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FIXED POINT THEOREMS FOR SELF MAPS ON A PARTIALLY ORDERED SET WITH A METRIC INVOLVING RATIONAL EXPRESSIONS

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ABSTRACT

In this paper, we obtain sufficient conditions for a self map on a partially ordered set with a metric controlled by a non decreasing function involving rational expressions, to have a fixed point. We also obtain condition for the uniqueness of the fixed point. An example, to show the non uniqueness of the fixed points is proved. We conclude that maps of the type discussed partition the partially ordered set with respect to the fixed points.

Key words: *ordered metric space, rational expression, partially ordered set.*

Mathematical subject classification (2010): *47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

Banach contraction principle was generalized by many authors in several ways. One among the ways of generalizations of Banach contraction mapping theorem was replacing the right hand side of the inequality by the related terms involving rational expressions.

In 1980, Jaggi and Dass [13] proved the existence of fixed point for a self map T on a complete metric space using rational expression.

Theorem: 1.1 (Jaggi and Dass [13]) Let T be a self map of a metric space (X, d) which satisfy

(i) for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)} + \beta d(x, y) \tag{1.1.1}$$

for all $x, y \in X, x \neq y$

(ii) there exists $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X Then z is the unique fixed point of T in X.

Further, Alber and Guerre-Delabrire [3] defined weakly contractive map on a Hilbert space and established a fixed point theorem for such map. This theorem was extended to metric spaces by Rhoades [22]. Existence of fixed points of self maps satisfying weakly contractive type inequalities have been studied by several authors. For example we refer [3, 5, 6, 11, 12, 14, 22].

Definition: 1.2 [22] Let (X, d) be a complete metric space. A mapping $T: X \to X$ is said to be a weakly contractive map if $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X$ where $\varphi: [0, \infty) \to [0, \infty)$ is continuous and non decreasing function with $\varphi(t) = 0$ iff $t = 0$.

Theorem: 1.3 [22] Let (X, d) be a complete metric space and T be a weakly contractive map on X. Then T has a unique fixed point.

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Now a days a lot of research work is going on in the field of existence of fixed points in partially ordered metric spaces. The first result in this direction was given by Ran and Reurings [20]. Many other results on the existence of fixed points or common fixed points in ordered spaces were studied by several authors. For more literature on the existence of fixed points in ordered metric spaces we refer [1, 2, 4, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 20, 23].

Recently, Kameswari. M.V.R [14] proved the existence and uniqueness of fixed points for a weakly contractive map via rational expressions in partially ordered metric spaces.

Theorem: 1.4 [14] Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following inequality: there exists $\varphi: [0, \infty) \to [0, \infty)$ which is lower semi continuous and $\varphi(t) = 0$ iff $t = 0$ such that $d(Tx, Ty) \leq M(x, y) - \varphi\big(M(x, y)\big)$ for all $x, y \in X, x \geq y, x \neq y$ where

$$M(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)}, \frac{d(x,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)+d(x,y)}, d(x,y)\right\}.$$

Suppose that either

(1) T is continuous

or (2) X has the following property:

if $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \leq x \ \forall\, n \geq 1$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

The purpose of this paper is to prove the existence and uniqueness of fixed points for a weakly contractive map via rational expressions in partially ordered metric spaces, by eliminating the lower semi continuity on φ in [14], however assuming φ to be non-decreasing.

2. MAIN RESULTS

Theorem: 2.1 Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following inequality: there exists $\varphi: [0, \infty) \to [0, \infty)$ which is increasing and $\varphi(t) > 0$ if $t > 0$ such that

$$d(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big) \tag{2.1.1}$$

for all $x, y \in X, x, y$ are comparable and $x \neq y$ where $m(x,y) = \frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)} + d(x,y)$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then the sequence $\{T^n(x_0)\}, n = 1,2, ...$is a Cauchy sequence in (X, d).

Proof: Let $x_0 \in X$ be such that $x_0 \leq Tx_0$.

Consider the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, n = 0,1,2, ...$

By induction, first we show that $x_n \leq x_{n+1} \ \forall\, n \geq 0$ …. (2.1.2)

By assumption, we have $x_0 \leq Tx_0 = x_1$. Thus (2.1.2) is true for $n = 0$.

