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DISCRETE BETA-FACTORIAL FUNCTION AND ITS SERIES INTERMS OF CLOSED FORM SOLUTION OF GENERALIZED DIFFERENCE EQUATION

G. Britto Antony Xavier[1*], S. U. Vasantha Kumar[2] and B. Govindan[3]

[1,2,3]Department of Mathematics, Sacred Heart College, Tirupattur - 635601, Vellore District- Tamil Nadu, S. India.

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ABSTRACT

We investigate numerical-closed form solution to higher order generalized difference equation to find the value of discrete beta series in terms of discrete gamma factorial function. Examples are inserted to illustrate our main results.

***Key words**: Closed form solution, Discrete Beta function, Discrete Gamma function, Generalized difference equation, Numerical solution.*

***AMS Subject classification**: 39A70, 47B39, 39A10.*

1. INTRODUCTION

The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermo dynamics), chemistry, biology, economics and control theory ([6], [7], [8], [9], [10]). In 1989, K.S.Miller and Ross [5] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The main definition of fractional difference equation (as done in [5]) is the ν fractional sum of $f(t)$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma_{(\nu)}} \sum_{s=a}^{t-\nu} \frac{\Gamma_{(t-s)}}{\Gamma_{(t-s-(\nu-1))}} f(s), \tag{1}$$

where $\nu > 0$. On the other hand, when $\nu = m$ is a positive integer, if we replace $f(t)$ by $u(k)$ and Δ by Δ_ℓ, (as given in definition 2.8 of [5]) then (1) becomes

$$\Delta_\ell^{-m} u(k) = \sum_{r=m}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(m-1)}}{(m-1)!} u(k-r\ell), \tag{2}$$

where $(r-1)^{(m-1)} = (r-1)(r-2)\cdots(r-m+1)$ and $\left[\frac{k}{\ell}\right]$ is the integer part of $\frac{k}{\ell}$ which is a numerical solution of the generalized higher order difference equation

$$\Delta_\ell^{\nu} v(k) = u(k), k \in [0,\infty), \ell > 0. \tag{3}$$

Now (2) is very useful to derive many interesting results in a different way, such as the sum of the m^{th} partial sums to the n^{th} powers and the products of n consecutive terms of arithmetic and geometric progressions [4].

In this paper, we define the discrete beta functions and obtain the value of beta series in terms of discrete gamma factorial function by numerical-closed form solution of (3).

Corresponding author: G. Britto Antony Xavier[1*],
[1,2,3]Department of Mathematics, Sacred Heart College, Tirupattur - 635601, Vellore District- Tamil Nadu, S. India.

2. PRELIMINARIES

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be useful for further subsequent discussions. Throughout this paper, let $\ell > 0, k \in [0,\infty), j = k - \left[\frac{k}{\ell}\right]\ell$ where $\left[\frac{k}{\ell}\right]$ denotes the integer part of $\frac{k}{\ell}$ and $\mathbb{N}(a) = \{a, a+1, a+2, \cdots\}$.

For $p \in \mathbb{N}(1)$, we denote $\Delta_\ell^{-p}u(k)\|_{(p-1)\ell+j}^k = \Delta_\ell^{-1}\left(\cdots\Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(k)|_j^k\right)|_{\ell+j}^k\cdots\right)|_{(p-1)\ell+j}^k$, where $\Delta_\ell^{-1}u(k)|_j^k = u_1(k)|_j^k = \Delta_\ell^{-1}u(k) - \Delta_\ell^{-1}u(j), \Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(k)|_j^k\right)|_{\ell+j}^k = u_2(k)|_{\ell+j}^k = \Delta_\ell^{-1}u_1(k) - \Delta_\ell^{-1}u_1(\ell+j)$, and so on are closed form solutions of (3) for $\nu = 1, 2, \cdots, p$ respectively.

