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# Geodomination, g-independence and g-irredundance 

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#### Abstract

The concept of geodetic set was introduced by F. Buckley and F.Harary in [1] and G.Chartrand , F.Harary and P.Zhang in [2]. In [4], the geodetic number was defined by F.Harary, E.Loukakis and C.Tsouros. In this paper, we introduce the concept of geodetic neighbourhood(g-neighborhood) and closed geodetic neighbourhood sets of a pair of vertices of a connected graph $G$ with atleast two vertices. And, we define the geodetic number of a graph using $g$-neighbourhood sets. Further, we introduce some new concepts such as $g$-isolated vertices, $g$-independence, $g$-independence number, $g$-connectedness of a graph, $g$-independent geodetic set, $g$-independent geodetic number, $g$-irredundance number etc. Results connecting the above defined parameters are developed.


Keywords: $g$-neighborhood , g-independence, $g$-connectedness, $g$-irredundance.

## I. INTRODUCTION

Through out this paper, we consider only finite, undirected, connected graphs with at least two vertices and with out loops and multiple edges. For graph theoretic representations, we refer [5]. Let $G=(V, E)$ be any graph and $u, v \in V(G)$ such that $u \neq v$. $d(u, v)$ is the length of the shortest path connecting $u$ and $v$. A $u-v$ geodesic is a $u-v$ path of length $d(u, v)$. For a pair $u, v$ of vertices in $G$, the closed interval $I[x, y]$ is defined as $I[x, y]=\{x, y\} \cup\{v: v$ is an internal vertex of an $x-y$ geosic in $G\}$. In this paper, we define the geodetic neighborhoood(g-neighborhood) set of a pair $x, y$ of vertices in $G$ as the open interval $\mathrm{I}(\mathrm{x}, \mathrm{y})=\{\mathrm{v}$ : v is an internal vertex of an $\mathrm{x}-\mathrm{y}$ geodesic in G$\}$. We denote it by $\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y})$. Correspondingly, we call $\mathrm{I}[\mathrm{x}, \mathrm{y}]$ as the closed geodetic neighborhood set of $\mathrm{x}, \mathrm{y}$ and we denote it as $\mathrm{N}_{\mathrm{g}}[\mathrm{x}, \mathrm{y}]$. That is, $\mathrm{N}_{\mathrm{g}}[\mathrm{x}, \mathrm{y}]=\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y}) \cup\{\mathrm{x}, \mathrm{y}\}$. For $\mathrm{S} \subseteq \mathrm{V}, \mathrm{I}[\mathrm{S}]$ is defined as $\mathrm{I}[\mathrm{S}]=\underset{x, y \in S}{\cup} \mathrm{I}[\mathrm{x}, \mathrm{y}]=\underset{x, y \in S}{\cup} \mathrm{~N}_{\mathrm{g}}[\mathrm{x}, \mathrm{y}]$. We denote it as $\mathrm{N}_{\mathrm{g}}[\mathrm{S}]$. A set $S$ of vertices in $G$ is called a geodominating (or geodetic) set of $G$ if $I[S]=V$. Equivalently, a set $S$ of vertices in $G$ is a geodetic set of G if $\mathrm{N}_{\mathrm{g}}[\mathrm{S}]=\mathrm{V}$. A geodetic set of minimum cardinality is called a minimum geodetic set in [2] (or a geodetic basis in [1]). A geodetic set $S$ is said to be a minimal geodetic set of $G$ if no proper subset of $S$ is a geodetic set of G . The minimum cardinality of all minimal geodetic sets of G is the geodomination(or geodetic) number of G . It is denoted as $g(G)[1,2,4]$.

Definition: 1.1 Let $G=(V, E)$ be any graph and $v \in V(G)$. The neighborhoood of $v$, written as $N_{G}(v)$ or $N(v)$ is defined by $N(v)=\{x \in V(G): x$ is adjacent to $v\}$. The closed neighborhood of of $v$ is defined as $N[v]=N(v) \cup\{v\}$.

Definition: 1.2 A vertex v in $G$ is an extreme vertex of $G$ if the sub graph induced by its neighbors is complete.
Definition: 1.3 Let $G=(V, E)$ be any graph. Then, $\mathrm{G}^{+}$is the graph obtained from G by attaching a pendant vertex to each vertex of $G$.

