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EMIGRATION EFFECT ON A SINGLE SPECIES ECO SYSTEM

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ABSTARCT

This paper deals with a single species ecosystem with emigration. This can be characterized by a mathematical model equation $\frac{dN}{dt} = bN(K-N) + f(N)$, *where N is the population strength at a time't', 'K' is the carrying capacity and 'Kb' is natural birth rate of species assumed to be constants. Further f (N) denotes the Emigration function. The birth rate and the carrying capacity of the species are effected by the emigrants. The equilibrium points are disturbed and also their stability criteria included Special cases of emigration function considered are when emigration function f (N) is*

(i) A constant (ii) linear in N (iii) proportional to N^{α} $(\alpha = 2,3)$

In each of the cases the effective birth rate, carrying capacity are computed, equilibrium points are identified, criteria of stability established, the trajectories are illustrated, the threshold results are stated.

Key words: *carrying capacity, equilibrium points, stability, trajectories (N vs. t).*

1. INTRODUCTION:

Mathematical biology is a fast growing and well recognized, but not clearly defined, interdisciplinary subject, which attracted life and Medical Scientists apart from applied Mathematicians. Ecology is basically the study of the inter-relationship between species in nature and environment, sharing common resources and habitats.It is a common observation that no species of same kind cannot grow in isolation in any environment. Coexistence of species of different nature most essential for the mutual growth. However detailed study of isolated population (of the same species) would facilitate in understanding ecological interactions between species with diverse habits. Further there are also higher practical applications of single species models in biomedical sciences –in particular physiology. Single species models are of relevance in laboratory studies and in real world reflecting telescopic effects which influence the dynamics of the population.

Studies on growth of single species were initiated in the year 1798 by Malthus [1]. N (t) represents numbers of individuals in a population at a time 't', and then the Malthus growth model is $\frac{dN}{dt} = aN$, the solution is of which $N = N_0 e^{at}$ where N_0 is the initial population and 'a' is the growth constant. The population growth, as given by solution may be valid for a short time, but it cannot hold forever. Malthus model was improved by Dutch mathematical biologist Verhulst[2] in the year 1837 based on the observation that relative growth rate is a decreasing function of the population .$\frac{dN}{dt} = bN(K-N)$ where 'K' is carrying capacity (K=a/b), b is constant. Several improvements are suggested by M. Svirezhev and D.O.Logofet [3] J. N Kapur [4] to make the model more realistic. Freedman [5], J. D. Murray [6] discussed some deterministic Mathematical Models in population Ecology.

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When this carrying capacity is abundant there would be an incentive for the species to emigrate from outside. In contrast to this when carrying capacity is meager there would be migration of the species from the present habitat to another in search of resources needed for growth and maintenance.

2. NOTATIONS ADOPTED:

N (t): The strength of the population at a time instant t
a : Natural birth rate(=bk)
K : carrying capacity (a/b)
N_0 : Initial population strength at time t=0
b : multiplicative parameter
c : coefficient of emigration
$\overline{N}_i$ (i=1, 2, 3): Equilibrium points
f (N) : Characteristic of emigration

3. EMIGRATION MODEL EQUATION ANALYSIS

The model equations for the single species with Emigration are given by first order non linear differential equation.

Let 'N' be the species number at a time't', the growth rate equation of the species is

$$\frac{dN}{dt} = bN(K-N) + f(N) \tag{1}$$

The equilibrium points are defined by taking $\frac{dN}{dt} = 0$

i;e; $bN(K-N) + f(N) = 0$ (2)

The positive roots of the equation (2) are equilibrium points.

Let $\overline{N}$ be an equilibrium point i;e; $b\overline{N}(K-\overline{N}) + f(\overline{N}) = 0$ (3)

To examine the stability of this equilibrium point, consider small perturbation around equilibrium point

Let $N = \overline{N} + u$ (4)

where 'u' is a first order small quantity, Substituting (4) in (1), we get after neglecting higher powers of 'u', the differential equation for u :

$$\frac{du}{dt} = \left[bK - 2b\overline{N} + f'(\overline{N}) \right] u \tag{5}$$

$$\Rightarrow u = u_0 e^{\lambda t} \tag{6}$$

where u_0 is the initial value perturbation and $\lambda = \left[bK - 2b\overline{N} + f'(\overline{N}) \right]$

When λ is real the equilibrium state is

STABLE if $\lambda < 0$, i,e; $f'(N) < b(2\overline{N} - K)$

UNSTABLE if $\lambda > 0$, i,e; $f'(N) > b(2\overline{N} - K)$

NEUTRALLY STABLE if $\lambda = 0$, i,e; $f'(N) = b(2\overline{N} - K)$

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When λ is a complex the criterion for the stability is that the real part of λ is negative.

