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TRANSIENT SOLUTION OF $M^{[X]}/G/1$ RETRIAL QUEUE WITH TWO STAGE HETEROGENEOUS SERVICE, EXTENDED VACATION, NON-PERSISTENT CUSTOMERS AND SETUP TIME

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ABSTRACT

This paper investigates a single server retrial queuing system with two stages heterogeneous service.Customers arrive in batches in accordance with compound Poisson processes. After the completion of first stage service, the second stage service starts with probability 1. In addition to this, the server takes Bernoulli vacation and setup times. We assume that the retrial time, the service time, the repair time, the vacation time and the setup time of the server are all arbitrarily distributed. We obtain the time dependent probability generating functions in terms of their Laplace transforms and the corresponding steady state results explicitly. Also we derive the average number of customers in the queue and the average waiting time in closed form with numerical illustration.

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Keywords: *Batch Arrival, Bernoulli vacation, Non- persistent customers, Retrial time, Setup time, Two stages service.*

1. INTRODUCTION

Retrial queues are characterized by the feature that a customer who finds the server is busy or on vacation he/she may join the group of blocked customers (called orbit) for trying their demand or request after some time or leave the system immediately. The study of retrial queuing system in queuing theory has become an indispensable area because of its vast applicability in telephone switching networks, telecommunication networks and computer networks packet switching networks, collision avoidance star local area networks and transportation networks. Most generally in telecommunication networks, telephone callers may break contact when the line is busy and retry for connection later. In packet switching networks, if the bus is idle, then one packet is chosen for transmission automatically so that the rest are stored in buffer on the other hand, if the bus is busy then all packets must be stored in the buffer and the station will try the transmission later on.

There is a vast literature on retrial queuing models with many dimensions. Kulkarni (1983), Farahmand (1990) Choi (1990) studied queuing systems with retrials. For recent bibliographies and survey we can refer Yang and Templeton (1990), Fallin and Templeton (1997) and Artalejo (1999). Artalejo and Fallin (2002) discussed a comparison between standard and retrial queuing models. Krishna kumar *et al.* (2002) studied an M/G/1 retrial queue with two phase service and pre emptive resume. Retrial queues with vacations has also been concentrated by many authors include Artalejo (1997, 1999), Krishna kumar (2002) and Atencia (2005) and Zhou (2005). Arumuganathan (2008) discussed single server batch arrival two phases of heterogeneous service, retrial queue under Bernoulli schedule and the same author (2009) performed analysis on two phase heterogenous service and different vacation policies of an M/G/1 retrial queue with non-persistent calls . Wang (2008) studied repairable retrial queue with setup time and Bernoulli vacation. Yang *et al.* (1990) studied M/G/1 retrial queue with impatient customers. Choudhry (2009) studied both single and batch arrival of retrial queue with two phase of service and general retrial times. Kasthuri Ramnath and Kalidass (2010) studied single server non-Markovian retrial queue with second optional service and different vacation policies for non-persistent customers. The same authors (2010) extended their results for two phase service of the same model. Arivudainambi and Godhandaraman (2012) studied batch arrival retrial queue with two phases of service, feedback and K Optional Vacations. Sumitha and Udaya Chandrika (2012) studied about starting failure, single vacation policy and orbital search of M/G/1 retrial queuing model. Vishwa Nath Maurya (2013) analysed the maximum entropy of batch arrival two phase retrial queuing system with second phase optional service and Bernoulli vacation.

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In this paper, a single server retrial queue with impatient customers, two stages of heterogeneous service, extended vacation is considered. In addition to this after the vacation is over the server must take some time to set up. By using supplementary variable technique, the prescribed model is analysed. The rest of the paper is organised as follows: section 2 describes the mathematical model, section 3 discusses the steady state of the model, section 4 gives the performance measures of the system, section 5 deals with numerical analysis and conclusion is given in section 6.