Suppose that (2.1.2) is true for some $n = m$. i.e. $x_m = x_{m+1}$.

Now, by increasing property of T, we have $Tx_m \leq Tx_{m+1}$

So that $x_{m+1} \leq x_{m+2}$

Therefore (2.1.2) is true for all $n \geq 1$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = Tx_n = x_{n+1}$.

Hence $x_{n+2} = Tx_{n+1} = Tx_n = x_n$.

Then $x_n = x_{n+1} = x_{n+2} = \cdots$

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n \neq x_{n+1}\ \forall\, n$.

Since $x_n \geq x_{n-1}\ \forall\, n \geq 1$ from (2.1.1), we have

$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq m(x_n, x_{n-1}) - \varphi\big(m(x_n, x_{n-1})\big) < m(x_n, x_{n-1})$

Now $m(x_n, x_{n-1}) = \frac{d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})}{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)+d(x_n,x_{n-1})} + d(x_n, x_{n-1})$
$= 0 + d(x_n, x_{n-1}) = d(x_n, x_{n-1})$

$\therefore\ m(x_n, x_{n-1}) = d(x_n, x_{n-1}), n = 1, 2, 3, \ldots$ (2.1.3)

Now $d(x_{n+1}, x_n) \leq m(x_n, x_{n-1}) - \varphi(m(x_n, x_{n-1}))$
$= d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1}))$ (from (2.1.3)) (2.1.4)
$< d(x_n, x_{n-1})$

$\therefore\ \{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence converges, say, to $\delta \geq 0$.

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$.

We shall now show that $\delta = 0$.

From (2.1.4), we have $\varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n) \to \delta - \delta = 0$ as $n \to \infty$

But $\delta \leq d(x_n, x_{n+1})\ \forall\, n$

$\Rightarrow \varphi(\delta) \leq \varphi\big(d(x_n, x_{n+1})\big)$ ($\because \varphi$ is increasing)

$\Rightarrow \varphi(\delta) \leq \lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0$

$\therefore\ \varphi(\delta) = 0$, consequently $\delta = 0$. ($\because \delta > 0 \Rightarrow \varphi(\delta) > 0$)

i.e. $\lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0$ (2.1.5)

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Otherwise there exists an $\varepsilon > 0$ and sequences $\{m(k)\}, \{n(k)\}$ with $m(k) > n(k) > k$ such that

$d\big(x_{m(k)}, x_{n(k)}\big) \geq \varepsilon$ and $d\big(x_{n(k)}, x_{m(k)-1}\big) < \varepsilon$

$\therefore\ \varepsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)}\big)$

Letting $k \to \infty$, using (2.1.5) and (2.1.6), we have $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$ (2.1.7)

Also $d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big)$

$d\big(x_{m(k)-1}, x_{n(k)-1}\big) \leq d\big(x_{m(k)-1}, x_{m(k)}\big) + d\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{n(k)}, x_{n(k)-1}\big)$

Letting $k \to \infty$, using (2.1.5), (2.1.6) and (2.1.7), we have

$\varepsilon = \lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) \leq \lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$

So that $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \varepsilon$ (2.1.8)

Now $d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big)$

Letting $k \to \infty$, using (2.1.5) and (2.1.8), we have

$\varepsilon \leq \lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) \leq \lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$

Therefore $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) = \varepsilon$ (2.1.9)

We have $\varepsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)}\big)$
$< d\big(x_{m(k)}, x_{m(k)-1}\big) + \varepsilon$

Letting $k \to \infty$, using (2.1.5) and (2.1.6), we have $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)}\big) = \varepsilon$

Since $m(k) > n(k)$, we have $x_{m(k)-1} > x_{n(k)-1}$.

