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TOPOLOGIES GENERATED BY THE α-CUTS OF A FUZZY SET

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ABSTRACT

Zadeh introduced the concept of fuzzy sets in 1965. The α-cuts of a fuzzy sub set A of a non-empty set X may generate a topology. Such a topology is called the topology generated by the α-cuts of the fuzzy set A of X. The purpose of this paper is to characterise such topologies.

1. INTRODUCTION

Fuzzy sets were introduced by Zadeh in 1965 as a generalisation of the classical sets. One important concept in fuzzy set theory is a notion of α- cuts that play a dominant role in studying fuzzy concepts using crisp concepts. The collection of α-cuts of fuzzy set A of X may generate a topology on the underlying set X. Such a topology is called the topology generated by the α-cuts of the fuzzy subset A. Richard P. Stanely [4], Stephen [5], Moussa Benoumhani [3], Krishnamurthy [2] discussed the topologies on finite sets as well as finite topologies. In this paper, the topologies generated by the α-cuts of the fuzzy subsets of finite sets are discussed. Some basic concepts of fuzzy set theory are given in section 2 and some new elementary properties of α-cuts are discussed in section 3. In section 4, the topologies generated by the α-cuts of fuzzy subsets of X are investigated where $|X| = 1, 2, 3, 4$.

2. PRELIMINARIES

Let X be the universal set. Then a fuzzy subset A of X is defined by its membership function $\mu_A: X \to [0, 1]$ which assigns a real number $\mu_A(x)$ in the interval [0, 1] to each element $x \in X$ where the value of μ_A at x shows the membership grade of x in A. Throughout this paper A denotes a fuzzy subset of X in which case we write A(x) for $\mu_A(x)$.

Given a fuzzy set A in X and any real number $\alpha \in [0, 1]$. The α-cut of A, denoted by ${}^{\alpha}A$ is the crisp set

${}^{\alpha}A = \{x \in X: A(x) \geq \alpha\}$.

The height h(A), of a fuzzy set A is the supremum of the values of A and $\ell(A)$ is the infimum of the values of A. That is $\sup_{x\in X} A(x) = h(A)$ and $\operatorname{Inf}_{x\in X} A(x) = \ell(A)$. A fuzzy set A is called normal when h(A) = 1 and is called subnormal when h(A) < 1.

The properties of α-cuts are listed below.

Let A, B are any two fuzzy subsets of X and $\alpha, \beta \in [0, 1]$. Then

i) $\alpha \leq \beta$ implies ${}^{\alpha}A \supseteq {}^{\beta}A$.

ii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$ and ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$.

iii) $\bigcup_{i\in I} {}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I} A_i\right)$ and $\bigcap_{i\in I} {}^{\alpha}A_i = {}^{\alpha}\left(\bigcap_{i\in I} A_i\right)$.

iv) $A \subseteq B$ if and only if ${}^{\alpha}A \subseteq {}^{\alpha}B$.

v) $A = B$ if and only if ${}^{\alpha}A = {}^{\alpha}B$.

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vi) ${}^{\alpha}A = \bigcap_{\beta \prec \alpha} {}^{\beta}A$.

vii) ${}^{\alpha}A = \bigcup_{\alpha \prec \beta} {}^{\beta}A$.

3. PROPERTIES

Proposition: 3.1 Let A be a fuzzy subset of X and $0 \le \alpha \le 1$, $0 \le \beta \le 1$. Then

$${}^{\alpha}A \cup {}^{\beta}A = \begin{cases} {}^{\alpha}A & if\ \alpha \le \beta \\ {}^{\beta}A & otherwise \end{cases} \text{ and } {}^{\alpha}A \cap {}^{\beta}A = \begin{cases} {}^{\beta}A & if\ \alpha \le \beta \\ {}^{\alpha}A & otherwise \end{cases}.$$

Proof: Suppose $\alpha \le \beta$. Then ${}^{\alpha}A = \{x \in X: A(x) \ge \alpha\}$ and ${}^{\beta}A = \{x \in X: A(x) \ge \beta\}$. If $x \in {}^{\beta}A$ then $A(x) \ge \beta$ that implies $A(x) \ge \alpha$ so that $x \in {}^{\alpha}A$. Therefore ${}^{\beta}A \subseteq {}^{\alpha}A$.