2. MATHEMATICAL MODEL

In this paper, we consider a single server retrial queuing system in which the primary customers arrive in batches according to Poisson process with rate λ. Let C_j, j=1, 2,…represent number of customers with probability distribution $P(C_j=n)=c_n$, n=1,2,…and probability generating function C(z). If the server is available, it begins the service to one of the customers immediately and the remaining customers leave the service area and hence join the orbit. Also upon arrival if the customer finds the server busy or on vacation joins the orbit with probability p or leaves the service area with probability 1-p, being impatient. The retrial time that is time between successive repeated attempts of each customer in orbit is assumed to be generally distributed with distribution function A(x), density function a(x), the mean value a, and Laplace transform $\bar{a}(s)$. Assuming that retrial times begin either at the completion instants of service or setup times so that the distribution of the remaining retrial time is $A_e(x) = \frac{1}{a}\int_0^{\infty}(1-A(x))dx$ and $a_e(x) = \frac{1-A(x)}{a}$.

The conditional completion rate time for retrials is given by $\eta_e(x) = \frac{a_e(x)}{(1-A_e(x))}$. The server provides two stages of service in succession to all customers. Let $B_1(x)$, b_1 (x) and $B_2(x)$, b_2 (x) be the distribution function and the density function of the service time of first and second stage respectively. The service times of both stages are independent to each other. Let $\mu_i(x)$ be the conditional probability density function of service completion of i^{th} service during the interval (x, x+dx] given that the elapsed time is x, so that $\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$; i=1,2. After the completion of second stage service, the server may go for a vacation with probability θ or staying back in the system to provide service to new customer if any, with probability 1-θ. After the completion of first phase vacation the server may extend its vacation by going to second phase vacation with probability r (>0) or return back to the system with probability 1-r. Let V_i (x) and $v_i(x)$ be the distribution function and density function respectively and its Laplace transform is $\bar{v}_i(s)$. Let $\gamma_i(x)$ be the conditional probability density function of service completion of i^{th}phase vacation during the interval (x, x+dx] given that the elapsed time is x, so that $\gamma_i(x) = \frac{v_i(x)}{1-V_i(x)}$; i=1, 2. After the vacation period is over, the server spend some time for setup and the setup time is arbitrarily distributed. Let D(x), d(x) be its distribution function, density function respectively and its Laplace transform is $\bar{d}(s)$ respectively. Let δ(x) be the conditional probability density function of setup time during the interval (x, x+dx] given that the elapsed time is x, so that $\delta(x) = \frac{d(x)}{1-D(x)}$.

All the stochastic processes are independent to each other.

Let N (t) be denote the number of customers in the orbit at time t and C(t) be the state of the server and which is given by:

$$C(t) = \begin{cases} 0 \text{ if the server is idle} \\ 1 \text{ if the server is busy with first stage service} \\ 2 \text{ if the server is busy with second stage service} \\ 3 \text{ if the server is on first phase vacation} \\ 4 \text{ if the server is on second phase vacation} \\ 5 \text{ if the server is in set up} \end{cases}$$

So that the supplementary variables are introduced as $Q^0(t)$ = elapsed retrial time of the customer at the head of the orbit at time t, $P_i^0(t)$ = elapsed service time of the i^{th} stage service time, $V_i^0(t)$ = elapsed vacation time of the i^{th} phase vacation j= 1,2 and $S^0(t)$ = elapsed set up time. The process {C(t), N(t), t >0} is a continuous time Markov process.

We define the following probability functions:

$Q_0(t)$ =Pr{ N(t)= 0,C(t)=0},

$Q_n(x, t)dt$ = Pr{N(t)= n, C(t)=0}, n>0,

$P_n^{(i)}(x, t)dt$ = Pr{N(t)= n, C(t)=i}, n≥0; i=1,2

$V_n^{(i)}(x, t)dt$ = Pr{N(t)= n, C(t)=i}, n≥0; i=3,4

$S_n(x, t)dt$= Pr{N(t)= n, C(t)=5}, n≥0

2. TIME DEPENDENT SOLUTION

The differential – difference equations governing the queuing model are

$$\frac{d}{dt}Q_0(t) = -\lambda Q_0 + (1-\theta)\int_0^\infty p_0{}^{(2)}(x,t)\mu(x)dx + \int_0^\infty S_0(x,t)\delta(x)dx; n \geq 1 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + +\eta_e(x)\right)Q_n(x,t) = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + +\mu_1(x)\right)P_n^{(1)}(x,t) = p\lambda\sum_{i=1}^{n} c_i P_{n-i}^{(1)}(x,t); n \geq 0 \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + +\mu_2(x)\right)P_n^{(2)}(x,t) = p\lambda\sum_{i=1}^{n} c_i P_{n-i}^{(2)}(x,t); n \geq 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \gamma_1(x)\right)V_n^{(1)}(x,t) = p\lambda\sum_{i=1}^{n} c_i V_{n-i}^{(1)}(x,t); n \geq 0 \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \gamma_2(x)\right)V_n^{(2)}(x,t) = p\lambda\sum_{i=1}^{n} c_i V_{n-i}^{(2)}(x,t); n \geq 0 \tag{6}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \delta\ (x)\right)S_n(x,t) = p\lambda\sum_{i=1}^{n} c_i S_{n-i}(x,t); n \geq 0 \tag{7}$$

Subject to the boundary conditions, the above equations (1) - (7) could be solved

$$Q_n(0,t) = (1-\theta)\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty S_n(x,t)\delta(x)dx; n \geq 1 \tag{8}$$

$$P_O^{(1)}(0,t) = \int_0^\infty Q_1(x,t)\eta_e(x)dx + \lambda c_1 Q_0 \tag{9}$$

$$P_n^{(1)}(0,t) = \int_0^\infty Q_{n+1}(x,t)\eta_e(x)dx + \lambda\sum_{i=1}^{n} c_i\int_0^\infty Q_{n-i}(x,t)dx; + \lambda c_{n+1}Q(t); n \geq 1 \tag{10}$$

$$P_n^{(2)}(0,t) = \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx; n \geq 0 \tag{11}$$

$$V_n^{(1)}(0,t) = \theta\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx; n \geq 0 \tag{12}$$

$$V_n^{(2)}(0,t) = r\int_0^\infty V_n^{(1)}(x,t)\gamma_1(x)dx;; n \ge 0 \tag{13}$$

$$S_n(0,t) = (1-r)\int_0^\infty V_n^{(1)}(x,t)\gamma_1(x)dx + \int_0^\infty V_n^{(2)}(x,t)\gamma_2(x)dx; n \ge 0 \tag{14}$$

The initial conditions and Normalising condition are given by respectively

$$Q_n(0) = 1; P^{(i)}{}_n(x,0) = 0; V_n^{(i)} = 0; n \succ 0, x \succ 0, i = 1,2$$

$$1 = Q_0 + \sum_{n=1}^\infty \int_0^\infty Q_n(x,t)dx + \sum_{n=0}^\infty \int_0^\infty P_n^{(1)}(x,t)dx + \sum_{n=0}^\infty \int_0^\infty P_n^{(2)}(x,t)dx + \sum_{n=0}^\infty \int_0^\infty V_n^{(1)}(x,t)dx + \sum_{n=0}^\infty \int_0^\infty V_n^{(2)}(x,t)dx + \sum_{n=0}^\infty \int_0^\infty S_n(x,t)dx \tag{15}$$

Now, Define Laplace transform

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{16}$$

Taking Laplace transforms for equations (1) – (14) we have,

$$(s+\lambda)\bar{Q}_0(s) = 1 + (1-\theta)\int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar{S}_0(x,s)\delta(x)dx; n \ge 1 \tag{17}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + +\eta_e(x)\right)\bar{Q}_n(x,s) = p\lambda\sum_{i=1}^n c_i \bar{Q}_{n-i}(x,s); n \ge 0 \tag{18}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + +\mu_1(x)\right)\bar{P}_n^{(1)}(x,s) = p\lambda\sum_{i=1}^n c_i \bar{P}_{n-i}^{(1)}(x,s); n \ge 0 \tag{19}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + +\mu_2(x)\right)\bar{P}_n^{(2)}(x,s) = p\lambda\sum_{i=1}^n c_i \bar{P}_{n-i}^{(2)}(x,s); n \ge 0 \tag{20}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + \gamma_1(x)\right)\bar{V}_n^{(1)}(x,s) = p\lambda\sum_{i=1}^n c_i \bar{V}_{n-i}^{(1)}(x,s); n \ge 0 \tag{21}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + \gamma_2(x)\right)\bar{V}_n^{(2)}(x,s) = p\lambda\sum_{i=1}^n c_i \bar{V}_{n-i}^{(2)}(x,s); n \ge 0 \tag{22}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + \delta(x)\right)\bar{S}_n(x,s) = p\lambda\sum_{i=1}^n c_i \bar{S}_{n-i}(x,s); n \ge 0 \tag{23}$$