Definition: Generalized Hajo’s graph: 1.4[6] For $k \geq 3$, the generalized Hajos graph $H_{k}$ is a graph on $n=k+\binom{k}{2}$ vertices with vertex set $\mathrm{V}\left(\mathrm{H}_{\mathrm{k}}\right)=\left\{\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{k}}\right\} \cup\left\{\mathrm{y}_{\mathrm{i}, \mathrm{j}}: 1 \leq \mathrm{i}<\mathrm{j} \leq \mathrm{k}\right\}$ where $\left\langle\left\{\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{k}}\right\}>=\mathrm{K}_{\mathrm{k}}\right.$ and each $\mathrm{y}_{\mathrm{i}, \mathrm{j}}$ has degree two with $N\left(y_{i, j}\right)=\left\{x_{i}, x_{j}\right\}, \operatorname{deg}\left(x_{i}\right)=2 k-2$.

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## Definition: Peterson graph: 1.5



## 2. GEODETIC NEIGHBORHOOD SETS

Definition: 2.1 Let $G=(V, E)$ be any graph and $x, y \in V(G)$ such that $x \neq y$. The geodetic neighborhood $(g$-neighborhood) set of the pair $\mathrm{x}, \mathrm{y}$ is defined as $\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y})=\{\mathrm{v}: \mathrm{v}$ is an internal vertex of an $\mathrm{x}-\mathrm{y}$ geodesic in G$\}$ and the closed geodetic neighborhood set of $\mathrm{x}, \mathrm{y}$ is defined as $\mathrm{N}_{\mathrm{g}}[\mathrm{x}, \mathrm{y}]=\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y}) \cup\{\mathrm{x}, \mathrm{y}\}$.

## Remark: 2.2

1.If x and y are adjacent, then $\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y})=\phi$. In particular, if $\mathrm{y} \in \mathrm{N}(\mathrm{x})$, then $\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y})=\phi$.
2. Let G be a complete graph. Then, $\mathrm{N}_{\mathrm{g}}(\mathrm{x}, \mathrm{y})=\phi$ for every $\mathrm{x}, \mathrm{y} \in \mathrm{V}(\mathrm{G})$.

Definition: 2.3 A pair of vertices $x, y \in G$ is said to geodominate a vertex $v \in G$ if $v \in N_{g}[x, y]$. Similarly, a set $S \subseteq V(G)$ is said to geodominate a vertex $v \in V(G)$ if $v \in N_{g}[x, y]$ for some $x, y \in S$.

Definition: 2.4 A vertex $v \in V(G)$ is said to be a geodetically isolated(g-isolated) vertex of $G$ if $v \notin N_{g}(x, y)$ for every pair of vertices $\mathrm{x}, \mathrm{y} \in \mathrm{G}$.
Extreme vertices are g-isolated vertices of G.

## Example: 2.5



G Figure 2.1
The vertices $\mathrm{v}_{5}$ and $\mathrm{v}_{6}$ are g-isolated vertices of $G$, as they do not belong to the open g-neighborhood set of any pair of vertices of $G$.

## Remark: 2.6

1.The set of all g-isolated vertices of $G$ is denoted by $\mathrm{I}_{\mathrm{g}}(\mathrm{G})$. The set of all extreme vertices of $G$ is a subset of $\mathrm{I}_{\mathrm{g}}(\mathrm{G})$. 2.If G is a connected graph without $g$-isolates, then G has no extreme vertices and $\delta(\mathrm{G}) \geq 2$.

Proposition: 2.7 For any graph $G, \mathrm{I}_{\mathrm{g}}(\mathrm{G})$ is equal to the set of all extreme vertices of G .
Proof: Let $S=$ The set of all extreme vertices of $G$. By Remark 2.6, $S \subseteq I_{g}(G)$. Let $x \in V-S$. Then, $N(x)$ contains at least two non-adjacent vertices, say, u,v. Further, the uxv geodesic from $u$ to $v$ contains v. So, $x \notin \mathrm{I}_{\mathrm{g}}(\mathrm{G})$. Hence the result.

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Definition: 2.8 Let $S \subseteq V(G)$. A vertex $v \in S$ is said to be a geodetically isolated(g-isolated) vertex of $S$ if $v \notin N_{g}(x, y)$ for every pair $x, y$ of vertices in $S$.

That is, v is not geodominated by the vertices of $\mathrm{S}-\{\mathrm{v}\}$.

## Example: 2.9



G Figure 2.2
Consider $S=\left\{v_{1}, v_{4}, v_{5}, v_{7}, v_{9}\right\}$. Every vertex of $S$ other than $v_{4}$ is a g-isolated vertex of $S$. For $S^{\prime}=\left\{v_{2}, v_{3}, v_{4}\right\}, v_{2}$ and $v_{4}$ are $g$-isolated vertices.