$$\begin{aligned}\text{We have } d\big(x_{m(k)}, x_{n(k)}\big) &= d\big(Tx_{m(k)-1}, Tx_{n(k)-1}\big)\\ &\leq m\big(x_{m(k)-1}, x_{n(k)-1}\big) - \varphi\Big(m\big(x_{m(k)-1}, x_{n(k)-1}\big)\Big)\\ &= \frac{d\big(x_{m(k)-1},x_{m(k)}\big)d\big(x_{n(k)-1},x_{n(k)}\big)}{d\big(x_{m(k)-1},x_{n(k)}\big)+d\big(x_{n(k)-1},x_{m(k)}\big)+d\big(x_{m(k)-1},x_{n(k)-1}\big)} + d\big(x_{m(k)-1}, x_{n(k)-1}\big)\\ &\quad -\varphi\left(\frac{d\big(x_{m(k)-1},x_{m(k)}\big)d\big(x_{n(k)-1},x_{n(k)}\big)}{d\big(x_{m(k)-1},x_{n(k)}\big)+d\big(x_{n(k)-1},x_{m(k)}\big)+d\big(x_{m(k)-1},x_{n(k)-1}\big)} + d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right)\end{aligned}$$

$$\begin{aligned}\text{Therefore } \varphi\Big(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\Big) &\leq \varphi\Big(m\big(x_{m(k)-1}, x_{n(k)-1}\big)\Big) \ (\because \varphi \text{ is increasing})\\ &\leq m\big(x_{m(k)-1}, x_{n(k)-1}\big) - d\big(x_{m(k)}, x_{n(k)}\big)\\ &\to \varepsilon - \varepsilon = 0 \text{ as } k \to \infty\end{aligned}$$

$\therefore \lim_{n\to\infty} \varphi\Big(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\Big) = 0$

From (2.1.8), $d\big(x_{m(k)-1}, x_{n(k)-1}\big) > \frac{\varepsilon}{2}$ for large k.

$\therefore \varphi\left(\frac{\varepsilon}{2}\right) \leq \varphi\Big(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\Big) \to 0$ as $k \to \infty$

Therefore $\varphi\left(\frac{\varepsilon}{2}\right) = 0$, again a contradiction.

$\therefore \varepsilon = 0$

Therefore $\{x_n\}$ is a Cauchy sequence.

Hence $\{T^n x_0\}$ is a Cauchy sequence.

Theorem: 2.2 In addition to the hypothesis of Theorem 2.1, suppose (X, d) is complete and T is continuous. Then T has a fixed point.

Proof: Suppose $x_0 \leq Tx_0$. Let $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$. Then by (2.1.2), $\{x_n\}$ is an increasing sequence.

Now, by Theorem 2.1, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

Since T is continuous, $Tx_n \to Ty$ i.e $x_{n+1} \to Ty$

But $x_{n+1} \to y$.

Therefore $Ty = y$.

$\therefore y$ is a fixed point of T.

That is $y = \lim_{n\to\infty} T^n x_0$ is a fixed point of T.

Theorem: 2.3 In addition to hypothesis of Theorem 2.1, suppose (X, d) is complete and X has the following property:

If $\{z_n\}$ is an increasing sequence with $z_n \to z$, then $z_n \geq z\ \forall\ n$ (2.3.1)

Then T has a fixed point.

Proof: Suppose $x_0 \leq Tx_0$. Let $x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$..

Then by (2.1.2), $\{x_n\}$ is an increasing sequence.

Now, by Theorem 2.1, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

We may assume that $x_n \neq y\ \ \forall\ n$.

By (2.3.1), $y \geq x_n\ \forall\ n$

$\therefore$ By (2.1.1), we have

$$d(x_{n+1}, Ty) = d(Tx_n, Ty) \leq m(x_n, y) - \varphi\big(m(x_n, y)\big) \leq m(x_n, y) \quad (2.3.2)$$

Now, $m(x_n, y) = \frac{d(x_n,Tx_n)d(y,Ty)}{d(x_n,Ty)+d(y,Tx_n)+d(x_n,y)} + d(x_n, y) \to 0$ as $n \to \infty$. ($\because Tx_n = x_{n+1} \to y$)

From (2.3.2), letting $n \to \infty$, we get $d(y, Ty) = 0$

$\therefore Ty = y$

Therefore y is a fixed point of T.

Combining Theorem 2.1, Theorem 2.2 and Theorem 2.3, we have the following theorem.