This proves that ${}^{\alpha}A \cup {}^{\beta}A = {}^{\alpha}A$ and ${}^{\alpha}A \cap {}^{\beta}A = {}^{\beta}A$. The proof for the case $\alpha \ge \beta$ is analog.

Proposition: 3.2

(i) ${}^{\alpha_1}A \cup {}^{\alpha_2}A \cup {}^{\alpha_3}A \cup \ldots\ldots\ldots \cup {}^{\alpha_n}A = {}^{\alpha}A$ where $\alpha = \min\{\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n\}$.

(ii) ${}^{\alpha_1}A \cap {}^{\alpha_2}A \cap {}^{\alpha_3}A \cap \ldots\ldots \cap {}^{\alpha_n}A = {}^{\beta}A$ where $\beta = \max\{\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n\}$.

Proof: We can rearrange $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n$ in such a way that $\alpha_1 \le \alpha_2 \le \ldots\ldots\ldots \le \alpha_n$.

By using Proposition 3.1 we see that ${}^{\alpha_1}A \supseteq {}^{\alpha_2}A \supseteq {}^{\alpha_3}A \supseteq \ldots\ldots \supseteq {}^{\alpha_n}A$.

This proves that ${}^{\alpha_1}A \cup {}^{\alpha_2}A \cup {}^{\alpha_3}A \cup \ldots\ldots\ldots \cup {}^{\alpha_n}A = {}^{\alpha_1}A$ where $\alpha_1 = \min\{\alpha_1, \alpha_2, \ldots\ldots \alpha_n\}$ and ${}^{\alpha_1}A \cap {}^{\alpha_2}A \cap {}^{\alpha_3}A \cap \ldots\ldots\ldots \cap {}^{\alpha_n}A = {}^{\alpha_n}A$ where $\alpha_n = \max\{\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n\}$.

Proposition: 3.3

(i) $\bigcup_{\alpha \in \Delta} {}^{\alpha}A = {}^{\alpha_0}A$ where $\alpha_0 = \inf \Delta$.

(ii) ${}^{\beta}A \subseteq \bigcap_{\alpha \in \Delta} {}^{\alpha}A$ where $\beta = \sup \Delta$ and $\Delta \subseteq [0, 1]$.

Proof: Let $\Delta \subseteq [0, 1]$ and $\alpha_0 = \inf \Delta$. Let $x \in \bigcup_{\alpha \in \Delta} {}^{\alpha}A$. Then $x \in {}^{\alpha}A$ for some $\alpha \in \Delta$. Therefore $A(x) \ge \alpha$ for some $\alpha \in \Delta$. This implies $A(x) \ge \alpha_0$, proving $x \in {}^{\alpha_0}A$ that implies $\bigcup_{\alpha \in \Delta} {}^{\alpha}A \subseteq {}^{\alpha_0}A$. Conversely let $x \in {}^{\alpha_0}A$. Then for all $\epsilon > 0$, there exist $\alpha \in \Delta$ with $\alpha < \alpha_0 + \epsilon$. Since $x \in {}^{\alpha_0}A$, $A(x) \ge \alpha_0 > \alpha - \epsilon$ that implies $A(x) > \alpha - \epsilon$. This proves $A(x) \ge \alpha$ so that $x \in {}^{\alpha}A$, proving $x \in \bigcup_{\alpha \in \Delta} {}^{\alpha}A$. Therefore ${}^{\alpha_0}A \subseteq \bigcup_{\alpha \in \Delta} {}^{\alpha}A$.

This proves (i). Now let $\beta = \sup \Delta$. Then $\alpha \le \beta$ for all $\alpha \in \Delta$ that implies ${}^{\alpha}A \supseteq {}^{\beta}A$ for all $\alpha \in \Delta$ proving

${}^{\beta}A \subseteq \bigcap_{\alpha \in \Delta} {}^{\alpha}A$. The reverse inclusion in Proposition 3.3 (ii) is not true as shown below.