Definition: 2.10 Let $S \subseteq V(G)$ and $v \in S$. A vertex $w \in V(G)$ is said to be a private g-neighbor of $v$ with respect to $S$ if w is geodominated by the vertices of S and it is not geodominated by the vertices of $\mathrm{S}-\{\mathrm{v}\}$.

Definition: 2.11 Let $\mathrm{S} \subseteq \mathrm{V}(\mathrm{G})$ and $\mathrm{v} \in \mathrm{S}$. The private g-neighbor set of v with respect to S is defined as $\mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S})=\{\mathrm{w}$ : w is a private g-neighbor of $v$ with respect to $S\}$

Example: 2.12 Condider $G$ in figure 2.2. Let $S=\left\{\mathrm{v}_{1}, \mathrm{v}_{5}, \mathrm{v}_{7}, \mathrm{v}_{9}\right\}$. The vertices $\mathrm{v}_{3}, \mathrm{v}_{4}, \mathrm{v}_{6}$ and $\mathrm{v}_{7}$ are g-private neighbors of $\mathrm{v}_{7}$ and $\mathrm{v}_{8}, \mathrm{v}_{9}$ are g-private neighbors of $\mathrm{v}_{9}$. The vertex $\mathrm{v}_{1}$ is the unique g-private neighbor of itself. Similarly, $\mathrm{v}_{5}$ is a only g-private neighbor of $\mathrm{v}_{5}$. Therefore, $\mathrm{pgn}_{\mathrm{gn}}\left(\mathrm{v}_{7}, \mathrm{~S}\right)=\left\{\mathrm{v}_{3}, \mathrm{v}_{4}, \mathrm{v}_{6}, \mathrm{v}_{7}\right\}, \mathrm{pgn}\left(\mathrm{v}_{9}, \mathrm{~S}\right)=\left\{\mathrm{v}_{8}, \mathrm{v}_{9}\right\}$, $\mathrm{pgn}\left(\mathrm{v}_{1}, \mathrm{~S}\right)=\left\{\mathrm{v}_{1}\right\}$ and $\mathrm{pgn}_{\mathrm{gn}}\left(\mathrm{v}_{5}, \mathrm{~S}\right)=\left\{\mathrm{v}_{5}\right\}$.

Remark: 2.13 If $v$ is a geodetically isolated(or g-isolated) vertex of $S$, then $v \in p_{g n}(v, S)$ and $p_{g n}(v, S) \neq \phi$.
Definition: 2.14 Let $\mathrm{S} \subseteq \mathrm{V}(\mathrm{G})$. The g-private neighbour set of S denoted by $\mathrm{pgn}_{\mathrm{gn}}(\mathrm{S})$ is defined as $\mathrm{pgn}_{\mathrm{gn}}(\mathrm{S})=\left\{\mathrm{v} \in \mathrm{S}: \mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S})\right.$ $\neq \phi\}$.

Definition: 2.15 The g-private neighbour count of $S$ is the cardinality of the g-private neighbour set of S. It is denoted


Remark: 2.16 For $\mathrm{S}=\left\{\mathrm{v}_{1}, \mathrm{v}_{5}, \mathrm{v}_{7}, \mathrm{v}_{9}\right\}$ in example 2.12, $\mathrm{p}_{\mathrm{gn}}(\mathrm{S})=\mathrm{S}$ and $\mathrm{p}_{\mathrm{gnc}}(\mathrm{S})=|\mathrm{S}|=4$.
Definition: $2.17[1,2,3]$ A set $S$ of vertices of $G$ is said to be a geodominating(or geodetic) set of $G$ if every vertex of $G$ is geodominated by the vertices of $S$.

Equivalently, a set $S$ of vertices of $G$ is a geodetic set of $G$ if for every $v \in V-S, v \in N_{g}(x, y)$ for some pair of vertices $\mathrm{x}, \mathrm{y} \in \mathrm{S}$ or $\mathrm{V}=\mathrm{I}[\mathrm{S}]=\underset{x, y \in S}{\cup} \mathrm{I}[\mathrm{x}, \mathrm{y}]=\underset{x, y \in S}{\cup} \mathrm{~N}_{\mathrm{g}}[\mathrm{x}, \mathrm{y}]=\mathrm{N}_{\mathrm{g}}[\mathrm{S}]$.

A geodetic set $S$ is said to be a minimal geodetic set of $G$ if no proper subset of $S$ is a geodetic set of $G$.
A geodetic set of minimum cardinality is called a minimum geodetic set (or a geodetic basis) of $G$ and the minimum cardinality of a minimal geodetic set of $G$ is called the geodetic number of $G$ and it is denoted as $g(G)$.

