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GENERALIZED LEFT-DERIVATION ON SEMIRINGS

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1. INTRODUCTION

A semiring is an algebric system (S, +,·) such that (S,+) and (S, ·) are semigroups such that the two distributive laws hold. It has been formally introduced by H. S. Vandiver in 1934[6]. A natural example of a semiring is the set of natural numbers under usual addition and multiplication. The notion of derivations on rings is quite old and is useful in the study of integration of analysis, algebraic geometry and algebra. The study of derivation in rings, got interested only after Posner who established two striking results on derivation in prime rings [5]

In [3] Jonathan S. Golan in his book introduced the notion of derivations in semiring, but nothing has been discussed in detail. This motivated Chandramouleeswaran and Thiruveni to study the notion of derivations on semirings [2]. In 1990 Braser and Vukman [1] introduced the notion of left- derivation in a ring. Mohammed Ashraf and Shakir Ali introduced the notion of generalized Jordan left derivation in rings[4]. Motivated by this, in this paper, we introduce the notion of generalized left derivation on semirings and prove some simple but interesting results.

2. PRELIMINARIES

In this section we recall some basic definitions on semirings and derivations on it that are needed for our work.

Definition 2.1: A semiring (S, +, ·) is an algebraic system with a nonempty set S together with two binary operations '+' and '·' such that
(i) (S, +) is a semi group
(ii) (S, ·) is a semi group
(iii)For all a, b, c $\in$ S, $a\cdot(b + c) = a\cdot b + a\cdot c$ and $(a +b)\cdot c = a\cdot c + b\cdot c$ hold.

Definition 2.2: A semiring (S, +, ·) is said to be additively commutative if (S , +) is a commutative semi group. A semiring (S, +, ·) is said to be multiplicatively commutative if (S , ·) is a commutative semi group. It is said to be commutative if both (S, +) and (S, ·) are commutative.

Definition 2.3: The semiring (S, +, .) is said to be a semiring with zero, if it has an element 0 in S such that $x + 0 = x = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x \quad \forall x \in S$

Definition 2.4: A semiring (S, +, ·) is said to be a semiring with an identity element 1, if there exists an element $1\neq 0\in S$ such that $1\cdot x = x = x\cdot 1 \ \forall x \in S$.

Definition 2.5: Let (S, +, ·) be a semiring. An element α of S is called additively left cancellative if for all $\alpha, \beta, \gamma \in S$

$$\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$$

If every element of a semiring S is additively left cancellative, it is called an additively left cancellative semiring.

Analogously, we define an additively right cancellative semiring.

Definition 2.6: A semiring (S, +, ·) is additively cancellative if it both additively left and right cancellative.

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Definition 2.7: Let (S, +, ·) be a semiring. A derivation on S is a map d: S $\rightarrow$X satisfying the following conditions
(i) $d(x + y) = d(x) + d(y)\ \forall\ x, y \in S$
(ii) $d(xy) = d(x)y + xd(y)\ \forall\ x, y \in S$

Example 2.8: Let S be a semiring. $M_2(S) = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \Big/ a,b,c \in S \right\}$ Define d: $M_2(S) \rightarrow M_2(S)$ is given by

$d\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$. Then d is derivation on $M_2(S)$.

Definition 2.9: Let S be a semiring with centre Z(S)
- S is said to be prime if $asb = 0 \Rightarrow a = 0$ or $b = 0$
- S is said to be semiprime if $asa = 0 \Rightarrow a = 0$
- S is said to be 2 – torsion free if $2a = 0,\ a \in S \Rightarrow a = 0$

Definition 2.10: Let S be a semiring. A left S – module is a commutative monoid (M, +, 0_M) in which scalar multiplication S X M $\rightarrow$M, denoted by (s, m) $\rightarrow$ sm, satisfies rhe following conditions
(i) (ss')m = s(s′m)
(ii) s(m+m') = sm + sm'
(iii) (s + s')m = sm + s'm
(iv) $1_S m = m$
(v) $_S0_M = 0_M = 0_S m.\ \forall s, s' \in S\ \&\ m, m' \in M$

If V(M) = M then M is an S – module where V(M) is the set of all elements of M having additive inverse.

