ by;

$$C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}, for\ \ j = 0,1,2, \dots n-2\ , i = 1,2, \dots., n-1, \quad (19)$$

Definition: 3: Let's consider polynomial P(x) of degree n;

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_o, \quad (1)$$

Where $a_n \neq 0, a_{n-1}, a_{n-2}, \dots\dots, a_3, a_2\ \ , a_1\ and\ a_0$ real coefficients of $P(x)$. Let $b_{1,i}$ be vertex points, for $i = 1,2,..,n-1$ and $-b_{j,i}, for\ i = 1,2,..,n-1\ and\ j = 2,..,n-1$ are degenerate points of P(x). Then P(x) can be written as;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,i}\right), for\ i = 1,2,3 \dots., n-1, \quad (2)$$

The functional values of P(x) either getting its least values or large values at the turning points are called the vertex values of P(x) at the vertex points defined by;

$$P(\,b_{1,i}) = a_0 - a_n b_{1,i}{}^2 \prod_{j=2}^{n-1} b_{j,i}, for\ i = 1,2, \dots., n-1, (3)$$

Therefore, the vertex values of polynomials getting constant values at small neighborhoods of the vertex points.

Theorem: 3 Let $b_{1,i}, for\ i = 1,2,..n-1$ be vertex points of a monic polynomial P(x) of degree n. Then there exist unique degenerate factors $G_i(x)$ and unique vertex values $K_i = P\left(b_{1,i}\ \right), for\ i = 1,2,..n-1$ of P(x) such that (n-1) unique vertices differentiation polynomials of P(x) defined by;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,i}\right), for\ i = 1,2,3 \dots., n-1, (1)$$

where $b_{1,i}$ vertex points, $P\left(b_{1,i}\right)$ vertex values and $G_i(x) = a_n \prod_{j=2}^{n-1}\left(x + b_{j,i}\right)$ degenerate polynomials of P(x), $for\ i = 1,2, \dots., n-1, (2)$

Proof: suppose there exist degenerate quotient factors $G_{1,m}(x)$ and $G_{2,m}(x)$ and constant reminder values $K_{1,m} = P\left(b_{1,m}\right)$ and $K_{2,m} = P(b_{1,m}\ )$, for a particular index i=m, and vertex point $b_{1,m}$ such that;

$$P(x) = \mathrm{G}_{1,m}(x)\left(x - b_{1,m}\right)^2 + K_{1,m} \text{ and } P(x) = G_{2,m}(x)\left(x - b_{1,m}\right)^2 + K_{2,m}, (2)$$

With $degK_{1,m} < degQ(x)$ and $degK_{2,m} < degQ(x) = \left(x - b_{1,i}\right)^2$.

From (2), we have $K_{1,m} = P(x) - G_{1,m}(x)\left(x - b_{1,m}\right)^2 and\ K_{2,m} = P(x) - G_{2,m}(x)\left(x - b_{1,m}\right)^2$, (3).

Then $K_{1,m} - K_{2,m} = \left(x - b_{1,m}\right)^2\left[G_{2,m}(x) - G_{1,m}(x)\right]$, (4).

Therefore, from (4) $Q(x) = \left(x - b_{1,i}\right)^2$ factor to $K_{1,m} - K_{2,m}$ and by division theorem, as $degree(K_{1,m} - K_{2,m}) \leq Max\{degK_{1,m},\ degK_{2,m}\} < degQ(x)$, (5), it follows that $K_{1,m} - K_{2,m} = 0$ as a result $K_{1,m} = K_{2,m}$ and from (4); $\left(x - b_{1,m}\right)^2\left[G_{2,m}(x) - G_{1,m}(x)\right] = 0, \forall x \in \mathbb{R}$ and it suggests that $\forall x \in \mathbb{R} | b_{1,m}, (x - b1,m2 \neq 0$.so, $G2,mx - G1,mx = 0, \forall x \in \mathbb{R}$, it implies $G2,mx = G1,mx, \forall x \in \mathbb{R}$. Hence, we can uniquely define the vertices differentiation polynomials in (1).

Definition: 4 Vertices differentiation polynomial: a function of P(x) that is uniquely represents a particular vertex point with the corresponding vertex value of P(x) such that for any degree n;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,j}\right), for\ \mathrm{i}, j = 1,2,..,n-1, (1)$$

Thus (1) are called Vertices differentiation polynomials of P(x). Where G_i (x) $= a_n \prod_{j=2}^{n-1}\left(x + b_{j,i}\right), for\ i = 1,2,..,n-1$ are degenerate polynomials of degree (n-2) $, K_i = P(b_{1,i}\ )\ for\ i = 1,2,..n-1$ are vertices values of P(x) at the vertex points $b_{1,i}, for\ i = 1,2,..n-1$, and $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}, for\ i = 1,2, \dots, n-1$ are degenerate points.

PROPERTIES OF DEGENERATE POLYNOMIALS

Definition: 5 The points in real space D is said to be degenerate, if the polynomial P(x) of degree n has more than one points in space such that it satisfy the same functional value of P(x).

$$P\left(-b_{n-1,i}\right) = P\left(-b_{n-2,i}\right) \dots\dots = P\left(-b_{3,i}\right) = P\left(-b_{2,i}\right) = P\left(b_{1,i}\right) = K_i\ , for\ i = 1,2,.,n-1, (1)$$

Where K_i, $for\ i = 1,2,3 \dots., n-1$ are functional vertex values and the negative of the points $-b_{2,i}, \dots., -b_{n-2,i}\ and\ -b_{n-1,i}, for\ i = 1, \dots., n-1$, are the degenerate points.