$$\bar{Q}_n(0,s) = (1-\theta)\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx + \int_0^\infty \bar{S}_n(x,s)\delta(x)dx; n \ge 1 \tag{24}$$

$$\bar{P}_0^{(1)}(0,s) = \int_0^\infty Q_1(x)\eta_e(x)dx + \lambda c_1 Q_0 \tag{25}$$

$$\bar{P}_n^{(1)}(0,s) = \int_0^\infty \bar{Q}_{n+1}(x,s)\eta_e(x)dx + \lambda\sum_{i=1}^n c_i\int_0^\infty \bar{Q}_{n-i}(x,s)dx + \lambda c_{n+1}\bar{Q}(s); n \ge 1 \tag{26}$$

$$\bar{P}_n^{(2)}(0,s) = \int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx; n \ge 0 \tag{27}$$

$$\bar{V}_n^{(1)}(0,s) = \theta\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx; n \geq 0 \tag{28}$$

$$\bar{V}_n^{(2)}(0,s) = r\int_0^\infty \bar{V}_n^{(1)}(x,s),\gamma_1(x)dx;; n \geq 0 \tag{29}$$

$$\bar{S}_n(0,s) = (1-r)\int_0^\infty \bar{V}_n^{(1)}(x,s)\gamma_1(x)dx + \int_0^\infty \bar{V}_n^{(2)}(x,s)\gamma_2(x)dx; n \geq 0 \tag{30}$$

To solve the above equations, we define the following generating functions:

$$Q_q(x,t,z) = \sum_{n=1}^{\infty} Q_n(x,t)z^n; Q_q(t,z) = \sum_{n=1}^{\infty} Q_n(t)z^n; \tag{31}$$

$$P_q^{(i)}(x,t,z) = \sum_{n=0}^{\infty} P_n^{(i)}(x,t)z^n; P_q^{(i)}(t,z) = \sum_{n=0}^{\infty} P_n^{(i)}(t)z^n i = 1,2 \tag{32}$$

$$V_q^{(i)}(x,t,z) = \sum_{n=0}^{\infty} V_n^{(i)}(x,t)z^n; V_q^{(i)}(t,z) = \sum_{n=0}^{\infty} V_n^{(i)}(t)z^n i = 1,2 \tag{33}$$

$$S_q(x,t,z) = \sum_{n=0}^{\infty} S_n(x,t)z^n; S_q(t,z) = \sum_{n=0}^{\infty} S_n(t)z^n; C(z) = \sum_{i=1}^{n} c_n z^n \tag{34}$$

Multiply equations (18) – (23) by appropriate powers of z and apply the generating function defined above

$$\left(\frac{d}{dx} + s + \lambda + \eta_e(x)\right)\bar{Q}_q(x,z,s) = 0 \tag{35}$$

$$\left(\frac{d}{dx} + s + p\lambda(1-C(z)) + \mu_1(x)\right)\bar{P}_q^{(1)}(x,z,s) = 0 \tag{36}$$

$$\left(\frac{d}{dx} + s + p\lambda(1-C(z)) + \mu_2(x)\right)\bar{P}_q^{(2)}(x,z,s) = 0 \tag{37}$$

$$\left(\frac{d}{dx} + s + p\lambda(1-C(z)) + \gamma_1(x)\right)\bar{V}_q^{(1)}(x,z,s) = 0 \tag{38}$$

$$\left(\frac{d}{dx} + s + p\lambda(1-C(z)) + \gamma_2(x)\right)\bar{V}_q^{(2)}(x,z,s) = 0 \tag{39}$$

$$\left(\frac{d}{dx} + s + p\lambda(1-C(z)) + \delta(x)\right)\bar{S}_q(x,z,s) = 0 \tag{40}$$

And for boundary conditions, multiply equation (24) by appropriate powers of z and using equation (17) we get

$$\bar{Q}_q(0,z,s) = (1-\theta)\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx + \int_0^\infty \bar{S}_q(x,z,s)\delta(x)dx + 1 - (s+\lambda)Q_0(s) \tag{41}$$