$\lambda < 0$, the equilibrium point state is stable

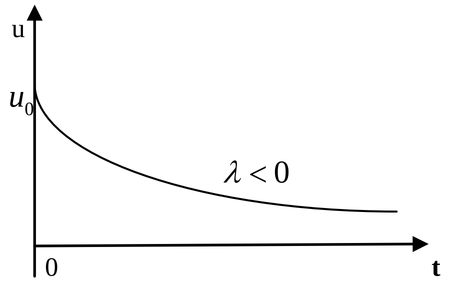


Fig (i): Stable

If $\lambda = 0$, $u = u_0$ for all 't' as such the $N = \overline{N} + u_0$

Hence equilibrium state is neutrally stable

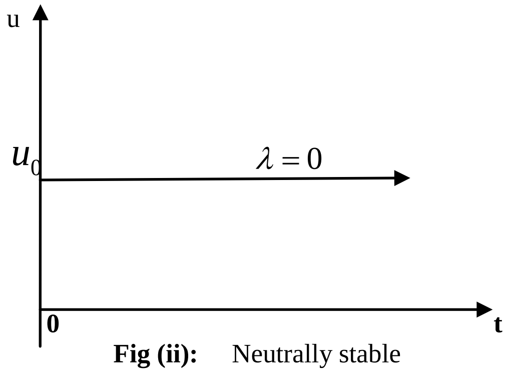


Fig (ii): Neutrally stable

If $\lambda > 0$, the equilibrium point is Unstable

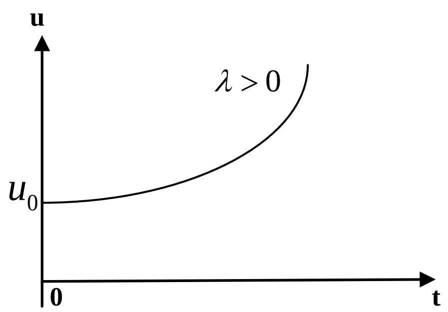


Fig (iii): Unstable

3. SPECIAL CASES

Case A: Emigration rate is a constant

Growth rate equation is $\dfrac{dN}{dt} = bN(K - N) + c$ (7)

The equilibrium states are given by $\dfrac{dN}{dt} = 0$

There exists only one equilibrium point $\overline{N} = \dfrac{bK + \sqrt{(bK)^2 + 4cb}}{2b}$ (> 0)

Stability of the equilibrium point:

The equation from (5) at equilibrium point $\overline{N}$ now reduces to

$$\frac{du}{dt} = \left[bK - 2b\left(\frac{bK + \sqrt{(bK)^2 + 4cb}}{2b} \right) \right] u \tag{8}$$

$$\Rightarrow u = u_0 e^{-\lambda t}, \text{ where } \lambda = \sqrt{(bK)^2 + 4cb} > 0 \quad (9)$$

Hence the equilibrium state is **STABLE**.

Now the solution is $N = \frac{K}{2} + q\left[\frac{1+pe^{-2bqt}}{1-pe^{-2bqt}}\right]$ (10)

where $p = \frac{2b(N_0 - q) - a}{2b(N_0 + q) - a}$ and $q = \sqrt{\frac{a^2 + 4bc}{4b^2}} = \sqrt{\frac{K^2}{4} + \frac{c}{b}}$

As $t \to \infty$, $N(t) \to \frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{c}{b}}$ is effective carrying capacity and this is more than the

Natural carrying capacity (K) (Vide Fig 1, 2, 3)

Trajectories (N vs. t)