Example: 3.4 Let $N = \{1, 2, 3, \ldots\ldots\}$ and $A(n) = \begin{cases} 0, & n\ is\ even \\ \dfrac{1}{(n+1)}, & n\ is\ odd \end{cases}$.

Here ${}^{0}A = N$ and ${}^{1}A = \varphi$. Let $\frac{1}{2} \le \alpha < 1$. Then

${}^{\alpha}A = \{n : n \text{ is odd}, A(n) \geq \alpha\} = \{n : n \text{ is odd}, \frac{1}{n+1} \geq \alpha\}$

$= \{n: n \text{ is odd}, n+1 \leq \frac{1}{\alpha}\} = \{n: n \text{ is odd}, n \leq \frac{1}{\alpha} - 1\}$

$= \{n: n \text{ is odd}, n \leq \frac{1}{\alpha} - 1 \text{ and } n \in N\}$.

Let $\Delta = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots 1-\frac{1}{n}\} = \left\{\frac{n-1}{n} : n \in N\right\}$.

When $\alpha = \frac{1}{2}$, ${}^{\alpha}A = \{1\}$. When $\alpha = \frac{2}{3}$, ${}^{\alpha}A = \emptyset$. When $\alpha = \frac{3}{4}$, ${}^{\alpha}A = \emptyset$ and so on.

Let $\alpha = \frac{n-1}{n}$, $n \in N$. It is easy to see that ${}^{\alpha}A = \{1\}$ for n = 2 and $= \emptyset$ for $n \geq 3$. Therefore $\bigcap_{\alpha\in\Delta} {}^{\alpha}A = \emptyset$

and ${}^{1}A = \emptyset$ where $1 = \sup \Delta$.

Example: 3.5 Let N = { 1 ,2 , 3, ……..} and $A(n) = \begin{cases} 0, & n \text{ is even} \\ \frac{1}{2}, & n = 1 \\ \frac{1}{n}, & \text{otherwise} \end{cases}$

Clearly ${}^{0}A = N$, ${}^{1}A = \emptyset$.

When $\alpha = \frac{1}{2}$, ${}^{\alpha}A = \{n : A(n) \geq \frac{1}{2}\} = \{1\} \cup \{n : \frac{1}{n} \geq \frac{1}{2} \text{ and n is odd}\}$

$= \{1\} \cup \{n: n \leq 2 \text{ and n is odd} = \{1\}$.

When $\alpha = \frac{2}{3}$, ${}^{\alpha}A = \{n: A(n) \geq \frac{2}{3}\} = \{1\} \cup \{n : \frac{1}{n} \geq \frac{2}{3} \text{ and n is odd}\}$.

$= \{1\} \cup \{n : n \leq \frac{3}{2} \text{ and n is odd}\} = \{1\}$.

Proceeding like this we get ${}^{\alpha}A = \{1\}$ where $\alpha = \frac{n-1}{n}$, $n \geq 2$.

Take $\Delta = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots \frac{n-1}{n}, \ldots\ldots\}$ Sup $\Delta = 1$

It is easy to see that $\bigcap_{\alpha\in\Delta} {}^{\alpha}A = \{1\} \neq {}^{1}A$ where $1 = \sup \Delta$.

Throughout this paper we use the following notations for any fuzzy subset A of X.

T (A) = Collection of all α-cuts of A.

From the propositions proved in this section, we see that T (A) may not be a topology on X. However they may generate a topology on X as illustrated in the following examples.

Example: 3.6 X = {a, b, c} and A = $\frac{.1}{a} + \frac{.2}{b} + \frac{.7}{c}$. T (A) = {X, {b, c}, {c}, ∅}. Clearly T (A) is a Topology on X.