Also multiply equation (25) by z, multiply equation (26) by z^{n+1} and adding them, we get

$$z\bar{P}_q^{(1)}(0,z,s) = \int_0^\infty \bar{Q}_q(x,z,s)\eta_e(x)dx + \lambda C(z)\int_0^\infty \bar{Q}_q(x,z,s)dx + \lambda C(z)\bar{Q}_0(s) \tag{42}$$

Similarly multiply equations (27), (28), (29) and (30) by appropriate powers of z, respectively

$$\bar{P}_q^{(2)}(0,z,s) = \int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx \tag{43}$$

$$\bar{V}_q^{(1)}(0,z,s) = \theta\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx \tag{44}$$

$$\bar{V}_q^{(2)}(0,z,s) = r\int_0^\infty \bar{V}_q^{(1)}(x,z,s)\gamma_1(x)dx \tag{45}$$

$$\bar{S}_q(0,z,s) = (1-r)\int_0^\infty \bar{V}_q^{(1)}(x,z,s)\gamma_1(x)dx + \int_0^\infty \bar{V}_q^{(2)}(x,z,s)\gamma_2(x)dx \tag{46}$$

0n solving equations (35) – (40), we have,

$$\bar{Q}_q(x,z,s) = \bar{Q}_q(0,z,s)e^{-(s+\lambda)x-\int_0^x \eta_e(t)dt} \tag{47}$$

$$\bar{P}_q^{(1)}(x,z,s) = \bar{P}_q^{(1)}(0,z,s)e^{-(s+\lambda p(1-C(z))x-\int_0^x \mu_1(t)dt} \tag{48}$$

$$\bar{P}_q^{(2)}(x,z,s) = \bar{P}_q^{(2)}(0,z,s)e^{-(s+\lambda p(1-C(z))x-\int_0^x \mu_2(t)dt} \tag{49}$$

$$\bar{V}_q^{(1)}(x,z,s) = \bar{V}_q^{(1)}(0,z,s)e^{-(s+\lambda p(1-C(z))x-\int_0^x \gamma_1(t)dt} \tag{50}$$

$$\bar{V}_q^{(2)}(x,z,s) = \bar{V}_q^{(2)}(0,z,s)e^{-(s+\lambda p(1-C(z))x-\int_0^x \gamma_2(t)dt} \tag{51}$$

$$\bar{S}_q(x,z,s) = \bar{S}_q(0,z,s)e^{-(s+\lambda p(1-C(z))x-\int_0^x \delta(t)dt} \tag{52}$$

We now multiply equation (47) by and $\eta_\varepsilon(x)$ integrate by part with respect to x

$$\int_0^\infty \bar{Q}_q(x,z,s)\eta_e(x)dx = \bar{Q}_q(0,z,s)\bar{Q}(s+\lambda) \tag{53}$$

where

$\bar{Q}(s+\lambda)$ is the Laplace - Stieltjes transform of the retrial time. Multiply equations (48) and (49) by $\mu_1(x)$ and $\mu_2(x)$ respectively and integrate with respect to x

$$\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+\lambda p(1-C(z))) \tag{54}$$

$$\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}_q^{(2)}(0,z,s)\bar{B}_2(s+\lambda p(1-C(z))) \tag{55}$$

where

$\bar{B}_1(s+\lambda p(1-C(z)))$ and $\bar{B}_2(s+\lambda p(1-C(z)))$ are the Laplace - Stieltjes transforms of first stage and second stage service time respectively.

Similarly multiply (50), (51) and (52) by

$\gamma_1(x)$, $\gamma_2(x)$ and $\delta(x)$ respectively and integrate with respect to x, we have

$$\int_0^\infty \bar{V}_q^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}_q^{(1)}(0,z,s)\bar{V}_1(s+\lambda p(1-C(z))) \tag{56}$$

$$\int_0^\infty \bar{V}_q^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}_q^{(2)}(0,z,s)\bar{V}_2(s+\lambda p(1-C(z))) \tag{57}$$

$$\int_0^\infty \bar{S}_q(x,z,s)\delta(x)dx = \bar{S}_q(0,z,s)\bar{S}(s+\lambda p(1-C(z))) \tag{58}$$

where

$\bar{V}_1(s+\lambda p(1-C(z)))$, $\bar{V}_2(s+\lambda p(1-C(z)))$ and $\bar{S}(s+\lambda p(1-C(z)))$ are the Laplace - Stieltjes transforms of first phase, second phase vacation time and setup time respectively.