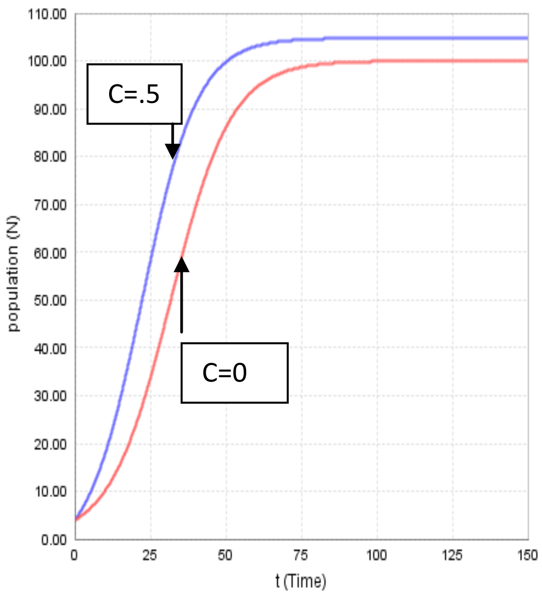


Fig. 1: b=.001, N (0) =4, K=100 $\left(N_0 < \frac{K}{2}\right)$

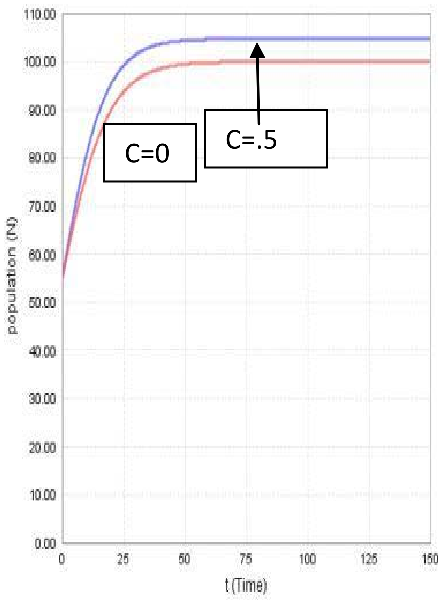


Fig. 2: K=100, b=.001, N (0) =55 $\left(N_0 > \frac{K}{2}\right)$

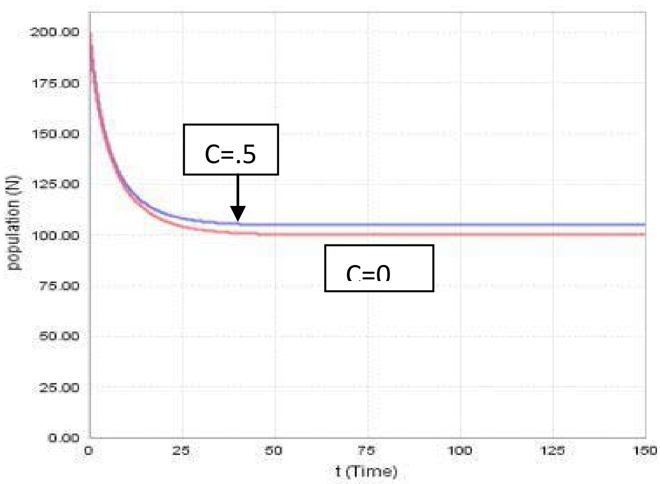


Fig. 3: K=100, N (0) =200, b=.001 $\left(N_0 > K\right)$

Case B: Emigration rate is proportional to population

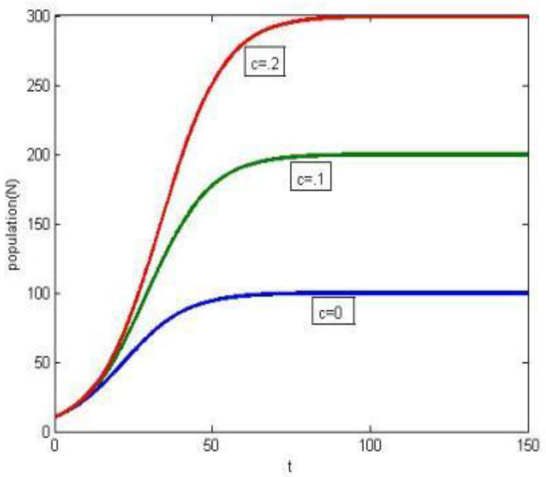
Growth rate equation is $\frac{dN}{dt} = bN(K-N) + cN$ (11)

$$= bN\left[(K + \frac{c}{b}) - N\right] \tag{12}$$

This is same as the logistic model with increased natural birth rate and carrying capacity is increased from K to $K + \frac{c}{b}$

Whose solution is $N = \dfrac{\left(K + \frac{c}{b}\right)N_0}{\left(K + \frac{c}{b} - N_0\right)e^{-Kbt} + N_0}$ (13)

Trajectories (N vs. t)