Theorem: 2.4 Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is complete. Suppose $T: X \to X$ is a non decreasing function satisfying the following:
(2) $\varphi: [0, \infty) \to [0, \infty)$ is increasing and $\varphi(t) > 0$ if $t > 0$
(3) $d(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big)$ (2.4.1)
for all $x, y \in X, x, y$ are comparable
(4) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$
(5) either T is continuous or X has the following property:

If $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \leq x\ \forall\ n \geq 1$. Then T has a fixed point.

(In fact, the sequence $\{x_n\}$ denoted by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$ with x_0 as in (3), is a Cauchy sequence and hence converges say to y which is a fixed point of T)

On similar lines as in the proof of Theorem 2.3, the following theorem can be proved.

Theorem: 2.5 Under hypothesis of Theorem 2.4, except (2) replaced by (4'): $d(Tx, Ty) \leq M(x, y) - \varphi\big(M(x, y)\big)$ for all $x, y \in X$ whenever x, y are comparable and $x \neq y$ where

$$M(x, y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)}, \frac{d(x,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)+d(x,y)}, d(x, y)\right\}.$$

Then T has a fixed point.

Note: Theorem 2.5 is proved by Kameswari. M.V.R [14], by assuming lower-semi continuity of φ, instead of increasing nature of φ.

Observation: Suppose φ in Theorem 2.5, is such that $t - \varphi(t)$ is an increasing function in t. Then Theorem 2.5 follows as a corollary from Theorem 2.4, since $M(x, y) \leq m(x, y)$.
The following theorem can be established as in Theorem 2.4

Theorem: 2.6 Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is complete. Suppose $T: X \to X$ is a non decreasing function satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is non decreasing and $\varphi(t) > 0$ for $t > 0$
(2) $d(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big)$ for all $x, y \in X,$ whenever x, y are comparable
(3) there exists $x_0 \in X$ such that $x_0 \geq Tx_0$
(4) either T is continuous or X has the following property:

If $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \geq x \; \forall \; n \geq 1$. Then T has a fixed point.

(In fact, the sequence $\{x_n\}$ denoted by $x_{n+1} = Tx_n,\; n = 0, 1, 2, \ldots$ with x_0 as in (3), is a Cauchy sequence and hence converges say to y which is a fixed point of T) Combining Theorems 2.4 and 2.6, we have the final version of our main theorem., which proves the existence of fixed point in a constructive way.

Theorem: 2.7 Let $(X, \leq)$ be a poset and (X, d) is complete metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is non decreasing and $\varphi(t) > 0$ for $t > 0$
(2) $d(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big)$ for all $x, y \in X,$ whenever x, y are comparable
(3) there exists $x_0 \in X$ such that x_0 are Tx_0 are comparable.
(4) either T is continuous or X has the following property:
 (i) If $\{z_n\}$ is an increasing sequence in X with $z_n \to z$, then $z_n \geq z$, then $z_n \leq z \; \forall \; n \geq 1$.
 (ii) If $\{z_n\}$ is a decreasing sequence in X with $z_n \to z$, then $z_n \geq z$, then $z_n \geq z \; \forall \; n \geq 1$.
Then T has a fixed point.

In fact, the sequence $\{x_n\}$ denoted by $x_{n+1} = Tx_n,\; n = 1, 2, \ldots$ is a Cauchy sequence and hence converges say to x and x is a fixed point of T.

Theorem: 2.8 Under the hypothesis of Theorem 2.4, suppose x is a fixed point of T. Suppose there exists $z \in X$ such that $z \leq x$. Then $x = \lim_{n\to\infty} z_n$ where $z_{n+1} = Tz_n, n = 0, 1, 2, \ldots$ where $z_0 = z$.

Proof: We have $z \leq x \Rightarrow Tz \leq Tx = x \Rightarrow z_1 \geq x$

By induction, we can show that $z_n \leq x \; \forall \; n = 0, 1, 2, \ldots$

$$\therefore d(z_{n+1}, x) = d(Tz_n, Tx) \leq d(z_n, x) - \varphi\big(d(z_n, x)\big)$$
$$\leq d(z_n, x) \qquad (\because z_n \text{ and } x \text{ are comparable}) \tag{2.8.1}$$

$$\text{There fore } \varphi(d(z_n, x)) \leq d(z_n, x) - d(z_{n+1}, x) \tag{2.8.2}$$

By (2.14.1), $\{d(z_n, x)\}$ is a decreasing sequence and hence tends to a limit say δ.