Example: 3.7 Let X = {q, r, s} and B= $\frac{.8}{q} + \frac{.9}{r} + \frac{1}{s}$.
T (B) = {X, {r, s}, {s}}. Clearly T (B) is not a topology on X.

Proposition: 3.8 ∅ ∈ T (A) if and only if A is subnormal.

Proof: Suppose ∅ ∈ T (A). Then ∅ = $^{\alpha}$A for some α, 0≤ α ≤ 1. Therefore A (x) < α, for all x ∈ X which implies sup A(x) < α, so that h (A) < α that implies h (A) < 1. This proves that A is subnormal.

Conversely let A be subnormal. Then h (A) < 1 that implies sup A (x) < 1. Choose ∈ > 0 with α = h (A) + ∈ < 1.

We know $^{\alpha}$A = {x∈ X: A (x) ≥ α} = {x ∈ X : A (x) > h(A)}= ∅ This completes the proof.

Proposition: 3.9 If ℓ (A) > 0 then X ∈ T(A).

Proof: ℓ (A) = inf {A (x): x ∈ X}. Suppose ℓ (A) > 0, Take ℓ (A) = α that implies ℓ (A) ≤ A (x) for all x.

Thus A(x) ≥ α for all x that implies $^{\alpha}$A = X, proving X ∈ T (A).

4. TOPOLOGY GENERATED BY THE α-CUTS OF A FUZZY SET

Throughout this section τ_A denotes the topology generated by the α-cuts of a fuzzy subset A of X.

We call τ_A as the **α-cut topology** of a fuzzy subset A of X.

Theorem: 4.1 If A is a fuzzy subset of X = {x} then τ_A = {X, ∅}.

Proof: X = {x}; Let A = $\frac{a}{x}$; $0 \le a \le 1$. Then T (A) = {X}.

Topology generated by T (A) = Topology generated by {X} = {X, ∅}. Therefore τ_A = {X, ∅}.

Theorem: 4.2 If A is a fuzzy subset of X = {x, y} then τ_A = {X, ∅} or τ_A = {X, ∅, λ } where λ ∈ {{x},{y}}.

Proof: Let X = {x, y}. There are 4 topologies on X.

They are given by τ_1 = {X, ∅}, τ_2 = {X, ∅, {x}}, τ_3 ={X, ∅, {y}}, τ_4 = {X, ∅, {x}, {y}}.

Case 1: If A = $\frac{a}{x} + \frac{a}{y}$ where $0 \le a \le 1$ then T(A) = {X}.
Topology generated by T (A) = Topology generated by{X} = {X, ∅}. τ_A={X, ∅}= τ_1.

Case 2: If A = $\frac{a}{x} + \frac{b}{y}$ where $0 \le b < a \le 1$ then T (A) = {X, {x}}.

Topology generated by T (A) =Topology generated by {X, {x}} ={X, {x}, ∅}.

τ_A = {X, {x}, ∅} = τ_2.

Case 3: If A = $\frac{a}{x} + \frac{b}{y}$ where $0 \le a < b \le 1$ then T (A) = {X, {y}}.
Topology generated by T (A) = Topology generated by {X, {y}} ={X, {y}, ∅}.

τ_A = { X , { y } , ∅} = τ_3.

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Therefore we conclude that only 3 topologies on {x, y} are generated by the α-cuts of the fuzzy subsets of {x, y}.

Theorem: 4.3 The discrete topology on {x, y} is not generated by the α-cuts of any fuzzy subset of {x, y}.

Proof: Let A be a fuzzy subset of {x, y}. Then A = $\frac{a}{x} + \frac{b}{y}$ where $0 \le a, b \le 1$.

If a = b then using Theorem 4.2, $\tau_A = \tau_1$ that is not discrete.

If a > b then using Theorem 4.2, $\tau_A = \tau_2$ that is not discrete.

If a < b then using Theorem 4.2, $\tau_A = \tau_3$ that is not discrete.