Lemma: 2 Let $G_i(x)$ be degenerate polynomials, $for\ i = 1,2, \dots\dots, n-1$ defined by;

$$G_i(x) = d_{n-2}x^{n-2} + d_{n-3}x^{n-3} + \cdots \dots . + d_2x^2 + d_1x + d_0, for\ i = 1,2, \dots\dots, n-1, (1)$$

Let $-b_{m,j}, for\ m = 2,3, \dots n-2$ and $j = 1,2, \dots n-2$ are degenerate points/roots of $G_i(x)$, only if $b_{m,j}, for\ m = 2,3, \dots n-2$ and $j = 1,2, \dots n-2$ roots of polynomials equation;

$C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \cdots \dots . + C_2x^2 + C_1x + C_0 = 0$, (2). Where C_j given by;

$C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}, for\ j = 0,1,2, \dots n-2, i = 1,2, \dots\dots, n-1,$ (3)

Then $d_j, for\ j = 0,1,2, \dots n-2$ of $G_i(x)$ in terms of coefficients C_j of (2) given;

$d_j = \sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}\, b_{1,k}{}^{i-(j+1)}, for\ j = 0,1,2, \dots n-2, i = 1,2, \dots\dots, n-1,$ (4)

Proof: let $G_i(x) = a_n \prod_{m=2}^{n-1}(x + b_{m,i})$ be degenerate polynomials $for\ i = 1,2, \dots\dots, n-1$, then by lemma (), there exists polynomial equations of degree (n-2) with roots $b_{m,j}, for\ m = 2,3, \dots n-2$ and $j = 0,1,2, \dots n-2$ at a particular vertex point $b_{1,k} = 1$;

$c_{n-2}\, b_{m,k}{}^{n-2} + c_{n-3}b_{m,k}{}^{n-3} + c_{n-4}b_{m,k}{}^{n-4} + \cdots + c_1b_{m,k} + c_0 = 0$, (5)

$$C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}, for\ j = 0,1,2, \dots n-2, (6)$$

For arbitrary x, then there exists a polynomial h (x) of degree (n-2) such that;

$$h(x) = c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + c_{n-4}x^{n-4} + \cdots + c_1x + c_0, (7)$$

where $C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}, for\ j = 0,1,2, \dots n-2$

From (5), at most (n-2) points of $b_{m,k}$ are roots of (5). But, we need to have negative roots $x = -b_{m,k}, for\ m = 0,1,2, \dots n-2$, of (5),to have the coefficients of $G_i(x) = a_n \prod_{m=2}^{n-1}(x + b_{m,i})$ so we should convert all coefficients to the same sign. Because, for each even index $j = 0,2,4,6 \dots, we\ have\ negative\ C_j$ and for each odd index $j = 1,3,5,7 \dots$, we have positive coefficients. Assume n an even degree, so by using odd power of $x = -b_{m,k}, m = 0,1,2, \dots n-2$, then we obtain the same sign, i.e;

$c_{n-2}\left(-b_{m,k}\right)^{n-2} + c_{n-3}\left(-b_{m,k}\right)^{n-3} + c_{n-4}\left(-b_{m,k}\right)^{n-4} + \cdots + c_1\left(-b_{m,k}\right) + c_0 = 0$, (8)

$c_{n-2}\, b_{m,k}{}^{n-2} - c_{n-3}b_{m,k}{}^{n-3} + c_{n-4}b_{m,k}{}^{n-4} - c_{n-5}b_{m,k}{}^{n-5} + \cdots + -c_1b_{m,k} + c_0 = 0$, (9)

Then, we have $d_{n-2}\, b_{m,k}{}^{n-2} + d_{n-3}b_{m,k}{}^{n-3} + d_{n-4}b_{m,k}{}^{n-4} + \cdots + d_1b_{m,k} + d_0 = 0$, (10).

Where $d_j = (-1)^j C_j = (-1)^j \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}, \ for\ j = 0,1, \dots n-2,$ (11)

Therefore, $d_j = \sum_{i=j+1}^{n-1}(-1)^{2j+1}(i-j)a_{i+1}, for\ j = 0,1,2, \dots n-2$ coefficients of $G_i(x), for\ i = 1,2, \dots n-1$

Theorem: 4 let's consider a polynomial of degree n, such that;

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_o, \quad (1)$$

Where $a_n \neq 0, a_{n-1}, a_{n-2}, \dots\dots, a_3, a_2, a_1\ and\ a_0$ are the real coefficients. Then the vertices differentiation polynomial of (1) is given by;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + K_i, for\ i = 1,2, \dots, n-1, (2)$$

Where G_i(x) $= a_n \prod_{j=2}^{n-1}(x + b_{j,i})$, $b_{1,i}$ vertex points, $K_i =$ P$(b_{1,i})$ vertex values, $for\ i = 1, 2, \dots\dots n-1$ and $-b_{n-1,i}, -b_{n-2,i}, \dots\dots\dots, -b_{3,i}\ and\ -b_{2,i}, for\ i = 1,2, \dots n-1$ degenerate points of P(x). Suppose a particular vertices differentiation polynomial for some i =m, $K_m =$ P$(b_{1,m}) = 0$, if and only if the vertex point is the double root and the degenerate points are the corresponding (n-2) single roots of P(x).

Proof: (→) Since P(x) we have at most (n-1) vertex points, vertex values and vertices differentiation polynomials. From (2);

$P(x) = a_n\left(x + b_{n-1,i}\right)\left(x + b_{n-2,i}\right) \dots\dots \left(x + b_{2,i}\right)\left(x - b_{1,i}\right)^2 + K_i,$ (3)

$For\ i = 1, 2, \dots\dots n-1$, are the vertices differentiation polynomials of P(x).

where $b_{1,i}, \ for\ i = 1, 2, \dots\dots n-1$ are vertex points of; $k_i =$ P$(b_{1,i})$, $for\ i = 1, 2, \dots\dots n-1$ are the vertex values and $-b_{n-1,i}, -b_{n-2,i}, \dots\dots\dots, -b_{3,i}\ and\ -b_{2,i}, for\ i = 1, 2, \dots\dots n-1$ are degenerate points of P(x). suppose for a particular index i =m and $K_m =$ P $(b_{1,m}) = 0$, (4).Thus, from (3) and (4), we obtained for some fixed index m;