Again integrate (47) – (50) by parts with respect to x respectively

$$\bar{Q}_q(z,s) = \bar{Q}_q(0,z,s)\left[\frac{1-\bar{Q}_e(s+\lambda)}{s+\lambda}\right] \tag{59}$$

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{60}$$

$$\bar{P}_q^{(2)}(z,s) = \bar{P}_q^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{61}$$

$$\bar{V}_q^{(1)}(z,s) = \bar{V}_q^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{62}$$

$$\bar{V}_q^{(2)}(z,s) = \bar{V}_q^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{63}$$

$$\bar{S}_q(z,s) = \bar{S}_q(0,z,s)\left[\frac{1-\bar{S}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{64}$$

Let $H(z,s) = \bar{B}_1(s+\lambda p(1-C(z)))\bar{B}_2(s+\lambda p(1-C(z)))$ and

$$M(z,s) = [(1-r)\bar{V}_1(s+\lambda p(1-C(z))) + \bar{V}_2(s+\lambda p(1-C(z)))]\bar{S}(s+\lambda p(1-(Cz)))$$

Now equation (41) becomes

$$Q_q(0,z,s) = [1-(s+\lambda)Q_0(z,s)] + [((1-\theta)H(z,s)+\theta M(z,s))H(z,s)]P_q^{(1)}(0,z,s) \tag{65}$$

$$P_q^{(1)}(0,z,s) = \frac{\lambda C(z)\bar{Q}_0(s) + \left[1-(s+\lambda)\bar{Q}_0(s)\right]\left[\lambda C(z)\left[\frac{1-\bar{Q}_e(s+\lambda)}{(s+\lambda)}\right] + \bar{Q}_e(s+\lambda)\right]}{D(z,s)} \tag{66}$$

where D (z, s)is given by

$$D(z,s) = z - [(1-\theta) + \theta M(z,s)]H(z,s)\left[\lambda C(z)\left[\frac{1-\bar{Q}_e(s+\lambda)}{s+\lambda}\right] + \bar{Q}_e(s+\lambda)\right] \tag{67}$$

Sub the values for

$P_q^{(1)}(0,z,s)$ in equations (60) - (64) with help of equations (43) - (46), we get the probability generating functions of retial time, first stage service, second stage service, first phase vacation, second phase vacation and setup time respectively.

3. STEADY STATE DISTRIBUTION

In this section we shall derive the steady state probability distribution for our model. To define the steady state probabilities, suppressthe argument 't' where ever it appears in the time dependent analysis. By using well known Tauberian property in the probability generating functions of retial time, first stage service, second stage service, first phase vacation, second phase vacation and setup time respectively in the transient state.

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\,\bar{f}(s) \tag{68}$$

$$P_q^{(1)}(z) = \frac{Q_0\,\bar{Q}_e(\lambda)(1-\bar{B}_1(\lambda p(1-C(z))))}{pDr} \tag{69}$$

$$P_q^{(2)}(z) = \frac{Q_0\,\bar{Q}_e(\lambda)\,\bar{B}_1(\lambda p(1-C(z)))(1-\bar{B}_2(\lambda p(1-C(z)))}{pDr} \tag{70}$$

$$V_q^{(1)}(z) = \frac{\theta Q_0\,\bar{Q}_e(\lambda)\,\bar{B}_1(\lambda p(1-z))\,\bar{B}_2(\lambda p(1-z))(1-\bar{V}_1(\lambda p(1-z)))}{pDr} \tag{71}$$

$$V_q^{(2)}(z) = \frac{r\theta Q_0\,\bar{Q}_e(\lambda)\,\bar{B}_1(\lambda p(1-C(z)))\,\bar{B}_2(\lambda p(1-C(z)))\bar{V}_1(\lambda p(1-C(z)))(1-\bar{V}_2(\lambda p(1-C(z)))}{pDr} \tag{72}$$