The maximum cardinality of a minimal geodetic set of $G$ is called its upper geodetic number and it is denoted as $\mathrm{g}^{+}(\mathrm{G})$.
Remark: 2.18 The set $\mathrm{I}_{\mathrm{g}}(\mathrm{G})$ is a subset of every geodetic set of G .

## 3. g-INDEPENDENT SETS

Definition: 3.1 A subset $S$ of $V(G)$ is said to be a g-independent set of $G$ if every $v \in S$ is such that $v \notin N_{g}(x, y)$ for every $\mathrm{x}, \mathrm{y} \in \mathrm{S}-\{\mathrm{v}\}$.

If S is g -independent then every vertex of S is a g-isolate of S .

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## Observation: 3.2

1. For any graph $G$, every two element subset of $V(G)$ is a g-independent set.
2. For the complete graph $G=(V, E), V(G)$ is a g-independent set.
3. Let G be a non-complete connected graph. If S is a g-independent set of G , then $2 \leq|\mathrm{S}| \leq \mathrm{p}-1$.
4. If $G$ is any graph, then the set of all end vertices of $\mathrm{G}^{+}$is a g-independent set of $\mathrm{G}^{+}$.

Definition:3.3 A g-independent set $S$ of $G$ is said to be maximal if no super set of $S$ is a g-independent set of $G$.

## Example: 3.4

1. For $C_{5}$, the set $S=\left\{\mathrm{v}_{1}, \mathrm{v}_{3}\right\}$ is a g-independent set whereas $S^{\prime}=\left\{\mathrm{v}_{1}, \mathrm{v}_{3}, \mathrm{v}_{4}\right\}$ is a maximal g-independent set.

Definition: 3.5 The maximum cardinality of all maximal g-independent sets of $G$ is called its $g$-independence number. It is denoted as $\beta_{\mathrm{g}}(\mathrm{G})$. A g-independent set of cardinality $\beta_{\mathrm{g}}(\mathrm{G})$ is called a $\beta_{\mathrm{g}}$-set of G .

## Example: 3.6



G figure 3.1

1. For $\mathrm{G},\left\{\mathrm{v}_{1}, \mathrm{v}_{3}, \mathrm{v}_{6}, \mathrm{v}_{8}\right\},\left\{\mathrm{v}_{2}, \mathrm{v}_{4}, \mathrm{v}_{5}, \mathrm{v}_{7}\right\},\left\{\mathrm{v}_{1}, \mathrm{v}_{7}\right\}$ and $\left\{\mathrm{v}_{3}, \mathrm{v}_{5}\right\}$ are maximal g-independent sets.

Observation: 3.7

1. For a complete graph $G$ on $p$ vertices, $\beta_{g}(G)=p$.
2. For a non-complete connected graph $G, 2 \leq \beta_{\mathrm{g}}(\mathrm{G}) \leq \mathrm{p}-1$.
3. For $n \geq 2, \beta_{g}\left(P_{n}\right)=2$.
4. For $n \geq 4, \beta_{g}\left(C_{n}\right)=\left\{\begin{array}{l}2 \text { if } n \text { is even } \\ 3 \quad \text { otherwise }\end{array}\right.$
5. $\beta_{\mathrm{g}}\left(\mathrm{K}_{\mathrm{m}, \mathrm{n}}\right)=\left\{\begin{array}{lc}2 & \text { if } m=n=1 \\ \max \{m, n\} & \text { otherwise }\end{array}\right.$

6 . For any graph $G$ on $p$ vertices, $\beta_{\mathrm{g}}\left(\mathrm{G}^{+}\right)=\mathrm{p}$.
Definition: 3.8 A geodetic set $S$ is said to be a g-independent geodetic set of $G$ if $S$ is g-independent. A g-independent geodetic set S is maximal if no super set of S is a g-independent geodetic set of G . That is, $\mathrm{S} \cup\{\mathrm{v}\}$ is not a g-independent geodetic set for every $\mathrm{v} \in \mathrm{V}-\mathrm{S}$. The minimum cardinality of all maximal g-independent geodetic sets is called g-independent geodetic number of G. It is denoted as gig(G). A g-independent geodetic set of cardinalilty gig(G) is called a gig-set of G.