$P(x) \ = a_n(x+b_{n-1,m})(x+b_{n-2,m}) \ldots\ldots (x+b_{2,m})(x-b_{1,m})^2 + K_m$, (5)

Since $K_m = 0$ and from (5); $a_n(x+b_{n-1,m})(x+b_{n-2,m})(x+b_{2,m})(x-b_{1,m})^2 = P(x) = 0$, (6)

Thus, (6) zeros of P(x) and $(x+b_{n-1,m}),(x+b_{n-2,m}),..,(x+b_{2,m})$ and $(x-b_{1,m})^2$, (7). By lemma (5); $x_{n-1} = -b_{n-1,m}$, $x_{n-2} = -b_{n-2,m}$, …….., $x_3 = -b_{3,m}$ and $x_2 = -b_{2,m}$ are the single roots and $x_1 = b_{1,m}$ is the double roots of P(x). Hence $x_{n-1} = -b_{n-1,m}$, $x_{n-2} = -\mathrm{b}_{n-2,m}$, …….., $x_3 = -b_{3,m}$ and $x_2 = -b_{2,m}$ are single roots and $x_1 = b_{1,m}$ is the double roots of P(x).

Suppose: (←) P(x) of degree n has (n-2) single roots and one double root such that;

$$P(x) = a_n(x-r_1)(x-r_2) \ldots (x-r_{n-3})(x-r_{n-1})(x-r_n), for\ a_n \neq 0, (1)$$

Where $r_1, r_2, r_3 \ldots r_{n-2}$ single roots and r_n a double root of P(x). Then we need to show r_n is the vertex point and single roots $r_1, r_2, r_3 \ldots r_{n-3}\ and\ r_{n-2}$ are the corresponding degenerate points and the extreme value for some index j=m, $K_m = 0$. Now, (1) can be written as;

$$P(x) = a_n(x-r_{n-2})(x-r_{n-3}) \ldots (x-r_2)(x-r_1)(x-r_n)^2, (2)$$

Since P(x) may have at most (n-1) number of vertices differentiation polynomials such that;

$$P(x) \ = a_n(x+b_{n-1,i})(x+b_{n-2,i}) \ldots\ldots (x+b_{2,i})(x-b_{1,i})^2 + K_i,\ i = 1,2,..n-1, (3)$$

Where $b_{1,i}$, $for\ i = 1,2,\ldots\ldots n-1$ are the vertex points, $k_i =$ P $(b_{1,i})$, $for\ i = 1,2,\ldots\ldots,n-1$, are the extreme values and $-b_{n-1,i}, -b_{n-2,i}$, …….,$-b_{3,i}\ and\ -b_{2,i}$, $for\ i = 1,2,\ldots\ldots n-1$ are the corresponding degenerate points of P(x). Thus, from those (n-1) number of vertices differentiation polynomials and from uniqueness of polynomial factorization, equation (2) is one of the vertices differentiation polynomial of P(x) such that for some index j=m, equate with the corresponding vertices differentiation polynomial;

$$P(x) = a_n(x-r_{n-2,m})(x-r_{n-3,m}) \ldots (x-r_{2,m})(x-r_{1,m})(x-r_{n,m})^2, (4) \text{ and}$$

$$P(x) = a_n(x+b_{n-1,m})(x+b_{n-2,m}) \ldots\ldots (x+b_{2,m})(x-b_{1,m})^2 + K_m, \quad (5)$$

Where the value K_m for some fixed index j=m, we have $K_m = 0$, for some fixed index m, (6). Hence from (4) and (5), we can conclude in particular the double root $x = r_{n,m}$ is the vertex point and the single roots $r_{n-2,m} = -b_{n-1,i}, r_{n-3,m} = -b_{n-2,i}$, ….,$r_{3,m} = -b_{3,i}\ and\ r_{1,m} = -b_{2,m}$ are the corresponding degenerate points $-b_{n-1,m}, -b_{n-2,m}$, …….,$-b_{3,m}\ and\ -b_{2,m}$.

Lemma: 3 let's consider a polynomial of degree (n-2);

$$C_{n-2}{b_{i,k}}^{n-2} + C_{n-3}{b_{i,k}}^{n-3} + \cdots \ldots + C_2{b_{i,k}}^2 + C_1 b_{i,k} + C_0 = 0, (1)$$

Where $C_j, for\ j = 0,1,2,\ldots,n-2$ given by;

$$C_j = \sum_{i=j+1}^{n-1}(-1)^{j+1}(i-j)a_{i+1}\,{b_{1,k}}^{i-(j+1)}, for\ i = 1,2,\ldots,n-2\ and\ a\ fixed\ index\ k, (2)$$

Then $C_j, for\ j = 0,1,2,\ldots\ldots,n-2$ of (1) are polynomials of degree (n-2-j), $for\ j = 0,1,,\ldots\ldots,n-2$ at the vertex point $b_{1,k}, for\ i = k$.

Proof: Suppose $C_j, for\ j = 0,1,2,\ldots\ldots,n-2$ of (1) expanded in to the finite sums;

$$C_j = (-1)^{j+1}a_{j+2} + (-1)^{j+1}2a_{j+3}b_{1,k} + (-1)^{j+1}3a_{j+4}{b_{1,k}}^2 + .. + (-1)^{j+1}(n-1)a_n{b_{1,k}}^{n-2}, (3)$$

$For\ j = 0,1,2,\ldots\ldots,n-2,\ for\ a\ particular\ index\ k$ and $b_{1,k}$ a vertex point for a fixed index k.