Thus we conclude that the discrete topology on {x, y} is not generated by the α-cuts of any fuzzy subset of {x, y}

Theorem: 4.4 If A is a fuzzy subset of X = {x, y, z} then τ_A = {X, ∅} or {X, ∅, λ_1} or {X, ∅, λ_2} or {X, ∅, λ_1, λ_2} with $\lambda_1 \subseteq \lambda_2$ where $\lambda_1 \in \{\{x\},\{y\},\{z\}\}$ and $\lambda_2 \in \{\{x, y\},\{y, z\}, \{x, z\}\}$.

Proof: Let X = {x, y, z}. There are 29 possible topologies on X. Let A be a fuzzy subset of X.

Case 1: Let A = $\frac{a}{x} + \frac{a}{y} + \frac{a}{z}$ where $0 \le a \le 1$. Then T (A) = {X}.

Topology generated by T (A) = Topology generated by {X} = {X, ∅}.

τ_A = {X, ∅}. In this case only one topology on X is generated by the α-cuts of A.

Case 2: Let A = $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$; $0 \le a, b, c \le 1$

Consider the case a = b < c. Then T (A) = {X, {z}}.

Topology generated by T (A) = Topology generated by {X, {z}} = {X, {z}, ∅}.

τ_A = {X, {z}, ∅}.

Consider the case a = b > c. Then T (A) = {X, {x, y}}.

Topology generated by T (A) = Topology generated by {X, {x, y}} ={X, {x, y},∅}.

τ_A = {X, {x, y}, ∅}.

Other cases are analogously discussed as follows.

If a = c < b then τ_A = {X, {y}, ∅}. If a = c > b then τ_A = {X, {x, z}, ∅}.

If b = c < a then τ_A = {X, {x}, ∅}. If b = c >a then τ_A ={X, {y, z}, ∅}.

In this case only 6 topologies on X are generated by the α-cuts of respective fuzzy sets.

Case 3: Let A = $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$; $0 \le a, b, c \le 1$.

Consider the cases a < b < c. Then T (A) = {X, {z}, {y, z}}.

Topology generated by T(A) = Topology generated by {X, {z}, {y, z}} ={X, {z}, {y, z}, ∅}.

τ_A = {X, {z}, {y, z}, ∅}.

Other cases are analogously discussed as follows.

If $b < c < a$ then $\tau_A = \{X, \{x\}, \{x, z\}, \emptyset\}$. If $c < b < a$ then $\tau_A = \{X, \{x\}, \{x, y\}, \emptyset\}$.

If $a < c < b$ then $\tau_A = \{X, \{y\}, \{y, z\}, \emptyset\}$. If $c < a < b$ then $\tau_A = \{X, \{y\}, \{x, y\}, \emptyset\}$.

If $b < a < c$ then $\tau_A = \{X, \{z\}, \{x, z\}, \emptyset\}$

In this case only 6 topologies on X are generated by the α-cuts of fuzzy subsets of X. Therefore we conclude that only 13 topologies on X = {x, y, z} are generated by the α-cuts of the fuzzy subsets of {x, y, z}.

Theorem: 4.5 If A is a fuzzy subset of X = {x, y, z, w} then $\tau_A = \{X, \emptyset\}$ or $\{X, \emptyset, \lambda_1\}$ or $\{X, \emptyset, \lambda_2\}$ or $\{X, \emptyset, \lambda_3\}$ or $\{X, \emptyset, \lambda_1, \lambda_2\}$ with $\lambda_1 \subseteq \lambda_2$ or $\{X, \emptyset, \lambda_2, \lambda_3\}$ with $\lambda_2 \subseteq \lambda_3$ or $\{X, \emptyset, \lambda_1, \lambda_3\}$ with $\lambda_1 \subseteq \lambda_3$ or $\{X, \emptyset, \lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 \subseteq \lambda_2 \subseteq \lambda_3$ where $\lambda_1 \in \{\{x\}, \{y\}, \{z\}, \{w\}\}$, $\lambda_2 \in \{\{x, y\}, \{x, z\}, \{x , w\}, \{y, z\}, \{y, w\}, \{z, w\}\}$ and $\lambda_3 \in \{\{x, y, z\}, \{x, z, w\}, \{x, y, w\}, \{y, z, w\}\}$.