## Observation:3.9

1. For a complete graph on $p$ vertices, $\operatorname{gig}(G)=p$, as the unique geodetic set $S=V(G)$ is g-independent.
2. For a non-complete connected graph $\mathrm{G}, 2 \leq \operatorname{gig}(\mathrm{G}) \leq \mathrm{p}-1$.
3. For $n \geq 2, \operatorname{gig}\left(P_{n}\right)=2$, as the set of two end vertices of $P_{n}$ is a g-independent geodetic set of $P_{n}$.
4. For $\mathrm{n} \geq 4, \operatorname{gig}\left(\mathrm{C}_{\mathrm{n}}\right)=\left\{\begin{array}{l}2 \text { if } n \text { is even } \\ 3 \quad \text { otherwise }\end{array}\right.$

Proposition: $3.10 \operatorname{gig}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}\right)= \begin{cases}2 & \text { if } m=n=1 \\ \min \{m, n\} & \text { ifm, } n \geq 2 \\ \max \{m, n\} & \text { otherwise }\end{cases}$
Proof: Let $\mathrm{U}, \mathrm{W}$ be the vertex partition of $\mathrm{K}_{\mathrm{m}, \mathrm{n}}$ with $|\mathrm{U}|=\mathrm{m}$ and $|\mathrm{W}|=\mathrm{n}$.
When $m=n=1, V\left(K_{m, n}\right.$ is the unique g-independent geodetic set. So, $\operatorname{gig}\left(K_{m, n}\right)=2$.

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If $\mathrm{m}=1$ and $\geq \mathbb{2}$ (or $\mathrm{n}=1$ and $\mathrm{m} \geq 2$ ), then W (or U ) is the unique g-independent geodetic set of $G$. Therefore, $\operatorname{gig}\left(K_{m, n}\right)=\max \{m, n\}$.

When $m, n \geq 2$, $U$ and $W$ are the only minimal $g$-independent geodetic set. Hence, $\operatorname{gig}\left(K_{m, n}\right)=\min \{m, n\}$.
Proposition:3.11 For any graph $G$ on $p$ vertices, $\operatorname{gig}\left(G^{+}\right)=p$.
Proof: In $\mathrm{G}^{+}$, the set of all end vertices attached to the vertices of G is the unique g-independent geodetic set. Hence the result.

Proposition: 3.12 Let $G=(V, E)$ be any graph. A g-independent set $S$ of $G$ is maximal if and only if it is g-independent and geodetic.

Proof: Let $S$ be a g-independent set of $G$. Suppose $S$ is maximal. Then, for every $v \in V-S, S \cup\{v\}$ is not g-independent. So, for every $v \in V-S, v \in N_{g}(x, y)$ for some $x, y \in S$. Therefore, $S$ is a geodetic set of $G$. Conversely, Suppose, $S$ is both gindependent and geodetic. Since $S$ is geodetic, retracing the above steps we find, $S \cup\{v\}$ is not g-independent for every $\mathrm{v} \in \mathrm{V}-\mathrm{S}$. But, S is g-independent. Hence, S is a maximal g-independent set of $G$.

Proposition: 3.13 Every maximal g-independent set in a graph $G$ is a minimal geodetic set of $G$.
Proof: Let $S$ be a maximal g-independent set of G. Proposition 3.12 asserts that S is a geodetic set. If S is not a minimal geodetic set, then $S-\{v\}$ is a geodetic set for some $v \in S$. So, for every $u \in V-(S-\{v\}), u \in N_{g}(x, y)$ for some $x, y \in S-\{v\}$. In particular, $v \in N_{g}(x, y)$ for a pair of vertices $x, y \in S-\{v\}$. This contradicts our assumption that $S$ is $g-$ independent. Therefore, $S$ is minimal geodetic set of $G$.

Corollary: 3.14 For any graph $G, g(G) \leq \operatorname{gig}(G) \leq \beta_{g}(G) \leq g^{+}(G)$.
Proof: We prove the theorem in three steps.
(i) $g(G) \leq \operatorname{gig}(G)$.

Let $S$ be a gig-set of $G$. Since every g-independent geodetic set is a geodetic set of $G, g(G) \leq|S|=\operatorname{gig}(G)$.
(ii) $\operatorname{gig}(G) \leq \beta_{\mathrm{g}}(\mathrm{G})$.

Let $S$ be a $\beta_{\mathrm{g}}$-set of G . That is, S is a maximal g-independent set. By proposition 3.13, S is a geodetic set of G .
Therefore, S is a g-independent geodetic set of G.But, gig-set is a g-independent geodetic set of minimum cardinality.
So, $\operatorname{gig}(G) \leq \beta_{\mathrm{g}}(\mathrm{G})$.
(iii) $\beta_{\mathrm{g}}(\mathrm{G}) \leq \mathrm{g}^{+}(\mathrm{G})$.