Then, from (3) compute the corresponding coefficients $C_{j,i}$ for all j=0,1,…,n-2 and particular i, such that 1≤ i ≤n-1;

For j =0; $C_0 = -a_2 - 2a_3b_{1,k} - 3a_4{b_{1,k}}^2 - \cdots - (n-2)a_{n-1}{b_{1,k}}^{n-3} - (n-1)a_n{b_{1,k}}^{n-2}$,

For j =1; $C_1 = a_3 + 2a_4b_{1,k} + 3a_5{b_{1,k}}^2 + \cdots . + (n-3)a_{n-1}{b_{1,k}}^{n-4} + (n-2)a_n{b_{1,k}}^{n-3}$,

For j =2; $C_2 = -a_4 - 2a_5b_{1,k} - 3a_6{b_{1,k}}^2 - \cdots . - (n-4)a_{n-1}{b_{1,k}}^{n-5} - (n-3)a_n{b_{1,k}}^{n-4}$,

.

For j =n-3; $C_{n-3} = (-1)^{n-2}a_{n-1} + (-1)^{n-2}2a_nb_{1,k}$,

For j =n-2; $C_{n-2} = (-1)^{n-1}a_n$. Therefore, the coefficients C_j, $for\ j = 1,2,\ldots\ldots,n-2$ are polynomials at the vertex point $b_{1,k}$ such that their degree decrease from (n-2) for C_0, (n-1) for C_1, (n-2) for C_2 …., 1 for C_{n-3} and 0 for the leading coefficients C_{n-2}.

Theorem:4 Let a vertex point $b_{1,k}$,for a particular index i=k and the corresponding degenerate points$-b_{m,k}$,for j=2,3,….,n-1 and for a particular index i=k and for m =2,3…n-1 satisfy;

$$\sum_{i=j+1}^{n-1}\sum_{j=1}^{n-2}(-1)^{j+1}\,(i-j)a_{i+1}{b_{1,k}}^{i-(1+j)}{b_{m,k}}^{j} = \sum_{i=1}^{n-1} i a_{i+1}\,{b_{1,k}}^{i-1}, (1)$$

If $\sum_{i=1}^{n-1} i a_{i+1}\,{b_{1,k}}^{i-1} = 0$, (2). Then at least one degenerate point could vanish or zero.

Proof: From (1) is reduced to a polynomial of degree (n-2). Suppose the right side of (1) vanish, if and only if by lemma (3), the constant coefficient $C_0 = 0$, then from (1), for a particular index k such that;

$$\sum_{i=j+1}^{n-1}\sum_{j=1}^{n-2}(-1)^{j+1}\,(i-j)a_{i+1}{b_{1,k}}^{i-(1+j)}{b_{m,k}}^{j} = 0, for\ m = 2,3..,n-1, (3)$$

From (3), we have a polynomial of degree n-2;), for m = 2,3..,n-1, i =1,2,..,n-1 and j =1,2,.,n-1; ;
$a_n{b_{j,i}}^{n-2} - [a_{n-1} + 2a_n b_{1,i}]{b_{j,i}}^{n-3} + [a_{n-2} + 2a_{n-1}b_{1,i} + 3a_n{b_{1,i}}^2]{b_{j,i}}^{n-4} \ldots\ldots + [(n-2)a_n{b_{1,i}}^{n-3} \ldots\ldots + 3a_5{b_{1,i}}^2 + 2a4b1,i + a3bj,i$=0,(4). Therefore, from (4), bj,i is a common factor and bj,k=0, for a particular index j=m, is root of (4).

Lemma: 4 let's consider a monic polynomial $P(x)$ of degree n with μ number of indexes of vertex values vanish $K_j = \mathrm{P}\,(b_{1,j}) = 0$. Then the vertex points are μ - double roots and degenerate points are (n-2μ) simple roots of P(x).

Proof: (→) Here let's see for a particular index i=m, at which $k_m = \mathrm{P}(b_{1,m}) = 0$, and since P(x) has at most (n-1) vertex points, vertex values and vertices differentiation polynomials. Then P(x) is can be written as;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + K_i, (1).$$

where $b_{1,i}$, $for\ i = 1,2,\ldots\ldots n-1$ vertex points; $k_i = \mathrm{P}(b_{1,i})$, $for\ i = 1,2,\ldots\ldots n-1$ vertex values and $-b_{n-1,i}, -b_{n-2,i}$,……-,$b_{3,i}$ $and\ -b_{2,i}$ $for\ i = 1,2,\ldots\ldots n-1$ degenerate points of (1).

Let $P(x) = a_n\left(x + b_{n-1,i}\right)\left(x + b_{n-2,i}\right)\ldots\ldots\left(x + b_{2,i}\right)\left(x - b_{1,i}\right)^2 + K_i$, (2) and suppose for a particular index i =m, $k_m = \mathrm{P}(b_{1,m}) = 0$, (3). Thus, from (2) and (3), for some fixed index m;

$$a_n\left(x + b_{n-1,m}\right)\left(x + b_{n-2,m}\right)\ldots\ldots\left(x + b_{2,m}\right)\left(x - b_{1,m}\right)^2 = P(x) = 0, (4).$$

Here (4) is zeros of P(x). Thus, $x_{n-1} = -b_{n-1,m}$, $x_{n-2} = -b_{n-2,m}$, …, $x_3 = -b_{3,m}$ and $x_2 = -b_{2,m}$ are simple roots and $x_1 = b_{1,m}$ double roots of P(x).Therefore, for μ number of vertex values vanish $K_m = \mathrm{P}\,(b_{1,m}) = 0$ such that there are 2μ - double roots and (n-2μ) simple roots of P(x).

Lemma: 5 Suppose a monic polynomial P(x) of degree n has m none zero multiple roots for $m \geq 3$, if and only if P'(x) and $G_k(x)$ have at least $u \geq 1$ number of common factors.