Proof: Let X = {x, y, z, w}. There are 355 possible topologies on X. Let A be a fuzzy subset of X.

Case 1: Let $A = \frac{a}{x} + \frac{a}{y} + \frac{a}{z} + \frac{a}{w}$ where $0 \le a \le 1$. Then T(A) = {X}.

Topology generated by T (A) = Topology generated by {X} = $\{X, \emptyset\}$.

$\tau_A = \{X, \emptyset\}$. Only one topology on X is generated by the α-cuts of A.

Case 2: Let $A = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w}$ where $0 \le a, b, c, d \le 1$.

Consider the case a = b = c < d. Then T(A) = {X, {w}}.

Topology generated by T(A) = Topology generated by {X, {w}} = $\{X, \{w\}, \emptyset\}$.

$\tau_A = \{X, \{w\}, \emptyset\}$.

Consider the case a = b = c > d. Then T(A) = {X, {x, y, z}}.

Topology generated by T(A) = Topology generated by {X, {x, y, z}} = $\{X, \{x, y, z\}, \emptyset\}$.

$\tau_A = \{X, \{x, y, z\}, \emptyset\}$.

Other cases are analogously discussed below.

If a = b = d < c then $\tau_A = \{X, \{z\}, \emptyset\}$. If a = b = d > c then $\tau_A = \{X, \{x, y, w\}, \emptyset\}$.

If a = c = d < b then $\tau_A = \{X, \{y\}, \emptyset\}$. If a = c = d > b then $\tau_A = \{X, \{x, z, w\}, \emptyset\}$.

If b = c = d < a then $\tau_A = \{X, \{x\}, \emptyset\}$. If b = c = d > a then $\tau_A = \{X, \{y, z, w\}, \emptyset\}$.

In this case only 8 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Case 3: Let $A = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w}$ where $0 \le a, b, c, d \le 1$.

Consider the cases a = b, c = d, a < c. Then T(A) = {X, {z, w}}.

Topology generated by T (A) = Topology generated by {X, {z, w}}= $\{X, \{z, w\}, \emptyset\}$.
$\tau_A = \{X, \{z, w\}, \emptyset\}$.

Consider the cases a = b, c = d, a > c. Then T(A) = {X, {x, y}}.

Topology generated by T(A) = Topology generated by {X, {x, y}} = {X, {x, y}, ∅}.

τ_A ={X, {x, y}, ∅}.

Other cases are analogously discussed as follows:

If a = c, b = d, a < b then τ_A = {X, {y, w}, ∅}. If a = c, b = d, a > b then τ_A = {X, {x, z}, ∅}

If a = d, b = c, a < b then τ_A = {X, {y, z}, ∅}. If a = d, b = c, a > b then τ_A = {X, {x, w}, ∅}

In this case only 6 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Case 4: Let A = $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}$ where 0 ≤ a, b, c, d ≤ 1.

Consider the case a = b, a < c, a < d, c < d. Then T(A) = {X, {z, w}, {w}} .

Topology generated by T(A) = Topology generated by {X, {z, w}, {w}} = {X, {z, w},{w}, ∅}.

τ_A = {X, {z, w}, {w}, ∅}.

Other cases are analogously discussed below:

If a = c, a < b, a < d, b < d then τ_A = {X, {y, w}, {w}, ∅}.

If a = d, a < b, a < c, b < c then τ_A = {X, {y, z}, {z}, ∅}.

If b = c, a < b, a < d, b < d then τ_A = {X, {y, z, w}, {w}, ∅}.

If b = d, a < b, a < c, b < c then τ_A = {X, {y, z, w}, {z}, ∅}.

If c = d, a < b, a < c, b < c then τ_A = {X, {y, z, w}, {z, w}, ∅}.

In this case only 6 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Case 5: Let A = $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}$ where 0 ≤ a, b, c, d ≤ 1.

Consider the case a = b, a > c, a > d, c > d. Then T(A) = {X, {x, y}, {x, y ,z}}.