Definition: 2.1 [3] *For a real valued function* $u(k)$, *the generalized difference operator* Δ_ℓ *and its inverse are respectively defined as*

$$\Delta_\ell u(k) = u(k+\ell) - u(k), k \in [0,\infty), \ell \in (0,\infty), \tag{4}$$

and if $\Delta_\ell v(k) = u(k)$, then $v(k) = \Delta_\ell^{-1}u(k) + c_j$ (5)

where c_j is a constant for all $k \in \mathbb{N}_\ell(j) = \{j, \ell+j, 2\ell+j, \cdots\}$.

Definition: 2.2 [6] *For* $k, n \in (0,\infty)$, *the* ℓ*-factorial function is defined by*

$$k_\ell^{(n)} = \ell^n \frac{\Gamma\left(\frac{k}{\ell}+1\right)}{\Gamma\left(\frac{k}{\ell}+1-n\right)}, \tag{6}$$

where Γ is the Euler gamma function and when $\ell = 1$, $k_1^{(n)} = k^{(n)}$ and we denote $k_\ell^{(-n)} = \frac{1}{k_\ell^{(n)}}$.

Remark: 2.3 When $n \in \mathbb{N}(1)$, (6) *and its* Δ_ℓ *difference becomes,*

$$k_\ell^{(n)} = \prod_{t=0}^{n-1}(k - t\ell) \text{ and hence } \Delta_\ell k_\ell^{(n)} = (n\ell)k_\ell^{(n-1)}. \tag{7}$$

Leemma: 2.4 [3] *Let* s_q^n *and* S_q^n *are the Stirling numbers of the first and second kinds respectively,* $s_0^0 = S_0^0 = 1$ *and* $s_q^0 = S_q^0 = 0 = s_0^q = S_0^q$ *if* $q \neq 0$. *Then for* $n \in \mathbb{N}(0)$,

$$k_\ell^{(n)} = \sum_{q=0}^{n} s_q^n \ell^{n-q} k^q, \; k^n = \sum_{q=0}^{n} S_q^n \ell^{n-q} k_\ell^{(q)} \text{ and } \Delta_\ell^{-\nu} k_\ell^{(q)} = \frac{k_\ell^{(q+\nu)}}{(q+\nu)^{(\nu)}\ell^\nu} \tag{8}$$

is a closed form solution of (3) for $u(k) = k_\ell^{(q)}$.

Definition: 2.5 [3] *Let* $\nu \in \mathbb{N}(1)$ *and* $u(k)$ *defined on* $[0,\infty)$ *be real valued function. Then,*

$$\Delta_\ell^{-\nu}u(k)\|_{(\nu-1)\ell+j}^k = \sum_{r=\nu}^{[\frac{k}{\ell}]} \frac{(r-1)^{(\nu-1)}}{(\nu-1)!} u(k - r\ell) \tag{9}$$

is a numerical solution of equation (3) and when $\nu = 1$, we have

$$\Delta_\ell^{-1}u(k)|_j^k = \sum_{r=1}^{[\frac{k}{\ell}]} u(k - r\ell). \tag{10}$$

Theorem: 2.6 [1] *Let* $k \in [0,\infty)$ *and* $\lim_{k\to\infty}\Delta_\ell^{-1}u(k) = 0$. *Then,*

$$\Delta_\ell^{-1}u(k)\,|_k^\infty = \sum_{r=0}^{\infty} u(k+r\ell). \tag{11}$$

Definition: 2.7 [1] *For* $n \in (-\infty,\infty)-\{0\}$ *and* $k \in [0,\infty)$, *the Discrete* k *- Gamma Factorial Function is defined as*

$$_k\Gamma_\ell((n)) = \ell\Delta_\ell^{-1}\left[k_\ell^{(n-1)}e^{-k}\right]_k^\infty = \ell\sum_{r=0}^{\infty}(k+r\ell)_\ell^{(n-1)}e^{-(k+r\ell)} \tag{12}$$

and the Discrete k - Gamma Function is defined as

$$_k\Gamma_\ell(n) = \ell\Delta_\ell^{-1}\left[k^{n-1}e^{-k}\right]_k^\infty = \ell\sum_{r=0}^{\infty}(k+r\ell)^{n-1}e^{-(k+r\ell)}. \tag{13}$$