Definition 2.11: Let S be a semiring, X be a left S module. An additive map d_L: S $\rightarrow$ X is said to be a left derivation on S, $d_L(xy) = xd_L(y) + y\, d_L(x)\quad \forall x, y \in S$.

Definition 2.12: Let S be a semiring, X be right S module. An additive map d_R: S$\rightarrow$ X is said to be a right-derivation, if $d_R(xy) = d_R(x)y + d_R(y)x\quad \forall x, y \in S$.

3. GENERALIZED LEFT-DERIVATION

Definition 3.1: Let S be a semiring, X be a left S module and d_L be a left derivation on S. An additive map
D_L: S $\rightarrow$X is said to be a generalized left derivation on S, $D_L(xy) = xd_L(y) + yD_L(x)$
if $d_L(xy) = xd_L(y) + yd_L(x)\quad \forall\ x, y \in S$.

Examples 3.2: Let S be a additively commutative semiring with charactric 2. Consider
$M_2(s) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} / a,b \in S \right\}$. Then $M_2(S)$ is a semiring. Define $D_L : M_2(S) \rightarrow M_2(S)$ is given by
$D_L\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ where d_L:$M_2(S)\rightarrow M_2(S)$ is given by $d_L \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$

Then D_L is a generalized left derivation on S.

Definition 3.3: Let S be a semiring, X be a right S module and d_R be a right-derivation on S. An additive map
D_R: S $\rightarrow$X is said to be a generalized right-derivation on S, $D_R(xy) = D_R(x)y + d_R(y)x$

If $d_R(xy) = d_R(x)y + d_R(y)x\quad \forall\ x, y \in S$.

Examples 3.4: Let S be a additively commutative semiring with charactric 2. Consider
$M_2(s) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} / a,b \in S \right\}$ then $M_3(S)$ is a semiring. Define $D_R : M_2(S) \rightarrow M_2(S)$ is given by

$D_R\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ Where d_R:$M_2(S) \rightarrow M_2(S)$ is given by $d_R \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$

Then D_R is a generalized right derivation on S.

Theorem 3.5: Sum of two generalized left-derivation on a additively commutative semiring is again a generalized left-derivation.

Proof:
$(D_L+F_L)(x+y) = D_L(x + y) + F_L(x + y)$

$= D_L(x) +D_L(y) + F_L(x) +F_L(Y)$

$= (D_L +F_L) (x) +(D_L + F_L)(y)$

$(D_L + F_L)(xy) = D_L(xy) +F_L(xy)$

$= xd_L(y) + yD_L(x) + xf_L(y) +yF_L(x)$

$= x(d_L(y) +f_L(y)) + y(D_L(x) + F_L(x))$

Thus sum of two generalized left-derivation is again a generalized left-derivation.

Analogously, we can proved that the Sum of two generalized right-derivation on a additively commutative semiring is again a generalized right-derivation.

Theorem 3.6: Let S be semiring and D_L: S $\rightarrow$X be a generalized left derivation. For any element a $\in$ S,

$D_L(a^n) = (n – 1) a^{n-1} d_L(a) + a^{n-1} D_L(a)$

Proof: We prove this theorem by induction hypothesis.

For n = 2,

$D_L(a^2) = D_L(a.a)$

$= a\, d_L(a) + a\, D_L(a)$

For n = 3

$D_L(a^3) = D_L(a^2.a)$

$= a^2d_L(a) + a\, D_L(a^2)$

$= a^2 d_L(a) + a(a\, d_L(a) + a\, D_L(a))$

$= 2a^2d_L(a) + a^2D_L(a)$

Assumed that $D_L(a^n) = (n – 1) a^{n-1}d_L(a) + a^{n-1} D_L(a)$

To prove that $D_L(a^{n+1}) = n\, a^n d_L(a) + a^n D_L(a)$

$D_L(a^{n+1}) = D_L(a^n.a)$

$= a^n d_L(a) + a\, D_L(a^n)$

$= a^nd_L(a) + a ((n – 1) a^{n-1}d_L(a) + a^{n-1} D_L(a))$

$= a^nd_L(a) + a(n-1)a^{n-1}d_L(a) + a^nD_L(a)$

$= na^nd_L(a) + a^nD_L(a)$
Thus $D_L(a^n) = (n – 1) a^{n-1}d_L(a) + a^{n-1} D_L(a)$.

