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STRUCTURE OF CERTAIN CLASSES OF SEMIRINGS

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ABSTRACT

*In this paper, we study the properties of semirings satisfying the identity **a + ab = a**. It is proved that, if (S, +, .) is a zerosumfree Semiring with additive identity 0, then a + ab = a, for all a, b in S if and only if ab = 0.*

Keywords: *Viterbi Semiring, E- inverse semigroup, zerosumfree semiring, left regular semigroup.*

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1. INTRODUCTION

The theory of rings and the theory of semigroups have considerable impact on the developments of theory of semirings. During the last three decades, there is considerable impact of semigroup theory and semiring theory. The study of rings, which are special semirings shows that the multiplicative structures are quite independent though their additive structures are abelian groups. However, in semirings it is possible to derive the additive structures from their special multiplicative structures. S.Gosh studied on the class of idempotent Semirings. He proved that an idempotent commutative semiring S is distributive lattice. If and only if it satisfies the absorption equality **a + ab = a** for all a, b in S.

2. PRELIMINARIES

A triple (S, +, .) is called a semiring if (S, +) is a semigroup; (S, .) is semigroup; a (b + c) = ab + ac and (b + c) a = ba + ca for every a, b, c in S. (S, +) is said to be band if a + a = a for all a in S. A (S, +) semigroup is said to be rectangular band if a + b + a = a for all a, b in S. A semigroup (S, .) is said to be a band if $a = a^2$ for all a in S. A semigroup (S, .) is said to be rectangular band if aba = a. A semiring (S, +, .) is said to be Mono semiring if a + b = ab for all a, b in S.

Definition 2.1: A semigroup (S, .) is said to be left (right) singular if ab = a (ab = b) for all a, b in S.

Definition 2.2: A semigroup (S, +) is said to be left (right) singular if a + b = a (a + b = b) for all a, b in S.

Theorem 2.3: Let (S, +, .) be a semiring satisfying the identity a + ab = a for all a, b in S. Then the following hold.

(i) If (S, .) is a band, then (S, +) is a band. Converse is also true if (S, +) is left cancellative.

(ii) If (S, .) is rectangular band and (S, +) is commutative then (S, .) is a band. Converse is true if (S, +) is left cancellative.

Proof: **(i)** a + ab = a for all a, b in S taking b = a

$$\Rightarrow a + a.\,a = a \;\Rightarrow\; a + a^2 = a \;\Rightarrow\; a + a = a \quad (\because\ (S, .) \text{ is a band})$$

$$\Rightarrow a + a = a \text{ for all a, b in S}$$

Hence (S, +) is a band

Conversely, we have to prove that (S, .) is a band

Consider a + ab = a for all a, b in S.

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Clearly, $a + a.a = a \Rightarrow a + a^2 = a \Rightarrow a + a^2 = a + a$ ($\because$ (S, +) is a band)

$\Rightarrow a^2 = a$ ($\because$ (S, +) is left cancellative)

Hence (S, .) is a b and

(ii) If (S, .) is rectangular b and

$a + ab = a \Rightarrow a^2 + aba = a^2$

$\Rightarrow a^2 + a = a^2$ (1)

And $a + a.a = a \Rightarrow a + a^2 = a$ (2)

(S, +) is commutative $\Rightarrow a = a^2$

i.e (S, .) is band

Again from (1) we get $a + a = a$ (3)

Conversely, $a + ab = a$

$\Rightarrow a^2 + aba = a^2 \Rightarrow a + aba = a$ ($\because$ (S, .) is band)

$\Rightarrow a + aba = a + a$ ($\because$ from (3))

Since, S is left cancellative

$\therefore$ $aba = a$

Theorem 2.4: If (S, +, .) is a semiring satisfying the identity a + ab = a for all a, b in S and (S, .) is left singular, then (S, +) is a band.

Proof: Consider $a + ab = a$ for all a, b in S (1)

Since (S, .) is left singular implies $ab = a$

$a + a = a$

$\therefore$ (S, +) is a b and

Example 2.5:

+	a	2a
a	a	a
2a	a	2a

.	a	2a
a	a	a
2a	2a	2a

Theorem 2.6: If (S, +, .) is a semiring satisfying the identity a + ab = a for all a, b in S and (S, +) is a right singular semigroup, then (S, +) is a rectangular band.

Proof: By hypothesis $a + ab = a$, for all a, b in S (1)

$\Rightarrow a + ab + b = a + b$

$\Rightarrow a + ab + b = b$ ($\because$ (S, +) is a right singular)

$\Rightarrow a + ab + b + a = b + a$

$\Rightarrow a + ab + b + a = a$ ($\because$ (S, +) is a right singular)

$\Rightarrow a + b + a = a$ ($\because$ (from (1))

which proves the theorem

i.e, (S, +) is a rectangular band

Definition 2.7: A semiring (S, +, .) is said to be zero square semiring if $x^2 = 0$ for all x in S.

Theorem 2.8: If (S, +, .) is a zero square semiring where 0 is the additive identity and S satisfies the identity a + ab = a for all a, b in S, then aba = 0 and bab = 0.