Lemma: 2.8 [1] *Let* $u(k)$ *and* $w(k)$ *be two real valued functions. Then,*

$$\Delta_\ell^{-1}[u(k)w(k)] = v(k)\Delta_\ell^{-1}w(k) - \Delta_\ell^{-1}[\Delta_\ell^{-1}w(k+\ell)\Delta_\ell u(k)] \tag{14}$$

is a solution of difference equation (3) for $\nu = 1$ and by considering $u(k)w(k)$ as $u(k)$.

Lemma: 2.9 [3] *Let* $k \in [\ell,\infty)$. *Then,*

$$\Delta_\ell^{-1}e^{-k}\,|_j^k = \frac{e^{-k}}{e^{-\ell}-1} - \frac{e^{-j}}{e^{-\ell}-1} \tag{15}$$

is a numerical solution of equation (3) for $\nu = 1$.

Theorem: 2.10 *For* $a > 0$ *and* $\nu = 1$, *the solution* $\Delta_\ell^{-1}u(k)$ *of equation (3) satisfies the relation*

$$\Delta_\ell^{-1}u(k)\Delta k\,|_0^a = \Delta_\ell^{-1}u(a-k)\Delta k\,|_0^a, \tag{16}$$

which yields

$$\sum_{r=1}^{[\frac{k}{\ell}]}u(k-r\ell) = \sum_{r=1}^{[\frac{k}{\ell}]}u(a-k+r\ell). \tag{17}$$

Proof: The L.H.S of (16) is coinciding with R.H.S of (16) by taking the new variable t as $k = a-t$, $\Delta k = -\Delta t$, and (17) follows from (10) and (16).

3. MAIN RESULTS

In this section, we introduce discrete $_k\beta(.)$ factorial function of four types and obtain formulas on discrete $_k\beta(.)$ series by *Numerical-Closed* form solution of difference equation (3).

The following example shows that a closed form solution of equation (3) need not be coinciding with the numerical solution (2) of that equation.

Example: 3.1 Taking $\nu = 2$, $u(k) = k_\ell^{(2)}$ *in (3),* $\sum_{r=1}^{[\frac{k}{\ell}]}(k-r\ell)_\ell^{(2)}$ *is a numerical solution of (3), and* $\Delta_\ell^{-1}k_\ell^{(2)} = \frac{k_\ell^{(3)}}{3\ell}$ *is a closed form solution of (3). When* $\ell = 0.8$, $k = 2.7$, *we find that* **1**

$$2.27 = \sum_{r=1}^{3}(2.7-0.8r)_{0.8}^{(2)} \neq \frac{(2.7)_{0.8}^{(3)}}{2.4} = 2.3512\,.$$

The closed form solution which is coincided with numerical solution of (2) can be called as *Complete solution* of (3).

Remark: 3.2 Hereafter $t\left(L_{\nu-1}\right)$ = *set of all subsets of size* t *in ascending order from the set* $L_{\nu-1} = \{1,2,\cdots,\nu-1\}$ *and* $\{m_1,\ldots,m_t\} \in t(L_{\nu-1})$ *is denoted as* $\{m_t\} \in t(L_{\nu-1})$.

Theorem: 3.3 [1] *Let* $\nu \in \mathbb{N}(2)$, $\ell \in (0,\infty)$ *be fixed and* $k \in [\nu\ell,\infty)$ *be variable. If* $\Delta_\ell^{-\nu}u(k)$ *is any closed form solution of equation (3), then*

$$u_\nu(k) = \Delta_\ell^{-\nu}u(k)\big|_{(\nu-1)\ell+j}^{k} + \sum_{t=1}^{\nu-1}\sum_{\{m_t\}\in t(L_{\nu-1})}(-1)\times(\Delta_\ell^{-m_1}u((m_1-1)\ell+j))\frac{k_\ell^{(\nu-m_t)}}{(\nu-m_t)!\ell^{\nu-m_t}} \prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}\Big|_{(\nu-1)\ell+j}^{k} \tag{18}$$

is the complete solution of (3).