Analogously, we can proved that, Let S be semiring and D_R: S →X be a generalized right derivation. For any element a ∈ S,

$D_R(a^n) = D_R(a)\, a^{n-1} + (n-1)d_R(a)a^{n-1}$.

Theorem 3.7: Let S be a 2 torsion free semiring, X be a S module and D_L: S→X be a generalized left-derivation with associated left-derivation d_L :S→X then
(1) $D_L(xy + yx) = xd_L(y) + y D_L(x) + yd_L(x) + xD_L(y)$
(2) $D_L(xyx) = x^2d_L(y) +2xyd_L(x) + xy D_L(x) - yx d_L(x)$
(3) $D_L(xyz + zyx) = xy D_L(z) +zyD_L(x) +2xyd_L(z) + 2 zyd_L(x) + 2xzd_L(y) – yzd_L(x) – yxd_L(z)$

Proof:
$$D_L(xy + yx) = D_L(xy) + D_L(yx) = xd_L(y) + yD_L(x) + yd_L(x) + xD_L(y) \quad (1)$$

Replace by y by xy + yx in (1)

$$D_L(x(xy + yx) + (xy + yx)x) = D_L(x^2y +xyx + xyx + yx^2)$$
$$= D_L(x^2y) + 2 D_L(xyx) + D_L(yx^2)$$
$$= x^2d_L(y) + yD_L(x^2) + 2 D_L(xyx) + y\, d_L(x^2) + x^2D_L(y)$$
$$= x^2d_L(y) + y(xd_L(x) + xD_L(x)) + 2 D_L(xyx) + 2yxd_L(x) + x^2D_L(y)$$
$$= x^2d_L(y) +3yxd_L(x) + 2D_L(xyx) + yxD_L(x) + x^2 D_L(y) \quad (2)$$

$$D_L(x(xy + yx) + (xy + yx)x) = D_L(x(xy + yx)) + D_L((xy + yx)x)$$
$$= xd_L(xy + yx) +(xy +yx) D_L(x) + (xy + yx)d_L(x) + x D_L(xy + yx)$$
$$= x[xd_L(y) + y\, d_L(x) + y\, d_L(x) + x\, d_L(y)] + xyD_L(x)+ yxD_L(x) + xy\, d_L(x) + yxd_L(x) + x\, [xd_L(y) + yD_L(x) + yd_L(x) + x\, D_L(y)]$$
$$= 2x^2d_L(y) + 2xyd_L(x) + xyD_L(x) + yxD_L(x) + xy\, d_L(x) + yxd_L(x) + x^2d_L(y) + xyD_L(x) +xyd_L(x) + x^2D_L(y)$$
$$= 3x^2d_L(y) + 4xyd_L(x) + 2xyD_L(x) + yxD_L(x) + yx\, d_L(x) + x^2 D_L(y) \quad (3)$$

From (2) and (3)

$$x^2d_L(y) +3yxd_L(x) + 2D_L(xyx) + yxD_L(x) + x^2 D_L(y) = 3x^2d_L(y) + 4xyd_L(x) + 2xyD_L(x) + yxD_L(x) + yx\, d_L(x) + x^2 D_L(y)$$

$$2D_L(xyx) = 2x^2d_L(y) – 2yxd_L(x) +4xyd_L(x) +2xyD_L(x)$$

$$D_L(xyx) = x^2d_L(y) – yxd_L(x) + 2xyd_L(x) + xyD_L(x) \quad (4)$$

Replace x to x + z in (4)

$$D_L((x + z)y(x + z)) = D_L((xy + zy)(x + z))$$
$$= D_L (xyx + xyz +zyx +zyz)$$
$$= D_L(xyx) + D_L(xyz +zyx) + D_L(zyz)$$
$$= x^2d_L(y) – yxd_L(x) + 2xyd_L(x) +xyD_L(x) +D_L(xyz +zyx) + z^2d_L(y) – yzd_L(z) +2zyd_L(z) +zyD_L(z) \quad (5)$$