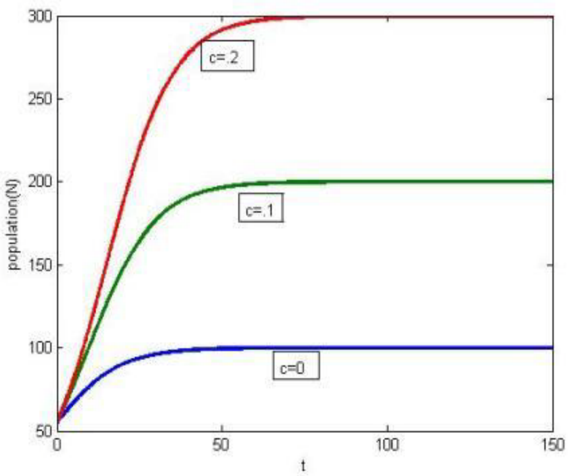


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Fig. 3: K=100, N (0) =10, b=.001 $\left(N_0 < \frac{K}{2}\right)$

Fig. 4: K=100, N (0) =55, b=.001 $\left(N_0 > \frac{K}{2}\right)$

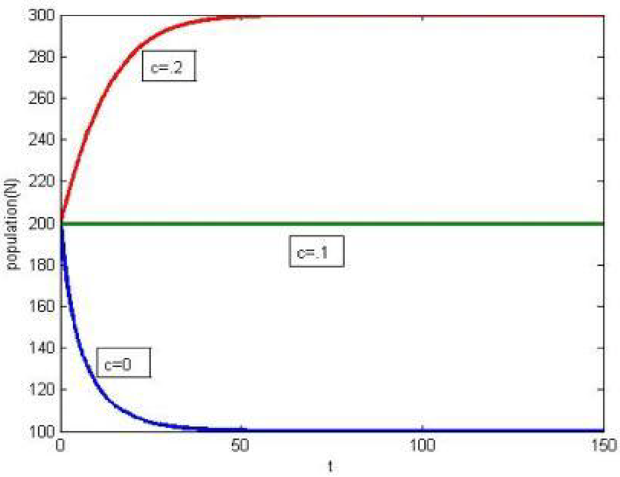

Fig. 6: K=100, N (0) =200, b=.001 $\left(N_0 > K\right)$

Case (C): Emigration rate is proportional to square of population

Growth rate equation is $\frac{dN}{dt} = bN(K-N) + cN^2$ (14)

$$= bN\left[K - \left(1 - \frac{c}{b}\right)N\right] \tag{15}$$

Which is same as the logistic model with no change in natural birth rate and carrying capacity is increased from K to $\dfrac{K}{1-\dfrac{c}{b}}$

The solution of (15) is $N = \dfrac{\left(\dfrac{bK}{b-c}\right)N_0}{\left(\dfrac{bK}{b-c} - N_0\right)e^{-Kbt} + N_0}$ (b>c) (16)

The Trajectories (N vs. t)