Corollary: 3.4 The Numerical-Closed form relation of (3) is

$$\Delta_\ell^{-\nu}u(k)\big\|_{(\nu-1)\ell+j}^{k} = u_\nu(k). \tag{19}$$

Proof: The proof follows by Theorem 3.3 and equating (9) and (18).

Theorem: 3.5 Let $0<\ell<a,\ m,n\in\mathbb{N}(1)$ *and* $j = k-\left[\frac{k}{\ell}\right]\ell$. *Then*

$$\begin{aligned}\sum_{r=\nu}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(\nu-1)}}{(\nu-1)!}\frac{(k-r\ell)^{m-1}}{(a-k+r\ell)^{1-n}} = {} & (-1)^{n-1}\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(-1)^i(n-1)^{(i)}a^i}{i!} \\ & \times\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}\Big\{\frac{k_\ell^{(r+\nu)}}{\ell^\nu(r+\nu)^{(\nu)}} + \sum_{t=1}^{\nu-1} \\ & \times\sum_{\{m_t\}\in t(L_{\nu-1})}\frac{(-1)^t((m_1-1)\ell+j)_\ell^{(r+m_1)}}{\ell^{m_1}(r+m_1)^{(m_1)}} \\ & \times\frac{k_\ell^{(\nu-m_t)}}{(\nu-m_t)!\ell^{\nu-m_t}} \\ & \times\prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}\Big\}\Big|_{(\nu-1)\ell+j}^{k}\end{aligned} \tag{20}$$

which is the Numerical-Complete solution of (3) for $u(k) = k^{m-1}(a-k)^{n-1}$.

Proof: Applying binomial expansion to $(k-a)^{n-1}$ and then converting polynomials into polynomial factorials by second term of (8), we find

$$k^{m-1}(a-k)^{n-1} = (-1)^{n-1}\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(-1)^i(n-1)^{(i)}a^i}{i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}k_\ell^{(r)}.$$

Taking Δ_ℓ^{-1} on both sides and by using the third term of (8), we arrive

$$\Delta_\ell^{-1}k^{m-1}(a-k)^{n-1} = \sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}\frac{k_\ell^{(r+1)}}{\ell(r+1)}.$$

Again taking Δ_ℓ^{-1} on both sides $\nu-1$ times and applying third term of (8) each time, we get

$$\Delta_\ell^{-\nu}k^{m-1}(a-k)^{n-1} = \sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}\frac{k_\ell^{(r+\nu)}}{\ell^\nu(r+\nu)^{(\nu)}}, \tag{21}$$

which is a closed form solution of (3) for $u(k) = k^{m-1}(a-k)^{n-1}$. Applying (21) in (18), we get a closed form solution mentioned in R.H.S of (20). Now the proof follows from (9) and (19).

Corollary: 3.6 Let $0<\ell<a,\ m,n\in\mathbb{N}(1)$ *and* $j=k-\left[\frac{k}{\ell}\right]\ell$. *Then,*

$$\sum_{r=\nu}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(\nu-1)}}{(\nu-1)!}\frac{(k-r\ell)_\ell^{(m-1)}}{(a-k+r\ell)^{1-n}}=(-1)^{n-1}\sum_{i=0}^{n-1}\sum_{r_1=0}^{m-1}\sum_{r_2=0}^{r_1+n-1-i}\frac{(-1)^i(n-1)^{(i)}a^i}{i!}\times\frac{s_{r_1}^{m-1}S_{r_2}^{r_1+n-1-i}}{\ell^{i+r_2+2-m-n}}\{\frac{k_\ell^{(r_2+\nu)}}{\ell^\nu(r_2+\nu)^{(\nu)}}.+\sum_{t=1}^{\nu-1}\sum_{\{m_t\}\in t(L_{\nu-1})}(-1)^t\times\frac{((m_1-1)\ell+j)_\ell^{(r_2+m_1)}k_\ell^{(\nu-m_t)}}{\ell^{m_1}(r_2+m_1)^{(m_1)}(\nu-m_t)!\ell^{\nu-m_t}}\times\prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}\}\Big|_{(\nu-1)\ell+j}^{k}\tag{22}$$