Let $S$ be a $\beta_{\mathrm{g}-\mathrm{set}}$ of G . By Proposition 3.13, S is a minimal geodetic set of G. Any g+-set is a minimal geodetic set of maximum cardinality. Therefore, $\beta_{\mathrm{g}}(\mathrm{G}) \leq \mathrm{g}^{+}(\mathrm{G})$.

The following theorem gives a necessary and sufficient condition for a geodetic set of a graph $G$ to be a minimal geodetic set of $G$.

Theorem: 3.15 A geodetic set $D$ of $G$ is a minimal geodetic set of $G$ if every $u \in D$ satisfies one of the following two conditions.
(i) $u$ is a g-isolated vetex of $D$.
(ii) There exists at least one vertex $w \in V-D$ such that $w$ is geodominated by $D$ and it is not geodominated by $D-\{u\}$.

Proof: Assume that D is a minimal geodetic set of $G$. Then, for every vertex $\mathrm{u} \in \mathrm{D}, \mathrm{D}-\{\mathrm{u}\}$ is not a geodetic set of G . Therefore, there exists a vertex $w \in V-(D-\{u\})$ such that $w$ is not geodominated by the vertices of $D-\{u\}$.

Case 1: w=u.
So, $u$ is not geodominated by the vertices of $D-\{u\}$. That is, $u \notin \operatorname{Ng}[x, y]$ for every $x, y \in D-\{u\}$. That is, $u \notin N_{g}(x, y)$ for every $x, y \in D$. That is, $u$ is a g-isolated vertex of $D$. So, $u$ satisfies (i).

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Case 2: $\mathrm{w} \neq \mathrm{u}$.
Now, $w \in V-D$ and it is not geodominated by the vertices of $D-\{u\}$. But, as $D$ is a geodetic set of $G$, $w$ is geodominated by the vertices of D. So, u satisfies (ii).

Conversely, assume that every $u \in D$ satisfies (i) or (ii). Let $C_{1}$ and $C_{2}$ denote the set of all vertices of $D$ satisfying (i) and (ii) respectively.
case a: Let $u \in C_{1}$. Then, $u$ is a g-isolated vertex of $D$. Therefore, $u \notin N_{g}(x, y)$ for every pair $x, y$ of vertices of $D$. That is, $u \in V-(D-\{u\})$ is not geodominated by the vertices of $D-\{u\}$.

Therefore, $\mathrm{D}-\{\mathrm{u}\}$ is not a geodetic set of G .
case b: Let $u \in C_{2}$. Then, there exists $w \in V-D$ such that $w$ is not geodominated by the vertices of $D-\{u\}$. So, $D-\{u\}$ is not a geodetic set of G .

By assumption, $D=C_{1} \cup C_{2}$. Therefore, $D-\{u\}$ is not a geodetic set of $G$ for every $u \in D$.
Therefore, $D$ is a minimal geodetic set of $G$.
Corollary: 3.16 A geodetic set $D$ of $G$ is a minimal geodetic set of $G$ if and only if $p_{g n}(v, S) \neq \phi$ for every $v \in D$.

## 4. g-IRREDUNDANT SETS

The irredundance and upper irredundance number were first defined by Cockayne, Hedetnienmi and Miller[3]. In this section, we extend these parameters with respect to the geodetic concept. Theorem 2.15 can be restated as "A geodetic set $S$ of $G$ is minimal if and only if for every vertex $v \in S$, there exists a vertex $w \in V-(S-\{v\})$ which is not geodominated by $S-\{v\}---(1)$. That is, for every vertex $v \in S, \mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S}) \neq \phi .--(2)$. We call a set S of vertices is g -irredundant if condition (2) is satisfied.

Definition: 4.1 A set $\mathrm{S} \subseteq \mathrm{V}(\mathrm{G})$ is said to be a g-irredundant set of $G$ if $\mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S}) \neq \phi$ for every $\mathrm{v} \in \mathrm{S}$.
Remark: 4.2 If S is a g-irredundant set, then $\mathrm{p}_{\mathrm{gnc}}(\mathrm{S})=|\mathrm{S}|$.
Proposition: 4.3 A geodetic set S is a minimal geodetic set if and only if it is geodetic and g-irredundant.
Proof: Let $S$ be a geodetic set of G. Suppose $S$ is minimal. By corollary 3.16, S is g-irredundant. Conversely, if a set $S$ is both geodetic and g-irredundant. Let $\mathrm{v} \in \mathrm{S}$. As S is g-irredundant, $\mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S}) \neq \phi$. Let $\mathrm{w} \in \mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S})$. Then, $\mathrm{w} \in \mathrm{V}-(\mathrm{S}-\{\mathrm{v}\})$ is not geodeominated by the vertices of $\mathrm{S}-\{\mathrm{v}\}$. Therefore, $\mathrm{S}-\{\mathrm{v}\}$ is not geodetic. As $\mathrm{v} \in \mathrm{S}$ is arbitrary, S is a minmal geodetic set of $G$.