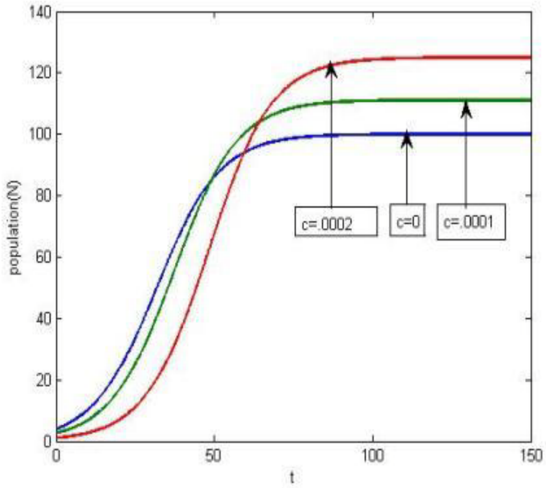

Fig. 7: K=100, N (0) = 4, b=.001 $\left(N_0 < \dfrac{K}{2}\right)$

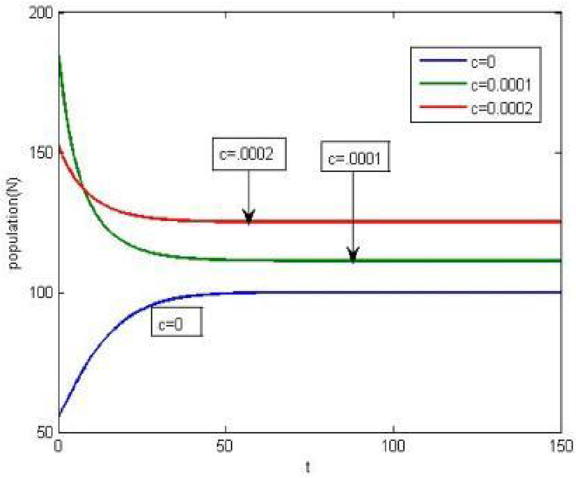

Fig. 8: K=100, N (0) = 55, b=.001 $\left(N_0 > \dfrac{K}{2}\right)$

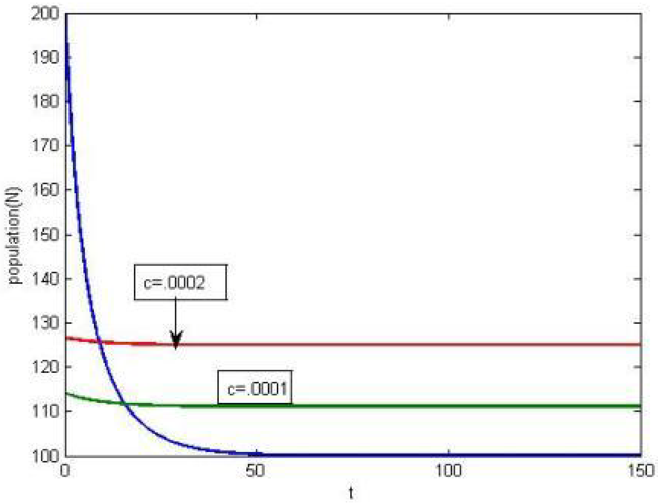

Fig. 9: K=100, N (0) =200, b=.001 $\left(N_0 > K\right)$

Case (D): Emigration rate is proportional to cube of population $f(N) = cN^3$

Growth rate equation $\dfrac{dN}{dt} = bN(K-N) + cN^3 = bN\left[(K-N) + \dfrac{c}{b}N^2\right]$ (17)

The equilibrium states are given by $\dfrac{dN}{dt} = 0$ i.e.; $N\left[\left(K-N\right) + \dfrac{c}{b}N^2\right] = 0$ (18)

The roots of the equation (18) are $\overline{N_1}, \overline{N_2}, \overline{N_3}$

Case (D.i): b < 4cK

There exists only 1equilibrium point E1: $\frac{du}{dt} = \left(bK - \frac{b^2}{4c}\right)u \Rightarrow u = u_0 e^{\left(bK - \frac{b^2}{4c}\right)t}$

Stability of equilibrium state E_1**:** $\overline{N_1} = 0$

Consider a small perturbation in u. Then $N = \overline{N_1} + u = u$

$$\frac{du}{dt} = bKu \Rightarrow \quad u = u_0 e^{Kbt} \text{ as } t \to \infty, \quad u \to \infty \tag{19}$$

This equilibrium state is **Unstable**

Case (D. ii): b = 4cK

There will be two equilibrium points E_1: $\overline{N_1} = 0$ and E_2: $\frac{b}{2c}$

As before the equilibrium state E_1 is unstable

Stability of equilibrium state E₂: $\overline{N_2} = \frac{b}{2c}$

Consider a small perturbation in u. Then $E_1 : \overline{N_1} = 0$

The linearized equation for 'u' is $\frac{du}{dt} = \left(bK - \frac{b^2}{4c}\right)u \Rightarrow u = u_0 e^{\left(bK - \frac{b^2}{4c}\right)t}$ (20)

Equilibrium state is	Stable	If	$c < \frac{b}{4K}$
	Unstable	If	$c > \frac{b}{4K}$
	Neutrally stable	If	$c = \frac{b}{4K}$

Case (D.iii): b > 4cK

There exits equilibrium points

$$E_1 : \overline{N_1} = 0,\ E_2 : \overline{N_2} = \frac{b + \sqrt{b^2 - 4cbK}}{2c},\ E_3 : \overline{N_3} = \frac{b - \sqrt{b^2 - 4cbK}}{2c}$$

Stability of equilibrium state E_1: $\overline{N_1} = 0$

As before equilibrium state E_1 is unstable

Stability of equilibrium state E_2: $\overline{N_2} = \frac{b + \sqrt{b^2 - 4cbK}}{2c}$