Proof: By using first term of (8), we have

$$k_\ell^{(m-1)}(a-k)^{n-1}=\sum_{r_1=0}^{m-1}s_{r_1}^{m-1}\ell^{m-1-r_1}k^{r_1}(a-k)^{n-1}.$$

Taking $\Delta_\ell^{-\nu}$ on both sides, applying theorem 3.5 and replacing $m-1$ by r_1, r by r_2, the proof is complete.

Corollary: 3.7 Let $0<\ell<a,\ m,n\in\mathbb{N}(1)$ *and* $j=k-\left[\frac{k}{\ell}\right]\ell$. *Then,*

$$\sum_{r=\nu}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(\nu-1)}}{(\nu-1)!}\frac{(k-r\ell)^{m-1}}{(a-k+r\ell)_\ell^{-(n-1)}}=(-1)^{n-1}\sum_{i=0}^{r_1}\sum_{r_1=0}^{n-1}\sum_{r_2=0}^{r_1+m-1-i}\frac{(-1)^i(r_1)^{(i)}a^i}{i!}\times\frac{s_{r_1}^{n-1}S_{r_2}^{r_1+m-1-i}}{\ell^{i+r_2+2-m-n}}\{\frac{k_\ell^{(r_2+\nu)}}{\ell^\nu(r_2+\nu)^{(\nu)}}+\sum_{t=1}^{\nu-1}\sum_{\{m_t\}\in t(L_{\nu-1})}(-1)^t\times\frac{((m_1-1)\ell+j)_\ell^{(r_2+m_1)}k_\ell^{(\nu-m_t)}}{\ell^{m_1}(r_2+m_1)^{(m_1)}(\nu-m_t)!\ell^{\nu-m_t}}\times\prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}\}\Big|_{(\nu-1)\ell+j}^{k}.\tag{23}$$

Proof: By using first term of (8), we have

$$k^{m-1}(a-k)_\ell^{(n-1)}=\sum_{r_1=0}^{n-1}s_{r_1}^{n-1}\ell^{n-1-r_1}k^{m-1}(a-k)^{r_1}.$$

Taking $\Delta_\ell^{-\nu}$ on both sides, applying theorem 3.5 and replacing $n-1$ by r_1, r by r_2, the proof is complete.

Corollary: 3.8 Let $0<\ell<a,\ m,n\in\mathbb{N}(1)$ *and* $j=k-\left[\frac{k}{\ell}\right]\ell$. *Then,*

$$\sum_{r=\nu}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(\nu-1)}}{(\nu-1)!}\frac{(k-r\ell)_\ell^{(m-1)}}{(a-k+r\ell)_\ell^{-(n-1)}}=(-1)^{n-1}\sum_{r_1=1}^{m-1}\sum_{r_2=1}^{n-1}\sum_{i=0}^{r_2}\sum_{r_3=1}^{r_1+r_2-i}\frac{(-1)^i(r_2)^{(i)}a^i}{i!}$$

$$\times\frac{s_{r_1}^{m-1}s_{r_2}^{n-1}S_{r_3}^{r_1+r_2-i}}{\ell^{i+r_3+2-m-n}}\{\frac{k_\ell^{(r_3+\nu)}}{\ell^\nu(r_3+\nu)^{(\nu)}}+\sum_{t=1}^{\nu-1}\sum_{\{m_t\}\in t(L_{\nu-1})}(-1)^t$$

$$\times\frac{((m_1-1)\ell+j)_\ell^{(r+m_1)}k_\ell^{(\nu-m_t)}}{\ell^{m_1}(r+m_1)^{(m_1)}(\nu-m_t)!\ell^{\nu-m_t}}\times\prod_{i=2}^{t}\frac{((m_i-1)\ell+j)_\ell^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!\ell^{m_i-m_{i-1}}}\}\Big|_{(\nu-1)\ell+j}^{k}.\quad(24)$$