Topology generated by T (A) = Topology generated by {X,{x, y},{x, y, z}}= {X, {x, y}, {x, y, z}, ∅}.

τ_A = {X, {x, y}, {x, y, z}, ∅}.

Other cases are analogously discussed as follows:

If a = c, a > b, a > d, b > d then τ_A ={X, {x, z}, {x, y, z}, ∅}.

If a = d, a > b, a > c, b > c then τ_A ={X, {x, w}, {x, y, w}, ∅}.

If b = c, a > b, a > d, b > d then τ_A ={X, {x}, {x, y, z}, ∅}.

If b = d, a > b, a > c, b > c then τ_A = {X, {x, y, w}, {x}, ∅}.

If c = d, a > b, a > c, b > c then τ_A = {X, {x}, {x, y}, ∅}.

In this case only 6 topologies on X are generated by the α-cuts of fuzzy subsets of X.

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Case 6: Let $A = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w}$ where $0 \leq a, b, c, d \leq 1$.

Consider the case a = b, a > c, c < d, a < d. Then T(A) ={X, {w}, {x, y ,w}} .

Topology generated by T (A) = Topology generated by {X, {w}, {x, y, w}} = {X, {w}, {x, y, w}, ∅}.

τ_A = {X, {w}, {x, y, w}, ∅}.

Consider the case a = b, a > c, c < d, a > d . Then T (A) = {X, {x, y}, {x, y, w}}.

Topology generated by T (A) = Topology generated by {X,{x, y}, {x, y, w} ={X, {x, y}, {x, y, w},∅}.

τ_A ={X, {x, y}, {x, y, w},∅}.

Other cases are analogously discussed.

If a = c, a > b, b < d, a > d then τ_A = {X, {x, z}, {x, z, w},∅}.

If a = c, a > b, b < d, a < d then τ_A = {X, {w}, {x, z, w},∅ }.

If a = d, a > b, b < c, a < c then τ_A = {X, {z}, {x, z, w}, ∅}.

If a = d, a > b, b < c, a > c then τ_A = {X, {x, w}, {x, z, w}, ∅}.

If b = c, a > b, b < d, a < d then τ_A = {X, {w}, {x, w}, ∅}.

If b = c, a > b, b < d, a > d then τ_A = {X, {x}, {x, w}, ∅}.

If b = d, a < b, b > c, a > c then τ_A = {X, {y, w}, {x, y, w}, ∅}.

If b = d, a < b, b > c, a < c then τ_A = {X, {y, w}, {y, z, w}, ∅}.

If c = d, a > b, b < c, a > c then τ_A = {X, {x}, {x, z, w}, ∅}.

If c = d, a > b, b < c, a < c then τ_A = {X, {z, w}, {x, z, w}, ∅}.

In this case only 12 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Case 7: Let $A = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w}$ where $0 \leq a, b, c, d \leq 1$.

Consider the cases a = b, a < c, c > d, a < d. Then T(A) = {X, {z}, {z ,w}} .

Topology generated by T (A) = Topology generated by{X, {z}, {z, w}} = {X, {z}, {z ,w}, ∅}.

τ_A = {X, {z}, {z ,w}, ∅}.

Consider the case a = b, a < c, c > d, a > d. Then T(A) = {X ,{z},{x, y, z}}.

Topology generated by T(A) = Topology generated by {X,{z}, {x, y, z}} ={X,{z}, {x, y, z},∅}.

τ_A = {X, {z} {x, y, z},∅}.

Other cases are analogously discussed as follows

If a = c, a < b, b > d, a < d then τ_A = {X, {y}, {y, w}, ∅}.

If a = c, a < b, b > d, a > d then τ_A = {X, {y}, {x, y, z},∅}.

If a = d, a < b, b > c, a < c then τ_A = {X, {y}, {y, z}, ∅}.

If a = d, a < b, b > c, a > c then τ_A = {X, {y}, {x, y, w}, Ø}.