Definition: 4.4 A g-irredundant set $S$ is said to be maximal g-irredundant if no super set of $S$ is a g-irredundant set of G.

That is, $\mathrm{S} \cup\{\mathrm{v}\}$ is not g-irredundant for every vertex $\mathrm{v} \in \mathrm{V}-\mathrm{S}$.
Remark: 4.5 By definition 4.1, S is a maximal g-irredundant set of G if and only if for every vertex $\mathrm{w} \in \mathrm{V}$-S, there exists a vertex $v \in S \cup\{\mathrm{w}\}$ for which $\mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S} \cup\{\mathrm{w}\})=\phi$.

Proposition: 4.6 Let $G=(V, E)$ be a connected graph and $S \subseteq V(G)$ is a g-irredundant set of $G$. Then, the following are equivalent.
(i). For every vertex $w \in V-S$, there exists a vertex $v \in S \cup\{w\}$ for which $p_{g n}(v, S \cup\{w\})=\phi$.
(ii). For every vertex $w \in V-S, p_{\text {gnc }}(S \cup\{w\}) \leq p_{\text {gnc }}(S)$.

Proof: (i) $\Rightarrow$ (ii)
By definitions 2.14 and 2.15, $\mathrm{pgnc}(\mathrm{S})=\left|\mathrm{pgn}_{\mathrm{gn}}(\mathrm{S})\right|=\left|\left\{\mathrm{v} \in \mathrm{S}: \mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S}) \neq \phi\right\}\right|$. As S is g-irredundant, by
Remark 4.2, $\mathrm{p}_{\mathrm{gnc}}(\mathrm{S})=|\mathrm{S}|$.
By (i), for every $w \in V-S,\left|\mathrm{pgn}_{\mathrm{gn}}(\mathrm{S} \cup\{\mathrm{w}\})\right|<|\mathrm{S} \cup\{\mathrm{w}\}|=|\mathrm{S}|+1$. Therefore, $\mathrm{pgnc}(\mathrm{S} \cup\{\mathrm{w}\})=\left|\mathrm{p}_{\mathrm{gn}}(\mathrm{S} \cup\{\mathrm{w}\})\right| \leq|\mathrm{S}|=\mathrm{p}_{\mathrm{gnc}}(\mathrm{S})$.
(ii) $\Rightarrow$ (i)

Since $S$ is g-irredundant, $\mathrm{pgnc}(\mathrm{S})=|\mathrm{S}|$. That is, for every $\mathrm{v} \in \mathrm{S}, \mathrm{pgn}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S}) \neq \phi$.

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(ii) implies for every $\mathrm{w} \in \mathrm{V}-\mathrm{S}$, $\left|\mathrm{pgnn}_{\mathrm{gn}}(\mathrm{S} \cup\{\mathrm{w}\})\right|<|S \cup\{\mathrm{w}\}|$. Therefore, for every $\mathrm{w} \in \mathrm{V}-\mathrm{S}$, there exists atleast one vertex v in $\mathrm{S} \cup\{\mathrm{w}\}$ such that $\mathrm{p}_{\mathrm{gn}}(\mathrm{v}, \mathrm{S} \cup\{\mathrm{w}\})=\phi$.

Hence the result.
Remark: 4.7 Let G be a connected graph. A g-irredundant set $S$ of $G$ is a maximal g-irredundant set of $G$ if and only if for every vertex $w \in V-S$, $p_{g n c}(S \cup\{w\}) \leq p_{g n c}(S)$.

Definition: 4.8 Let G be a connected graph. The minimum cardinality of a maximal g-irredundant set of G is called the g-irredundance number of G. It is denoted as gir(G). The maximum cardinality of a mamimal g-irredundant set is called its upper g-irredundance number. It is denoted by GIR(G).

Example: 4.9 For the graph $G$ in figure $3.1\left\{\mathrm{v}_{3}, \mathrm{v}_{6}, \mathrm{v}_{8}\right\}$ is a g-irredundant set whereas $\left\{\mathrm{v}_{1}, \mathrm{v}_{3}, \mathrm{v}_{6}, \mathrm{v}_{8}\right\}$ is a maximal g irredundant set.