Proof: By using first term of (8), we have

$$k_\ell^{(m-1)}(a-k)_\ell^{(n-1)}=\sum_{r_1=0}^{m-1}\sum_{r_2=0}^{n-1}s_{r_1}^{m-1}s_{r_2}^{n-1}\ell^{m+n-2-r_1-r_2}k^{r_1}(a-k)^{r_2}$$

Taking $\Delta_\ell^{-\nu}$ on both sides, applying theorem 3.5 and replacing $m-1$ by $r_1, n-1$ by r_2, r by r_3, the proof is complete.

When $\nu=1$, Theorem 3.5, Corollaries 3.6, 3.7 and 3.8 motivate us to define the following discrete $k-$ beta functions.

Definition: 3.9 Let $0<\ell<a,\ m>0,\ n>0,\ a>0,\ k\in[\ell,a)$ *and* $j=k-\left[\frac{k}{\ell}\right]\ell$. *Then the Discrete k-beta, k-beta factorial, k-beta left factorial and k-beta right factorial functions are respectively defined as*

(i) ${}_k\beta_\ell(m,n,a)=\ell\Delta_\ell^{-1}k^{m-1}(a-k)^{n-1}\Big|_j^k$

(ii) ${}_k\beta_\ell((m,n,a))=\ell\Delta_\ell^{-1}k_\ell^{(m-1)}(a-k)_\ell^{(n-1)}\Big|_j^k$

(iii) ${}_k\beta_\ell((m,n,a)=\ell\Delta_\ell^{-1}k_\ell^{(m-1)}(a-k)^{n-1}\Big|_j^k$

(iv) ${}_k\beta_\ell(m,n,a))=\ell\Delta_\ell^{-1}k^{m-1}(a-k)_\ell^{(n-1)}\Big|_j^k$.

In particular, $\beta_\ell(m,n)=\ell\Delta_\ell^{-1}k^{m-1}(1-k)^{n-1}\Big|_j^1$, where $j=1-\left[\frac{1}{\ell}\right]\ell$, is called discrete beta function.

Similarly the beta factorial $\beta_\ell((m,n))$, beta left factorial $\beta_\ell((m,n)$ and beta right factorial $\beta_\ell(m,n))$ functions are defined by putting $a=1$ in (ii), (iii) and (iv) respectively.

Theorem: 3.10 Let $0<\ell<a,\ m,n\in\mathbb{N}(1)$, $k\in[\ell,a)$. *Then,*

$${}_k\beta_\ell(m,n,a)=\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}\frac{k_\ell^{(r+1)}}{(r+1)}\Big|_j^k.\quad(25)$$

Proof: The proof follows by taking $\nu=1$ in Theorem 3.5 and then multiplying both sides by ℓ.

Corollary: 3.11 *When $k=a=1$, (25) becomes*

$$\beta_\ell(m,n)=(-1)^{n-1}\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(-1)^i(n-1)^{(i)}}{i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}}\frac{k_\ell^{(r+1)}}{(r+1)}\Big|_j^1.$$

Example: 3.12 Since $S_1^2 = S_2^2 = S_1^3 = S_3^3 = 1$ *and* $S_2^3 = 3$, *putting* $\ell = 0.55$, $m = 3$, $n = 2$ *and* $j = 0.45$ *in corollary 3.11, we get*

$$\beta_\ell(3,2) = (-1)\sum_{i=0}^{1}\sum_{r=0}^{3-i}\frac{(-1)^i(1)^{(i)}}{i!}\frac{S_r^{3-i}}{(0.55)^{i+r-3}}\frac{k_{0.55}^{(r+1)}}{(r+1)}\Big|_{0.45}^{1} = 0.0613.$$