So that $\delta \leq d(z_n, x) \; \forall \; n \geq 1$

Therefore $\varphi(\delta) \leq \varphi\big(d(z_n, x)\big) \; \forall \; n \geq 1$

$\therefore \; \varphi(\delta) \leq \varphi\big(d(z_n, x)\big) \leq d(z_n, x) - d(z_{n+1}, x) \to \delta - \delta = 0$ as $n \to \infty$

Hence $\varphi(\delta) = 0$

Therefore $\delta = 0$ (by (1) of Theorem 2.7)

$\therefore \; d(z_n, x) \to 0$ as $n \to \infty$

$\therefore \; z_n \to x$ as $n \to \infty$.

Theorem: 2.9 Under the hypothesis of Theorem 2.6, suppose x is a fixed point of T. Suppose there exists $z \in X$ such that $z \geq x$. Then $x = \lim_{n\to\infty} z_n$ where $z_{n+1} = Tz_n, n = 0, 1, 2, \ldots$ where $z_0 = z$.

The proof is similar to that of Theorem 2.8.

The following theorem establishes the uniqueness of the fixed point, while Theorem 2.7 provides the existence of fixed point of T.

Theorem: 2.10 Suppose the hypothesis of Theorem 2.7 holds. Further assume that, if x and y are fixed points of T, then there exists $z \in X$ such that x, z are comparable and y, z are comparable.

Then T has a unique fixed point.

Proof: By Theorem 2.7, T has a fixed point.

Suppose x, y are fixed points of T. By hypothesis there exists $z \in X$ such that z is comparable with x and z is comparable with y.

Case (i): $z \leq x \; and \; z \leq y$

Then by Theorem 2.8, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case (ii): $z \leq x \; and \; z \geq y$

Then by Theorem 2.8 and Theorem 2.9, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case (iii): $z \geq x \; and \; z \geq y$

Then by Theorem 2.9, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case (iv): $z \geq x \; and \; z \leq y$

Then by Theorem 2.8 and Theorem 2.9, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.
Hence the theorem is proved.

Note: If $(X, \leq)$ is a lattice, the condition (5) in Theorem 2.10 is automatically satisfied.

The following example shows that the absence of (5) in Theorem 2.10 may result in many fixed points.

Example: 2.11 Let $X = \{1, 2,3, ...\}$ with discrete ordering and discrete metric.

i.e. $x \leq y \; iff \; x = y$ and $d(x, y) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; x \neq y \end{cases}$

Then (X, d) is a complete metric space and x is comparable only with itself. Let $Tx = x \;\; \forall \, x \in X$ (i.e. T is the identity map)

$$m(x, y) = \frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)} + d(x, y = \; d(x, y) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; x \neq y \end{cases}$$

Define $\varphi(t) = \frac{t}{2}$ for $t \in [0, \infty)$

Then clearly $d(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big)$ for all $x, y \in X,$ whenever x, y are comparable.

But every point in X is a fixed point.

Here we observe that conditions of (1), (2), (3) and (4) of the Theorem 2.7 are satisfied, but condition of Theorem 2.10 is violated. (Since for any two distinct points in X, there is no point comparable with these two)

CONCLUSION

Under the hypothesis of Theorem 2.7 and in view of Theorem 2.10, the fixed point set $\mathfrak{F}$ of F decomposes the set X into pointwise disjoint sets $\{S_p \,/ p \in \mathfrak{F}\}$ in the following way:

For $p \in \mathfrak{F}$, write $S_p = \{a \in X : p \; is \; comparable \; with \; a\,\}$

Then (i) $S_p \neq \emptyset$, since $p \in S_p$
(ii) S_p and S_q are disjoint whenever $p, q \in \mathfrak{F}$ and $p \neq q$
(iii) $S = \bigcup_{p\in\mathfrak{F}} S_p$ may be a proper sub set of X

in which case $X - S$ does not contain any fixed point of F.

This can be done by suitably adjusting the map T in Example 2.12)

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