$$D_L((x + z)y(x + z)) = (x + z) yD_L(x +z) + 2(x +z)yd_L(x + z) + (x + z)^2 d_L(y) – y(x + z)\, d_L(x +z)$$
$$= (xy + zy) (D_L(x) + D_L(z)) + (2xy + 2zy)\, d_L(x + z) + (x^2 + 2xz + z^2)\, d_L(y) - yx\, d_L(x) – yz\, d_L(x) – yxd_L(z) – yz\, d_L(z)$$
$$= xyD_L(x) + xy\, D_L(z) + zy\, D_L(x) + zyD_L(z) + 2xy\, d_L(x)+ 2xy\, d_L(z) +2zyd_L(x) +2zyd_L(z) + x^2d_L(y)+ 2xzd_L(y)+ z^2d_L(y) – yx\, d_L(x) – yzd_L(x) – yxd_L(z) – yzd_L(z) \quad (6)$$

From (5) and (6)

$$x^2 d_L(y) - yxd_L(x) + 2xyd_L(x) + xyD_L(x) + D_L(xyz + zyx) + z^2 d_L(y) - yzd_L(z) + 2zyd_L(z) + zyD_L(z)$$
$$= xyD_L(x) + xy\,D_L(z) + zy\,D_L(x) + zyD_L(z) + 2xy\,d_L(x) + 2xy\,d_L(z) + 2zyd_L(x) + 2zyd_L(z) + x^2 d_L(y)$$
$$+ 2xzd_L(y) + z^2 d_L(y) - yx\,d_L(x) - yzd_L(x) - yxd_L(z) - yzd_L(z)$$

$$D_L(xyz + zyx) = xy\,D_L(z) + zyD_L(x) + 2xyd_L(z) + 2\,zyd_L(x) + 2xzd_L(y) - yzd_L(x) - yxd_L(z)$$

Analogously, we can prove the following theorem.

Theorem 3.8: Let S be a 2 torsion free semiring, X be a S module and D_R: S $\to$X be generalized right derivation with associated with right-derivation d_R:S$\to$X such that,

1) $D_R(xy + yx) = D_R(x)y + d_R(y)\,x + D_R(y)\,x + d_R(x)\,y$
2) $D_R(xyx) = D_R(x)\,yx + d_R(y)x^2 + 2d_R(x)yx - d_R(x)xy$
3) $D_R(xyz + zyx) = D_R(x)yz + D_R(z)yx + 2\,d_R(y)xz + 2d_R(x)yz + 2d_R(z)yx - d_R(x)zy - d_R(z)xy$

Theorem 3.9: Let S be a 2 torsion free prime semiring and X be a S module. If S admits a generalized left-derivation with associated left-derivation d_L, then either $d_L = 0$ or S is commutative.

Proof: Let D_L: S$\to$X be a generalized left-derivation with associated left-derivation d_L:S$\to$X
Then for any x, y $\in$ S

$$D_L(yx^2) = yd_L(x^2) + x^2\,D_L(y) = 2yxd_L(x) + x^2\,D_L(y) \qquad \to(1)$$

$$D_L(yx^2) = D_L((yx)x) = yxd_L(x) + xD_L(yx) = yxd_L(x) + xyd_L(x) + x^2 D_L(y) \quad \to(2)$$

From (1) and (2)

$$2yxd_L(x) + x^2\,D_L(y) = yxd_L(x) + xyd_L(x) + x^2 D_L(y)$$

$$yxd_L(x) = xyd_L(x) \Rightarrow [x, y]\,d_L(x) = 0 \quad \forall\, x, y \in S \qquad \to(3)$$

Replace y by yx in (3)

$[x, yx]\,d_L(x) = 0$

$(y[x, x] + [\,x, y\,]x)d_L(x) = 0$

$\Rightarrow [x, y]\,xd_L(x) = 0 \quad \forall\, x, y \in S$

$\Rightarrow [x, y]\,Sd_L(x) = 0$

$\Rightarrow [x, y] = 0$ or $d_L = 0$ (Since S is prime semiring)

$\Rightarrow$S is commutative or $d_L = 0$

Analogously we can prove the following theorem.

Theorem 3.10: Let S be a 2 torsion free prime semiring and X be a S module. If S admits a generalized right-derivation with associated right-derivation d_R, then either $d_R = 0$ or S is commutative.

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