If b = c, a < b, b > d, a < d then τ_A = {X, {y, z}, {y, z, w}, Ø}.

If b = c, a < b, b > d, a > d then τ_A = {X, {y, z}, {x, y, z}, Ø}.

If b = d, a > b, b < c, a > c then τ_A = {X, {x}, {x, z}, Ø}.

If b = d, a > b, b < c, a< c then τ_A = {X, {z}, {x, z}, Ø}.

If c = d, a < b, b > c, a > c then τ_A = {X, {y}, {x, y, w}, Ø}.

If c = d, a < b, b > c, a < c then τ_A = {X, {y}, {y, z, w}, Ø}.

In this case only 12 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Case 8: Let A = $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w}$ where 0 ≤ a, b, c, d ≤ 1.

Consider the case a < b < c < d. Then T(A) = {X,{w}, {z, w}, {y, z, w}} .

Topology generated by T (A) = Topology generated by {X, {w}, {z, w}, {y, z, w}}.

={X,{w},{z, w}, {y, z, w}, Ø}.

τ_A = {X, { w } , {z , w}, {y, z, w}, Ø}.

Other cases are analogously discussed.

If a < b < d < c then τ_A = {X, {z}, {z, w}, {y, z, w}, Ø}.

If a < d < b < c then τ_A = {X, {z}, {y, z}, {y, z, w}, Ø}.

If a < c < b < d then τ_A = {X, {w}, {y, w}, {y, z, w}, Ø}.

If a < c < d < b then τ_A = {X, {y}, {y, w}, {y, z, w}, Ø}.

If a < d < c < b then τ_A = {X, {y}, {y, z}, {y, z, w}, Ø}.

If b < c < d < a then τ_A = {X, {x}, {x, w}, {x, z, w }, Ø}.

If b < c < a < d then τ_A = {X, {w}, {x, w}, {x, z, w},Ø}.

If b < a < c < d then τ_A = {X, {w}, {z, w}, {x, z, w}, Ø}.

If b < d < c < a then τ_A = {X, {x}, {x, z}, {x, z, w}, Ø}.

If b < d < a < c then τ_A = {X, {z}, {x, z}, {x, z, w}, Ø}.

If b < a < d < c then τ_A = {X, {z}, {z, w}, {x, z, w}, Ø}.

If c < d < b < a then τ_A = {X, {x}, {x, y}, {x, y, w}, Ø}.

If c < d < a < b then τ_A = {X, {y}, {x, y}, {x, y, w}, Ø}.

If c < a < d < b then τ_A = {X, {y}, {y, w}, {x, y, w}, Ø}.

If c < b < d < a then τ_A = {X, {x}, {x, w}, {x, y, w}, Ø}.

If c < b < a < d then τ_A = {X, {w}, {x, z}, {x, y, w}, Ø}.

If c < a < b < d then τ_A = {X, {w}, {y, w}, {x, y, w}, Ø}.

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If d < a < b < c then τ_A = {X, {z}, {y, z}, {x, y, z}, ∅}.

If d < a < c < b then τ_A = {X, {y}, {y, z} {x, y, z}, ∅}.

If d < c < a < b then τ_A = {X, {y}, {x, y}, {x, y, z}, ∅}.

If d < b < a < c then τ_A = {X, {z}, {x, z}, {x, y, z}, ∅}.

If d < b < c < a then τ_A = {X, {x}, {x, z}, {x, y, z }, ∅}.

If d < c < b < a then τ_A = {X, {x}, {x, y}, {x, y, z}, ∅}.

In this case only 24 topologies on X are generated by the α-cuts of fuzzy subsets of X.

Therefore we conclude that only 75 Topologies on X = {x, y, z, w} are generated by the α-cuts of fuzzy {x, y, z, w}.

CONCLUSION

In this paper, the topologies on finite sets having one element, two elements, three elements and four elements are discussed. Moreover it has been established that the number of topologies generated by the α -cuts of fuzzy subsets of finite sets having 1, 2, 3 and 4 elements are 1, 3, 13 and 75 respectively.

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