Let $N = \overline{N_2} + u$ in the equation (5), the equation after neglecting higher powers of u

$$\frac{du}{dt} = \left[bK - 2b\left(\frac{b + \sqrt{b^2 - 4bcK}}{2c}\right) + 3c\left(\left(\frac{b + \sqrt{b^2 - 4bcK}}{2c}\right)^2\right)\right]u \tag{21}$$

Let $\Omega = \sqrt{b^2 - 4cbK}$

Then $u = u_0 e^{-\left(2bK - \frac{b^2}{4c} - \frac{b\Omega}{2c}\right)t}$ (22)

Equilibrium state is E_2

Stable If $c > \frac{b}{4K}$

Unstable If $c \le \frac{b}{4K}$

Stability of equilibrium state E_3: $\overline{N_3} = \dfrac{b-\sqrt{b^2-4cbK}}{2c}$

Substitute $N = \overline{N_3} + u$ in the equation (5)

$$\frac{du}{dt} = \left[bK - 2b\left(\frac{b-\sqrt{b^2-4bcK}}{2c} \right) + 3c\left(\left(\frac{b-\sqrt{b^2-4bcK}}{2c} \right)^2 \right) \right] u \tag{23}$$

$$\Rightarrow u = u_0 e^{-\left(2bK - \frac{b^2}{4c} + \frac{b\Omega}{2c}\right)t} \tag{24}$$

Then the equilibrium state is E_3

Stable If $c \ge \frac{b}{4K}$

Unstable If $c < \frac{b}{4K}$

Trajectories (N Vs t)

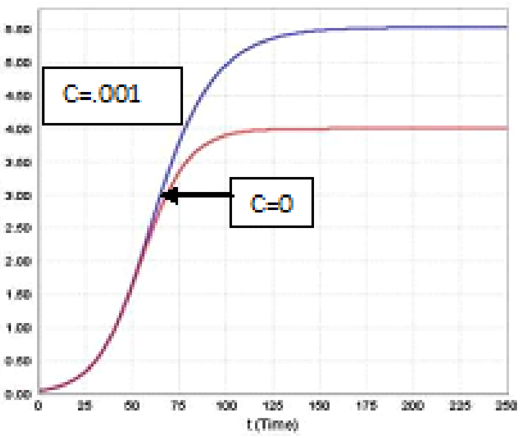

Fig 12: b=.02, N(0)=.05, K=4 $\left(N_0 < \frac{K}{2}\right)$

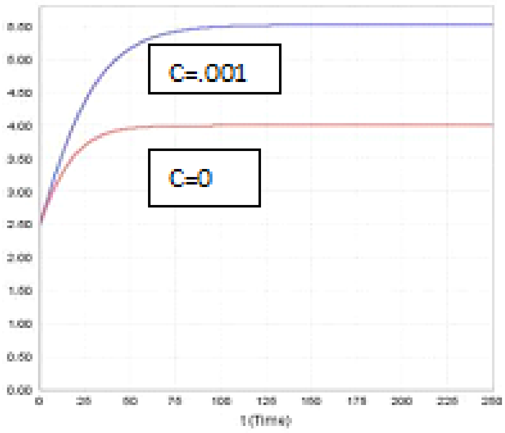

Fig.13: b=.02, N(0)=2.5, K=4 $\left(N_0 > \frac{K}{2}\right)$

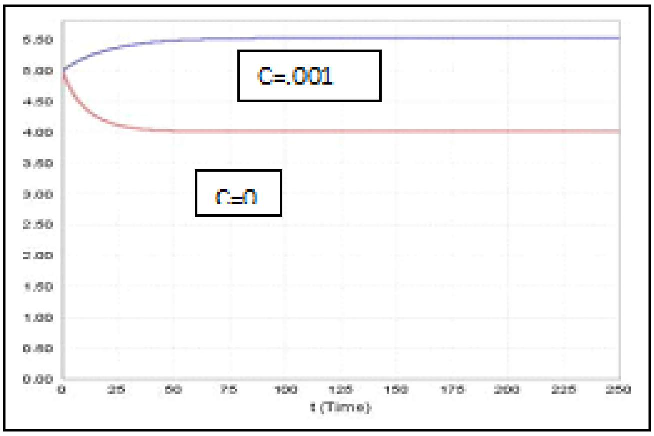

Fig. 14: b=.02, N (0) =2.5, K=5.5 $\left(N_0 > K\right)$

CONCLUSION

From all the trajectories (1 to14) it is observed that the carrying capacity is increased with variation of 'c'. A very small value of 'c' is identified for the case population is proportional to the square of the population.

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