Proof: Suppose P(x) has m none zero multiple roots for $m \geq 3$, then the vertex values vanish $K_k = \mathrm{P}\,(b_{1,k}) = 0$ such that we have the following unique vertices differentiation polynomial

$$P(x) = G_k(x)\left(x - b_{1,k}\right)^2, (1)$$

Where $G_k(x)$ degenerate polynomial $and\ b_{1,k}$ vertex point . Then by using the derivatives of P(x) with respect to variable x;

$$P'(x) = G_k'(x)\left(x - b_{1,k}\right)^2 + 2G_k(x)\left(x - b_{1,k}\right), (2)$$

By the above lemma (4), for some index k, $b_{1,k}$ the roots of P'(x) and double roots of P(x) such that;

$$P'(x) = G_k'(x)\left(x - b_{1,k}\right)^2 + 2G_k(x)\left(x - b_{1,k}\right) = 0, (3)$$

Since P(x) has m multiple root $at\ b_{1,k}$,$m \geq 3$ then $b_{1,k}$ also root of $P''(x)$ such that;

$$P''(x) = G_k''(x)\left(x - b_{1,k}\right)^2 + 2G_k'(x)\left(x - b_{1,k}\right) + 2G_k{}'(x)\left(x - b_{1,k}\right) + 2G_k(x) = 0, (4)$$

Then at $b_{1,k}$,(4) reduced to $P''(b_{1,k}) = G_k\left(b_{1,k}\right) = 0$, (5). Thus, $P''(x)\ and\ G_k(x)$ have a common factor $\left(x - b_{1,k}\right)$. Moreover, since $\left(x - b_{1,k}\right)$ a common factor of $P'(x)$ and $P''(x)$ such that by transitivity $\left(x - b_{1,k}\right)$ a common factor of $P'(x)$ and $G_k(x)$.

PERTURBATION AND LOCAL QUADRATIC APPROXIMATION OF N DEGREE POLYNOMIAL P(X)

Let's consider a polynomial of degree n;

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots\dots\dots + a_2x^2 + a_1x + a_0, \ (1)$$

Where $a_n \neq 0, a_{n-1}, \dots\dots, a_3, a_2, a_1 and\ a_0$ with at least two real coefficients of P(x) are none zero. To find the roots of (1), first construct at most (n-1) local quadratic approximation of nth degree polynomial P(x), which is called the vertices differentiation polynomials of (1). First construct at most (n-1) vertices differentiation polynomials of (1). And then, how can we construct (n-1) Local quadratic approximation of nth degree polynomial P(x), from (n-1) vertices differentiation polynomials?

Definition: 6 Local quadratic approximations: the approximation of 2nd degree polynomial by perturbation of nth degree polynomial about each local vertex points.
First define the Local quadratic correction coefficients of n degree polynomial P(x)

Definition: 7 Local quadratic correction coefficient is a coefficients used to corrects the perturbation of nth degree polynomial P(x) in to a local quadratic function at the vertex point.

Definition: 8 Perturbation around vertex point: Let G_i(x) be a degenerate polynomial of degree (n-2) with (n-1) degenerate points$-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}, for\ i = 1,2, \dots, n-1$ such that;

$$G_i(\mathrm{x}) = a_n\left(x + b_{n-1,i}\right)\left(x + b_{n-2,i}\right)\left(x + b_{n-3,i}\right)\left(x + b_{n-4,i}\right) \dots \left(x + b_{2,i}\right), (1)$$

For each vertex points $b_{1,i}, for\ i = 1,2,3, \dots\dots, n-1$, we can find the corresponding polynomial G_i(x), $for\ i = 1,2,3, \dots\dots, n-1$. Thus, the value of G_i(x) at vertex points $b_{1,k}$, for a particular index k, $1 \leq k \leq n-1$ is called Perturbation of polynomial defined by;

$$G_k(b_{1,k}) = \beta_{1,k} = a_n \prod_{j=2}^{n-1}\left(b_{1,k} + b_{j,k}\right), (2)$$

Hence, in (2); $\beta_{1,k} \neq 0$ is called local correction coefficient for particular local quadratic approximation of P(x) around the neighborhood of a vertex point $b_{1,k}$ for an index k**.**

Next we will see, the condition that the local correction coefficient $\beta_{1,k}$, for particular local quadratic approximation of P(x) around the neighborhood of a vertex point $b_{1,k}$ equal to the second derivatives of P(x) over two at the vertex point $b_{1,k}$ or $\frac{P''(b_{1,k})}{2}$ such that;

$$G_k(b_{1,k}) = \beta_{1,k} = a_n \prod_{j=2}^{n-1}\left(b_{1,k} + b_{j,k}\right) = \frac{P''(b_{1,k})}{2}, (3)$$

Theorem: 5 let P(x) be a polynomial of degree n and $G_i(x)$ degenerate polynomials,$for\ i = 1,2,..,n-1$. Then for any vertex points $b_{1,i}$ such that $P''\left(b_{1,i}\right) = G_i\left(b_{1,i}\right)\ and\ P^{(k)}\left(b_{1,i}\right) = k!\, G_i^{(k-2)}\left(b_{1,i}\right), for\ i = 1,2,..,n-1\ and\ k = 3, \dots, n$.

Proof: Suppose P(x) have the following unique vertices differentiation polynomials;

$$P(x) = G_i(x)\left(x - b_{1,i}\right)^2 + P\left(b_{1,i}\right), for\ i = 1,2,..,n-1, (1)$$

Where $G_i(x), for\ i = 1,2,..,n-1$ degenerate polynomials of P(x); $P\left(b_{1,i}\right), for\ i = 1,2,..,n-1$ are the vertex values of P(x) at the vertex points $b_{1,i}, for\ i = 1,2,..,n-1$.