Similarly,

$$\beta_\ell(2,3) = (-1)\sum_{i=0}^{2}\sum_{r=0}^{3-i}\frac{(-1)^i(2)^{(i)}}{i!}\frac{S_r^{3-i}}{(0.55)^{i+r-3}}\frac{k_{0.55}^{(r+1)}}{(r+1)}\Big|_{0.45}^{1} = 0.0749.$$

Remark: 3.13 *The above example shows that* $\beta_\ell(m,n)$ *need not be equal to* $\beta_\ell(n,m)$.

Lemma: 3.14 If $0 < \ell < k \le a$ *and* k *is a multiple of* ℓ, *i.e* $m(\ell) = k$, *then*

$${}_a\beta_\ell(m,n,a) = {}_a\beta_\ell(n,m,a).$$

In particular, when $a = 1$ and $1 = m(\ell)$, then $\beta_\ell(m,n) = \beta_\ell(n,m)$.

Proof: By Definition 3.9 (i), we have

$${}_a\beta_\ell(m,n,a) = \ell\Delta_\ell^{-1}k^{m-1}(a-k)^{n-1}\Big|_0^a$$

It can be rewritten as

$${}_a\beta_\ell(m,n,a) = \Delta_\ell^{-1}k^{m-1}(a-k)^{n-1}\Delta_\ell k\Big|_0^a$$

Taking $t = a - k$, we get the required result.

Example: 3.15 Taking $\ell = 0.5$, $m = 3$, $n = 2$ *and* $j = 0$ *in corollary 3.11,*

$\beta_\ell(3,2)$ *= 0.0625=* $\beta_\ell(2,3)$.

Theorem: 3.16 If $0 < \ell < k \le a$ *and* $m,n \in \mathbb{N}(1)$, *then*

$$(i)\ {}_k\beta_\ell((m,n,a) = \sum_{i=0}^{n-1}\sum_{r_1=0}^{m-1}\sum_{r_2=0}^{r_1+n-1-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{s_{r_1}^{m-1}S_{r_2}^{r_1+n-1-i}}{\ell^{i+r_2+2-m-n}}\frac{k_\ell^{(r_2+1)}}{(r_2+1)}\Big|_j^k$$

$$(ii)\ {}_k\beta_\ell(m,n,a)) = \sum_{i=0}^{r_1}\sum_{r_1=0}^{n-1}\sum_{r_2=0}^{r_1-i+m-1}\frac{(r_1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{s_{r_1}^{n-1}S_{r_2}^{r_1-i+m-1}}{\ell^{i+r_2+2-m-n}}\frac{k_\ell^{(r_2+1)}}{(r_2+1)}\Big|_j^k$$

$$(iii)\ {}_k\beta_\ell((m,n,a)) = \sum_{r_1=0}^{m-1}\sum_{r_2=0}^{n-1}\sum_{i=0}^{r_2}\sum_{r_3=0}^{r_1+r_2-i}\frac{(r_2)^{(i)}a^i s_{r_1}^{m-1}s_{r_2}^{n-1}S_{r_3}^{r_1+r_2-i}k_\ell^{(r_3+1)}}{(-1)^{n-1+i}i!\ell^{i+r_3+2-m-n}(r_3+1)}\Big|_j^k.$$

Proof: The proof of (i), (ii) and (iii) follows by taking $\nu = 1$ in Corollaries 3.6, 3.7 and 3.8 and then multiplying them both sides by ℓ.