Proposition: 4.10 Every minimal geodetic set in a connected graph G is a maximal g-irredundant set of G.
Proof: Let S be a minimal geodetic set of G. By Proposition 4.3, S is g-irredundant. Suppose it is not maximal girredundant. Then, there exists a vertex $u \in V-S$ such that $S \cup\{u\}$ is g-irredundant. So, for every $v \in S \cup\{u\}$, $\mathrm{pgn}_{\mathrm{gn}}(\mathrm{v}$, $S \cup\{u\}) \neq \phi$. In particular, $\mathrm{pgn}_{\mathrm{gn}}(\mathrm{u}, \mathrm{S} \cup\{\mathrm{u}\})=\phi$. Let $w \in \mathrm{pgn}_{\mathrm{gn}}(\mathrm{u}, \mathrm{S} \cup\{u\})$. Therefore, w is a private neighbor of $u$ in $S \cup\{u\}$ and w is not geodominated by the vertices of $S$. This is a contradiction to $S$ is a geodetic set of $G$. Hence $S$ is a maximal girredundant set of $G$.

Theorem: 4.11 For any graph G,

$$
\operatorname{gir}(\mathrm{G}) \leq \operatorname{g}(\mathrm{G}) \leq \operatorname{gig}(\mathrm{G}) \leq \beta \mathrm{g}(\mathrm{G}) \leq \mathrm{g}^{+}(\mathrm{G}) \leq \mathrm{GIR}(\mathrm{G}) .
$$

Proof: By Theorem 3.14, it is enough to prove $\operatorname{gir}(G) \leq g(G)$ and $g^{+}(G) \leq G I R(G)$.
(i) $\operatorname{gir}(\mathrm{G}) \leq \mathrm{g}(\mathrm{G})$.

Let $S$ be a g-set of $G$. By proposition 4.10, S is a maximal g-irredundant set of $G$. So, by definition of gir(G), we get $\operatorname{gir}(\mathrm{G}) \leq|\mathrm{S}|=\mathrm{g}(\mathrm{G})$.
(ii) $\mathrm{g}^{+}(\mathrm{G}) \leq \operatorname{GIR}(\mathrm{G})$.

Let S be a $\mathrm{g}^{+}$-set of G . Then, S is a minimal geodetice set of G . Therefore, by $3.10, \mathrm{~S}$ is a maximal g-irredundant set of G. Further, $\operatorname{GIR}(\mathrm{G})$ is the maximum cardinality of all maximal $g$-irredundant sets of G . Therefore, $\mathrm{g}^{+}(\mathrm{G}) \leq \mathrm{GIR}(\mathrm{G})$.

Hence the result.
Remark: 4.12 For a graph G, the above inequality is called the geodetic chain of G.

## 5. g-CONNECTIVITY OF A GRAPH

Definition: 5.1 A graph $G=(V, E)$ is said to be geodetically connected(or g-connected) if every vertex $u \in V(G)$ is an internal vertex of an $x-y$ geodesic for some $x, y \in V(G)$.

Remark: 5.2 If a graph $G$ is g-connected then

1. $G$ contains no extreme vertices.
2. $\delta(\mathrm{G}) \geq 2$.

## Examples: 5.3

1. Every cycle $C_{p}$ is a g-connected graph for $p \geq 4$.
2. Peterson graph is a g-connected graph.

Remark: 5.4 The following graphs are not g-connected.

1. Any complete graph is not g-connected.
2.The Hajo's graphs $\mathrm{H}_{\mathrm{k}}$ are not g-connected for $\mathrm{k} \geq 2$.
2. $P_{n}$ is not g-connected for $n \geq 2$.
3. Any graph $G$ with atleast one extreme vertex is not g-connected.

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Remark: 5.5 A graph G contains no extreme vertices is not a sufficent condition for G is g-connected. For example, the generalised Hajo’s graphs $\mathrm{H}_{\mathrm{k}}$ (see definition 1.4)contain no extreme vertices. It is not g-connected.

Problem: 5.6 Find out a sufficient condition for g-connectedness of a graph.
Conjecture: 5.7 Suppose G is a g-connected graph and $S$ is a geodetic set of $G$ with $|S| \leq n / 2$ then V-S contains a geodetic set of $G$.

Remark: 5.8 In the above conjecture, it is evident that the condition $|\mathrm{S}| \leq \mathrm{n} / 2$ cannot be ommited.

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