Then by using ordinary differential with respect to variable x;

$$P'(x) = G_i'(x)\left(x - b_{1,i}\right)^2 + 2G_i(x)\left(x - b_{1,i}\right), (2)$$

$$P''(x) = G_i''(x)\left(x - b_{1,i}\right)^2 + 2G_i'(x)\left(x - b_{1,i}\right) + 2G_i{}'(x)\left(x - b_{1,i}\right) + 2G_i(x), (3)$$

$$P'''(x) = G_i'''(x)\left(x - b_{1,i}\right)^2 + 2G_i''(x)\left(x - b_{1,i}\right) + 2G_i''(x)\left(x - b_{1,i}\right) + 2G_i'(x) + 2G_i''(x)\left(x - b_{1,i}\right) + 2G_i'(x) + 2G_i{}'(x), (4)$$

Let $b_{1,i}, for\ i = 1,2,..,n-1$ vertex points of P(x) such that from (2)-(4) we have;

$$P''\left(b_{1,i}\right) = 2!\, G_i\left(b_{1,i}\right)$$
$$P'''\left(b_{1,i}\right) = 3!\, G_i'\left(b_{1,i}\right)$$
$$P^{(4)}\left(b_{1,i}\right) = 4!\, G_i^{(2)}\left(b_{1,i}\right)$$
$$P^{(5)}\left(b_{1,i}\right) = 5!\, G_i^{(3)}\left(b_{1,i}\right)$$
$$P^{(6)}\left(b_{1,i}\right) = 6!\, G_i^{(4)}\left(b_{1,i}\right)$$

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$$P^{(7)}(b_{1,i}) = 7!\, G_i^{(5)}(b_{1,i})$$

Therefore, for any n[th] derivatives of P(x) at any vertex points $b_{1,i}, for\ i = 1,2,..,n-1$ related by;

$$P^{(n)}(b_{1,i}) = n!\, G_i^{(n-2)}(b_{1,i}), for\ i = 1,2,..,n-1$$

Corollary: 2 The ratio of kth derivatives of any polynomial P(x) and the corresponding degenerate polynomials $G_i(x), for\ i = 1,2,..,n-1$ of degree n-2, at any of the vertex points $b_{1,i}, for\ i = 1,2,..,n-1$ are constant (it's the factorial of kth order of $derivatives$)

$$\frac{P^{(k)}(b_{1,i})}{G_i^{(k-2)}(b_{1,i})} = k!, for\ k = 2,3,\dots,n \text{ and } i = 1,2,..,n-1, (1)$$

UNIQUENESS THEOREM

Theorem: 6 Suppose a monic polynomial P(x) of degree n has n distinct roots, if and only if, P(x) has distinct vertex values $K_i = P(b_{1,i}) \neq 0\ \ for\ all\ i = 1,2,..n-1$.

Proof: Suppose $K_m = P(b_{1,m}) = 0, for\ a\ particular\ index\ i = m$ and $K_i = P(b_{1,i}) \neq 0$, for other index $i \neq m$ and P(x) has n unique and distinct roots.

$$P(x) = G_i(x)(x - b_{1,i})^2 + K_i, for\ i = 1,2,..,n-1, (1)$$

Let G_i(x) be a monic polynomial of degree (n-2) with (n-2) possible degenerate points $-b_{n-1,i}, -b_{n-2,i}, \dots, -b_{2,i}, for\ i = 1,2,\dots,n-1$ such that;

$$G_i\ (x) = a_n \prod_{j=2}^{n-1}(x + b_{j,k}), for\ i = 1,2,..,n-1, (2)$$

For each vertex points $b_{1,i}, for\ i = 1,2,3,\dots,n-1$, we can find the corresponding polynomials G_i(x), $for\ i = 1,2,3,\dots,n-1$. Thus, the value of G_i(x) at a particular index i=k of vertex point $b_{1,i}$, is defined by; $G_k(b_{1,k}) = a_n \prod_{j=2}^{n-1}(b_{1,k} + b_{j,k}) = \beta_{1,k}, (3)$

Therefore, $\beta_{1,k} \neq 0$ is called local correction coefficient for a particular local quadratic approximation of P(x) around neighborhood of a particular vertex point $b_{1,k}$ is called perturbation of polynomial around the local vertex points. W.o.l.g P(x) and G_i(x), for i=1,2,3,…..n-1 is monic polynomials and since P(x) have (n-1) vertices points $b_{1,i}, for\ i = 1,2,\dots,n-1$, then we will have (n-1) local quadratic approximation with (n-1) local correction coefficients $\beta_{1,i}, for\ i = 1,2,\dots\dots,n-1$. For each local vertex points $b_{1,i}, for\ i = 1,2,\dots,n-1$ we can defined as;

$$q_i(x) = \beta_{1,i}(x - b_{1,i})^2 + P(b_{1,i}), for\ i = 1,2,\dots,n-1, (4)$$

Therefore, (4) are called the local quadratic approximation of P(x) with the local correction coefficients $\beta_{1,i}$ and $P(b_{1,i}), for\ i = 1,2,\dots,n-1$ vertex values of P(x) at a particular vertex point $b_{1,i}$. Simply, by using theorem (4) we can fix the local correction coefficients $\beta_{1,i} \neq 0$, by;

$$\beta_{1,i} = G_i(b_{1,i}) = \frac{P''(b_{1,i})}{2}, (5)\ and\ K_i = P(b_{1,i}), (6)$$

The roots square deviation from the vertex point $b_{1,i}$ for i=1,2, ….,n-1 are approximated by;

$$(x - b_{1,i})^2 = \frac{P(x) - P(b_{1,i})}{\frac{P''(b_{1,i})}{2}}, (7)$$

Since x root of P(x) such that $P(x) = 0$, then from (7), we have $(x - b_{1,i})^2 = \frac{-2P(b_{1,i})}{P''(b_{1,i})}, for\ i = 1,2,\dots,n-1, (8)$. Where $P''(b_{1,i}) \neq 0$ given in (5) and (6). Then, $for\ each\ i = 1,2,\dots,n-1$ by assuming the perturbation, we can find the corresponding root x_i such that;

$$x_i = b_{1,i} + sign(b_{1,i})\sqrt{\frac{-2P(b_{1,i})}{P''(b_{1,i})}}, for\ i = 1,2,\dots,n-1, (9)$$

$Provided\ P''(b_{1,i}) \neq 0$, at the vertex points $b_{1,i}, for\ i = 1,2,3,\dots,n-1$ and $x_n = \frac{(-1)^n a_0}{a_n \prod_{i=1}^{n-1} x_i}, (10)$