Proof: Consider a + ab = a for all a, b in S

$\Rightarrow a^2 + aba = a^2$

$\Rightarrow 0 + aba = 0$ ($\because$ S is a zero square semiring, $a^2 = 0$)

$\Rightarrow aba = 0$

Also, b + ba = b for all b, a in S

$\Rightarrow b^2 + bab = b^2$

$\Rightarrow 0 + bab = 0$ ($\because$ S is a zero square semiring, $b^2 = 0$)

$\Rightarrow bab = 0$

$\therefore$ aba = 0 and bab = 0

Definition 2.9: An element 'a' of 'S' is called E - inverse if there is an element 'x' of S such that ax + ax = ax,

i.e $ax \in E(S)$, where E (S) is the set of all idempotent elements of S.

→ A semigroup 'S' is called an E - inverse semigroup if every element of S is an E- inverse.

Definition 2.10: A semigroup (S, +) is said to be left regular if aba = ab.

Theorem 2.11: Let (S, +, .) be a Semiring satisfying the identity a + ab = a for all a, b in S.
(i) If (S, .) is left regular semigroup and (S, .) is commutative then S is an E – inverse semigroup.
(ii) If (S, .) is band, then S is an E – inverse semigroup.

Proof: **(i)** Consider a + ab = a for all a, b in S

$\Rightarrow (a + ab)\, b = ab \Rightarrow ab + ab^2 = ab \Rightarrow aba + ab^2a = aba$

$\Rightarrow ab + a.bb.a = ab$ ($\because$ S is left regular)

$\Rightarrow ab + (bab)\, a = ab$ ((S, .) is commutative)

$\Rightarrow ab + baa = ab \Rightarrow ab + aba = ab$ ((S, .) is commutative)

$ab + ab = ab$ ($\because$ S is left regular)

$\therefore$ S is an E – inverse semigroup

(ii) Consider a + ab = a for all a , b in S

$\Rightarrow (a + ab)\, b = ab \Rightarrow ab + ab^2 = ab$

$\Rightarrow ab + ab = ab$ ((S, .) is band)

$\therefore$ S is an E – inverse semigroup

Definition 2.12: A viterbi semiring is a semiring in which S is additively idempotent and multiplicatively subidempotent. i.e., a + a = a and $a + a^2 = a$ for all a in S.

Theorem 2.13: Let (S, +, .) be a semiring satisfying the identity a+ab=a for all a,b in S. If S contains the multiplicative identity 1, then S is viterbi semiring.

Proof: Consider $a + ab = a$ for all a, b in S

$\Rightarrow a + a.1 = a \Rightarrow a + a.1 = a$ for all a, 1 in S

$$\Rightarrow a + a = a \quad (1)$$

Again, $a + a^2 = a$ for all a in S (2)

From (1) & (2) S is viterbi semiring.

Note 2.14: If S is the semiring satisfying the identity $a + ab = a$ for all a, b in S. Then it reduces to multiplicatively sub idempotent semiring i.e $a + a^2 = a$ for all a in S.

Definition 2.15: A semiring (S, +, .) with additive identity zero is said to be zero sum free semiring if $x + x = 0$ for all x in S.

Theorem 2.16: If (S, +, .) is a zerosumfree Semiring with additive identity 0. Then $a + ab = a$, for all a, b in S If and only if $ab = 0$

Proof: Consider $a + ab = a$ for all a, b in S

$\Rightarrow a + a + ab = a + a$

$\Rightarrow ab = 0$ ($\because$ S is zerosumfree semiring, $a + a = 0$)

conversely, $ab = 0$ for all a, b in S

$\Rightarrow a + ab = a + 0 \Rightarrow a + ab = a$

$\Rightarrow a + ab = a$, for all a, b in S

Definition 2.17: A semiring (S, +) is said to be Additively Idempotent Semiring if $a + a = a$ for all a in S.

Theorem 2.18: If a, b, c and d are elements of an additively idempotent semiring S satisfying $a + c = b$ and $b + d = a$ and (S, +) is commutative, then $a = b$.

Proof: If S is additively Idempotent Semiring, i.e , $a = a + a$

Now, $a = a + b + d$ ($\because a = b + d$)

$= a + a + c + d$ ($\because b = a + c$)

$= a + c + d$ ($\because a = a + a$)

$= b + d + c + d$ ($\because a = b + d$)

$= b + d + d + c$ ($\because$ (S, +) is Commutative)

$= b + d + c$ ($\because d = d + d$)

$= a + c$

$= b$ ($\because b = a + c$)

Definition 2.19: A semiring S is said to be Positive Rational Domain (PRD) if and only if (S, .) is an abelian group.

Theorem 2.20: Let (S, +, .) be a PRD satisfying the identity $a + ab = a$ for all a, b in S. Then $a + b = a$, for all a in S.

Proof: $a + ab = a$, for all a in S (1)

$\Rightarrow$ $a + aa^{-1} = a$, for all a, a^{-1} in S

$a + 1 = a$, for all a in S ($\because$ S is PRD) (2)

$\Rightarrow ab + b = ab \quad \Rightarrow a + ab + b = a + ab$

$\Rightarrow a + b = a$ ($\because$ from (1))

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