$$S_q(z) = \frac{\theta Q_0\,\bar{Q}_e(\lambda)\,\bar{B}_1(\lambda p(1-C(z)))\,\bar{B}_2(\lambda p(1-C(z)))[(1-r)\bar{V}_1(\lambda p(1-C(z))) + \bar{V}_2(\lambda p(1-C(z)))](1-\bar{S}(\lambda p(1-C(z)))}{pDr} \tag{73}$$

$$Q_q(z) = \frac{Q_0 z(1-\bar{Q}_e(\lambda))\{1-[(1-\theta)+\theta M(z)]H(z)\}}{Dr} \tag{74}$$

where $H(z) = \bar{B}_1(\lambda p(1-C(z)))\,\bar{B}_2(\lambda p(1-C(z)))$

$$M(z) = [(1-r)\bar{V}_1(\lambda p(1-C(z))) + \bar{V}_2(\lambda p(1-C(z)))\,]\bar{S}(\lambda p(1-(Cz)))$$

To find Q_0 using the normalizing condition at z = 1

$$Q_0 + Q_q^{(1)}(1) + P_q^{(1)}(1) + P_q^{(2)}(1) + V_q^{(1)}(1) + V_q^{(2)}(1) + S_q(1) = 1 \tag{75}$$

$$Q_0 = \frac{[1-E(\bar{I})[1-Q_e(\lambda)] - p\rho]}{\bar{Q}_e(\lambda)[1+\rho(1-p)]} \tag{76}$$

where $\rho = \lambda\{(E(S_1) + E(S_2) + \theta[(E(V_1) + r\,E(V_2) + E(D)]\}$

The necessary and sufficient condition for the stability condition is

$$E(I)(1-\bar{Q}_e(\lambda))+p\rho<1 \tag{77}$$

4. PERFORMANCE MEASURES OF THE SYSTEM

In this section we give some system queuing measures of the system.

(1) Probability that the server is idle = $Q_0=\dfrac{[1-E(\bar{I})(1-Q_e(\lambda))-p\rho]}{\bar{Q}_e(\lambda)[1+\rho(1-p)]}$ (78)

(2) Probability that the server is idle but the orbit is not empty=

$$Q_q(1)=\frac{E(I)(1-\bar{\bar{Q}}_e(\lambda))p\rho}{\bar{Q}_e(\lambda)[1+\rho(1-p)]} \tag{79}$$

(3) Probability that the server is busy with either first stage or second stage

$$P_q^{(1)}(1)+P_q^{(2)}(1)=\frac{\lambda E(I)(E(S_1)+E(S_2))}{[1+\rho(1-p)]} \tag{80}$$

(4) Probability that the server is on first phase vacation = $V_q^{(1)}(1)=\dfrac{\lambda\theta E(I)E(V_1)}{[1+\rho(1-p)]}$ (81)

(5) Probability that the server is on second phase vacation= $V_q^{(2)}(1)=\dfrac{r\lambda\theta E(I)E(V_2)}{[1+\rho(1-p)]}$ (82)

(6) Probability that the server is on setup period $S_q(1)=\dfrac{\lambda\theta E(I)E(D)}{[1+\rho(1-p)]}$ (83)

(7) The average number of customers in the orbit

$$L_q=\left[\frac{(\lambda E(I))^2\,p\{E(S^2)+2E(S)\theta[E(V)+E(D)]+\theta[E(V^2)+E(D^2)]\}}{2[1-E(I)(1-Q_e(\lambda))-\rho p]}\right]$$
$$+\frac{[(1-\bar{Q}_e(\lambda))E(I)p\rho]}{[1-E(I)(1-\bar{Q}_e(\lambda))-p\rho]}+\frac{[E(I)(1-\bar{Q}_e(\lambda)+p\rho]}{[1-E(I)(1-\bar{Q}_e(\lambda))-p\rho]}C_{[R]} \tag{84}$$

Where $E(S) = E(S_1) + E(S_2)$; $E(S^2) = E(S_1^2) + 2E(S_1)E(S_2) + E(S_2^2)$;

$E(V)=E(V_1) + r\,E(V_2)$; $E(V^2)=E(V_1^2)+2r\,E(V_1)E(V_2)+ r\,E(V_2^2)$;