After the perturbation step, the next step is diagnostic recovery step to minimize existing total error in approximated roots during perturbation step such that $r_i = x_i - e_M(r_i), for\ i = 1,2,3,\dots n$.

where $e_M(r_i), for\ i = 1,2,3,\dots n$ maximum error recovery formula given by;

$$e_M(r_i) = \frac{P'(r_i) - sign(P'(r_i))\sqrt{P'(r_i)^2 - 2P(r_i)P''(r_i)}}{P''(r_i)}, (11)$$

But, from our supposition the vertex value $K_m = P(b_{1,m}) = 0$, $for\ i = m$, (12),Thus, x_m, for $i = m$ by lemma (2); x_m is a double root of P(x). Finally from (9) using (12) we can find the roots such that; $x_m = b_{1,m}$ double root,$x_i = b_{1,i} + sign(b_{1,i})\sqrt{\frac{-2P'(b_{1,i})}{P'''(b_{1,i})}}$, $for\ i = 2, 4 \ldots, n-2$ and x_n in (10) are distinct roots of P(x). Therefore, it contradicts the fact that the roots x_i, for i=1,2,3,…..n are distinct. Hence, P(x) has n distinct roots, if and only if the vertex points and vertex values $K_i = P(b_{1,i}) \neq 0, for\ all\ i = 1,2,..n-1$ are distinct and different from zero.

Lemma: 6 Let a monic polynomial P(x) of degree n has m none zero multiple root, for m≥ 3 , then there exist at least two points of P(x) such that; $b_{1,q} + b_{j,q} = 0$,(1), $for\ some\ j$ and particular index $1 \leq q \leq n-1$. Where $b_{1,q}$ and $-b_{j,q}$, $for\ j = 2,3, \ldots, n-1$, the vertex and degenerate points of P(x) respectively.

Proof: suppose vertices differentiation polynomials of monic polynomial P(x) of degree n given by;

$$P(x) = G_i(x)(x - b_{1,i})^2 + K_i, for\ i = 1,2,..,n-1, (2)$$

Where $b_{1,i}, for\ each\ i = 1, 2, 3, \ldots\ldots, n-1\ and$ vertex points; $K_i = P(b_{1,i}), for\ i = 1,2,..n-1$ the extreme value of P(x) and G_i(x) degenerate polynomial of degree (n-2), for i=1,2,..n-1 given by;

$$G_i(\mathrm{x}) = a_n(x + b_{n-1,i})(x + b_{n-2,i})(x + b_{n-3,i})(x + b_{n-4,i}) \ldots\ldots (x + b_{2,i})$$
$$= a_n \prod_{j=2}^{n-1}(x + b_{j,i}), for\ i = 1,2,..,n-1, \quad (3).$$

Where $a_n \neq 0$ and $-b_{n-1,i}, -b_{n-2,i}, \ldots, -b_{2,i}, for\ i = 1,2, \ldots, n-1$ are degenerate points. Since P(x) has at least m multiple roots at the point $x = b_{1,q}, for\ m \geq 3$, then from (2) the extreme value of P(x) for i= q is $K_q = P(b_{1,q}) = 0$, (4). Thus from (2);we have $P(x) = G_q(x)(x - b_{1,q})^2$, (5) From the given hypothesis the multiplicity of the root $x = b_{1,q}$ is $m \geq 3$ thus in order to (5) has a multiplicity of the root $m \geq 3$, we should have one common factor of $G_q(x)$ and $(x - b_{1,q})^2$, for at least one j and fixed index q. Hence, those points satisfy $b_{j,q} = -b_{1,q} = 0$, for at least one j and fixed index q, (6).

Theorem: 7 Suppose degenerate and vertex points of a monic nth degree polynomial P(x) satisfy $b_{1,q} = -b_{j,q}, and\ P(b_{1,q}) = P(-b_{j,q}), for\ all\ j = 2,3, \ldots\ldots, n-1$, and fixed q, (1). Then P(x) and P'(x) have n and (n-1) none zero multiple roots respectively. Where $b_{1,q}\ and\ -b_{j,q}, for\ all\ j = 2,3, \ldots, n-1$ are the vertex and degenerate points respectively.

Proof: The vertices differentiation polynomial of P(x), $for\ i = 1, 2, \ldots\ldots n-1$ given by;

$$P(x) = a_n(x + b_{n-1,i})(x + b_{n-2,i}) \ldots\ldots (x + b_{2,i})(x - b_{1,i})^2 + k_i, (1)$$

Where $b_{1,i}$, $for\ i = 1, 2, \ldots\ldots n-1$ vertex points, $k_i =$ P $(b_{1,i})$, $for\ i = 1, 2, \ldots\ldots n-1$ vertex values and $-b_{n-1,i}, -b_{n-2,i}$, …….,$-b_{3,i}\ and\ -b_{2,i}$, $for\ i = 1, 2, \ldots\ldots n-1$ are the degenerate points of P(x). Since those two points of P(x) satisfy (1), P(x) has multiple roots such that $K_k =$ P $(b_{1,k}) = 0$, for fixed k, (2). From (1), we can reduce;

$$P(x) = a_n(x + b_{n-1,k})(x + b_{n-2,k}) \ldots\ldots (x + b_{2,k})(x - b_{1,k})^2, (3)$$

From the given supposition for a particular vertex point $b_{1,k}$ and for all degenerate points; $b_{j,k} = -b_{1,k}, for\ all\ j = 2,3, \ldots\ldots, n-1$, (4). Hence, by using (3) and (4) we have; $P(x) = a_n(x - b_{1,k})(x - b_{1,k}) \ldots\ldots (x - b_{1,k})(x - b_{1,k})^2$ $P(x) = a_n(x - b_{1,k})^n$, (5). Similarly, from (3) the derivative of P(x) given by;

$$P'(x) = a_n(x + b_{n-2,k}) \ldots\ldots (x + b_{2,k})(x - b_{1,k})^2 + a_n(x + b_{n-1,k}) \ldots (x + b_{2,k})(x - b_{1,k})^2$$
$$+ a_n(x + b_{n-1,k})(x + b_{n-2,k}) \ldots\ldots (x - b_{1,k})^2 + 2a_n(x + b_{n-1,k})(x + b_{n-2,k}) \ldots\ldots (x + b_{2,k})(x - b_{1,k}), (6)$$