Lemma: 3.17 $\lim_{\ell\to 0} k_\ell^{(r)}(1-k)_\ell^{(t)} = \lim_{\ell\to 0} k_\ell^{(t)}(1-k)_\ell^{(r)}$

Lemma: 3.18 $\lim_{\ell\to 0}\beta_\ell((m,n)) = \lim_{\ell\to 0}\beta_\ell((n,m)) = \lim_{\ell\to 0}\beta_\ell(m,n) = \lim_{\ell\to 0}\beta_\ell(n,m)$

Theorem: 3.19 Let $0<\ell<k\le a\ m,n\in\mathbb{N}(1)$. *Then,*

$$\sum_{r=1}^{[\frac{k}{\ell}]}{}_{(k-r\ell)}\beta_\ell(m,n,a)=\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^iS_r^{m+n-2-i}k_\ell^{(r+2)}}{(-1)^{n-1+i}i!\ell^{i+r+3-m-n}(r+2)^{(2)}}\Big|_{\ell+j}^{k}\,. \tag{26}$$

Proof: Taking Δ_ℓ^{-1} on both sides of (25) and by using third term of (8) when $\nu=1$, the proof is complete.

Example: 3.20 When $a=7,\ k=6,\ \ell=2=m,\ n=3$ *and* $j=0$, *(26) gives*

$$\sum_{r=1}^{3}{}_{(6-2r)}\beta_2(2,3,7)=\sum_{i=0}^{2}\sum_{r=0}^{3-i}\frac{(-1)^i(2)^{(i)}7^i}{i!}\frac{S_r^{3-i}}{2^{i+r-2}}\frac{6_2^{(r+2)}}{(r+2)^{(2)}}\Big|_2^6=100.$$

The following theorem is the relation between Discrete Beta function and Discrete Gamma Factorial function.

Theorem: 3.21 Let $m,n\in\mathbb{N}(1)$ *and* $0<\ell<k\le a$, *then*

$$\begin{aligned}\ell\Delta_\ell^{-1}{}_k\beta_\ell(m,n,a)e^{-k}\Big|_k^\infty &=\ell\sum_{r=0}^{\infty}{}_{(k+r\ell)}\beta_\ell(m,n,a)e^{-(k+r\ell)}\\ &=\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}(r+1)}\times\left[{}_k\Gamma_\ell((r+2))+(j)_\ell^{(r+1)}\frac{\ell e^{-k}}{e^{-\ell}-1}\right].\end{aligned} \tag{27}$$

Proof: By theorem 3.10, we have

$${}_k\beta_\ell(m,n,a)=\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}(r+1)}\left[(k)_\ell^{(r+1)}-(j)_\ell^{(r+1)}\right]$$

$$\ell\Delta_\ell^{-1}{}_k\beta_\ell(m,n,a)e^{-k}\Big|_k^\infty=\sum_{i=0}^{n-1}\sum_{r=0}^{m+n-2-i}\frac{(n-1)^{(i)}a^i}{(-1)^{n-1+i}i!}\frac{S_r^{m+n-2-i}}{\ell^{i+r+2-m-n}(r+1)}\times\left[\ell\Delta_\ell^{-1}(k)_\ell^{(r+1)}e^{-k}-(j)_\ell^{(r+1)}\ell\Delta_\ell^{-1}e^{-k}\right]$$

The proof follows from equations (11), (12) and (15).

Remark: 3.22 $\lim\limits_{\ell\to0}\dfrac{\ell e^{-\ell}}{1-e^{-\ell}}=1$ *and* $\lim\limits_{\ell\to0}{}_k\Gamma_\ell((n))={}_k\Gamma(n)$.

The following corollary is the relation between Beta function and Gamma function.

Corollary: 2.23 $\displaystyle\int_{s=k}^{\infty}\left(\int_{t=0}^{s}{}_t\beta(m,n)dt\right)e^{-s}ds={}_k\Gamma(m+1)$

Proof: The proof follows by taking $\ell\to0$ on theorem 3.21 and using the remark 3.22.

Corollary: 3.24 $\displaystyle\int_{k=0}^{\infty}\left(\int_{t=0}^{k}{}_t\beta(m,n)dt\right)e^{-k}dk=\sum_{r=0}^{n-1}(-1)^r(m-n+r+1)!(n-1)^{(r)}$

Proof: The proof follows by taking $\ell\to0$ and then putting $k=0$ in theorem 3.21.

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