$E(D)$= Mean Setup time; $E(D^2)$ = second order moment of set up time

(8) The expected waiting time in the orbit $W_q=\dfrac{L_q}{\lambda pE(I)}$ (85)

(9) The blocking probability =1 - $[Q_0+Q_1]$ = $\dfrac{\rho}{[1+\rho(1-p)]}$ (86)

(10) The steady state availability of the server =A = $1-\left\{\dfrac{\lambda E(I)\theta[E(V_1)+rE(V_2)+E(D)]}{1+\rho(1-P)}\right\}$ (87)

5. NUMERICAL ANALYSIS

We present some numerical results in order to illustrate the effect of various parameters on the main performance of our model. The effect of parameters λ (arrival rate),θ , η(retrial rate), μ_1(first stage service rate), μ_2(second stage service rate), γ_1(first phase vacation completion rate), γ_2(second phase vacation completion rate),δ(set up rate) on mean queue in the orbit, availability, blocking probability, and ρ have been studied numerically. The calculations are carried out by considering the distributions of retrial time, service times, vacation times and setup times are exponential. By setting the default parameters, μ_1= 7, μ_2 = 8, γ_1 = 5, γ_2 = 7, δ= 5, η = 5, p = 0.5, r = 0.5. The following table shows the computed values of server's idle time and other queue characteristics for our model.

Table: Computed values server's idle time and other queue characteristics

λ Arrivalrate	Θ	ρ	Q_0	Availability	Blocking probability	L_q	W_q
0.1	0.2	0.0362	0.9993	0.9907	0.0356	0.0039	0.0785
0.1	0.4	0.0456	0.9990	0.9816	0.0446	0.0048	0.0959
0.1	0.6	0.0551	0.9987	0.9725	0.0536	0.0057	0.1135
0.1	0.8	0.0645	0.9983	0.9635	0.0625	0.0066	0.1314
0.1	1	0.0739	0.9979	0.9545	0.0713	0.0075	0.1493
0.3	0.2	0.0724	0.9972	0.9818	0.0699	0.0199	0.1326
0.3	0.4	0.0913	0.9960	0.9639	0.0873	0.0286	0.1906
0.3	0.6	0.1101	0.9946	0.9464	0.1044	0.0376	0.2504
0.3	0.8	0.1290	0.9931	0.9291	0.1212	0.0468	0.3123
0.3	1	0.1479	0.9914	0.9122	0.1377	0.0564	0.3762
0.5	0.2	0.1086	0.9936	0.9732	0.1030	0.0487	0.1946
0.5	0.4	0.1369	0.9909	0.9471	0.1282	0.0756	0.3024
0.5	0.6	0.1652	0.9878	0.9216	0.1526	0.1042	0.4167
0.5	0.8	0.1935	0.9843	0.8968	0.1764	0.1346	0.5382
0.5	1	0.2218	0.9803	0.8727	0.1996	0.1669	0.6676
0.7	0.2	0.1449	0.9885	0.9648	0.1351	0.0932	0.2663
0.7	0.4	0.1826	0.9837	0.9309	0.1673	0.1527	0.4362
0.7	0.6	0.2203	0.9781	0.8981	0.1984	0.2178	0.6224
0.7	0.8	0.2580	0.9717	0.8664	0.2285	0.2896	0.8275
0.7	1	0.2957	0.9646	0.8357	0.2576	0.3691	1.0545
0.9	0.2	0.1811	0.9819	0.9568	0.1660	0.1575	0.3501
0.9	0.4	0.2282	0.9743	0.9154	0.2048	0.2695	0.5988
0.9	0.6	0.2754	0.9654	0.8757	0.2420	0.3973	0.8828
0.9	0.8	0.3225	0.9553	0.8376	0.2777	0.5445	1.2100
0.9	1	0.3696	0.9439	0.8011	0.3120	0.7160	1.5911

6. CONCLUSION

In this paper, a batch arrival, single server retrial queue with two stage heterogeneous services, extended server vacation and setup time is considered. We studied the transient state distribution and steady state analysis with performance measures of system. Also numerical illustration has been given for the model. This model can be applied in the design of computer networks.

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