Since the relation between the degenerate and vertex points in (4), reduce (6) to;

$$P'(x) = a_n(x - b_{1,k})\left[(x - b_{1,k})^{n-2} + (x - b_{1,k})^{n-2} + (x - b_{1,k})^{n-2} + \cdots + 2(x - b_{1,k})^{n-2}\right]$$
$$P'(x) = a_n(x - b_{1,k})\left[(n-2)(x - b_{1,k})^{n-2} + \cdots + 2(x - b_{1,k})^{n-2}\right], (7)$$

Finally we have obtained; $P'(x) = a_n(x - b_{1,k})\left[(n - 2 + 2)(x - b_{1,k})^{n-2}\right]$

$$= na_n(x - b_{1,k})\left[(x - b_{1,k})^{n-2}\right]$$
$$P'(x) = na_n(x - b_{1,k})^{n-1}, (8)$$

Corollary: 3 Suppose a monic polynomial P(x) of degree n has at least k none zero multiple roots, then the points at which P'(x) vanishes has (k-1) multiple roots.

Proof: Suppose P(x) of degree n has k multiple roots and can be written as the product of its simple and multiple factors;

$$P(x) \ = a_n(x-r_n)(x-r_{n-1}) \ldots\ldots (x-r_{n-k})(x-r_i)^k, for\ i = 1,..,k, (2)$$

We need to show: P'(x) has (k-1) multiple root such that for $k \geq 2$. From (2), the derivative of P(x) given by; $P'(x) = anx-rik-1\ F(x)$, (3). Where F(x) a common factor polynomial of degree (n-(k-1)) such that approximated by; $F(x) = a_n(x-r_{n-1}) \ldots\ldots (x-r_{n-k})(x-r_i)^k + a_n(x-r_n) \ldots\ldots (x-r_{n-k})(x-r_i)^{k-1} \ldots\ldots + + a_n(x-r_n)(x-rn-1\ldots\ldots x-rik+kanx-rnx-rn-1\ldots\ldots x-rn-kx-rik-1$ and since $anx-rik-1$is the least (minimum) common factor of the derivative of P(x). Then from (3), P'(x) vanishes with (k-1) multiple root of P(x).

$a_n(x-r_i)^{k-1}\ F(x) = 0$, (4), $x = r_i$, for $i = 1,2,..,k-1$ are the (k-1) multiple root of P'(x).

Lemma: 7 Suppose vertices differentiation polynomials of P(x) of degree n has distinct vertex values P ($b_{1,j}$), for all j. Then there exist two points $b_{1,j}$ and P ($b_{1,j}$), for j=1,2,..3, that have none zero shortest distance $\delta > 0$ and $\epsilon > 0$ respectively such that $|b_{1,j} - b_{1,m}| < \delta \quad \rightarrow |P(b_{1,j}) - P(b_{1,m})| < \varepsilon\ for\ j \neq m$, (2), if and only if the points $b_{1,j}\ and\ P(b_{1,j}),\ are$ vertex points and vertex values of P(x) respectively.

Proof: Suppose $|b_{1,j} - b_{1,m}| = 0$, if and only if P(x) has multiple roots such that by theorem (2), $b_{1,m} = -b_{j,m}$ and $|P(b_{1,j}) - P(b_{1,m})| = 0$ for $j \neq m$. Thus, $-b_{j,m}$ is a degenerate point, and $P(b_{j,m})$ degenerate value, for particular index m. therefore, it contradicts the fact that $-b_{j,m}$ is a vertex point and $P(b_{j,m})$ vertex value of P(x).

Theorem: 8 Suppose a monic polynomial P(x) of degree n has n distinct roots, if and only if P(x) has (n-1) distinct vertex points $b_{1,i}$, for $i = 1,2, \ldots . n-1$.

Suppose: ($\rightarrow$) let P(x) of degree n has n distinct roots, then by theorem (2) P(x) has (n-1) distinct vertex values $P(b_{1,i}) = K_i \neq 0$, for all $i = 1,2, \ldots . n-1$.

We need to show: P(x) has distinct vertex points $b_{1,i}$ for all $i = 1,2, \ldots n-1$. Assume that K_i is local vertex value for each i=1,2,…,n-1, such that $K_i = P(b_{1,i}) \neq P(b_{1,j}) = K_j$ for all $i \neq j$, (1). From our supposition;$P(b_{1,i}) = K_i \neq 0$, for all $i = 1,2, \ldots . n-1$, (2). Since the vertex values $P(b_{1,i}) = K_i \neq 0$, for $i = 1,2, \ldots . n-1$ are distinct, then by lemma (4), there exists the smallest distance $\varepsilon > 0$, such that; $|P(b_{1,i}) - P(b_{1,j})| = \epsilon > 0$, for all $i \neq j$, (3). So, P(x) may not have the case at which there are the same vertex points $b_{1,m} = b_{1,j}$ for $m \neq j$ and $P(b_{1,m}) \neq P(b_{1,j})$, for $m \neq j$. Therefore, from (1) and (3), we can conclude that there exists smallest distance $\delta > 0$ such that $|b_{1,i} - b_{1,j}| = \delta > 0$ at the vertex points $b_{1,i}$ and the extreme values K_i, for i=1,2,…..,n-1; $K_i = P(b_{1,i}) \neq P(b_{1,j}) = K_j$ for all $i \neq j$, (4), and $b_{1,i} \neq b_{1,j}$, for all $i \neq j$, (5). Conversely, suppose $K_m = 0$, for only distinct vertex point $b_{1,m}$. Then, by theorem (1) the vertex point $b_{1,m}$, is the double roots of P(x). Hence it is a contradiction to the fact that P(x) has n distinct roots.

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