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FIXED POINT THEOREMS AND CONE BANACH SPACES

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ABSTRACT

In this paper we define Pseudo cone metric space and then prove some extensions of fixed point theorems to cone Banach spaces.

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1. INTRODUCTION

The concept of cone metric spaces has been introduced by Haung and Zhang [6] in 2007 by replacing real numbers with an ordering Banach space. Haung and Zhang gave an example of a function which is a contraction in the category of cone metric spaces but not contraction if considered over metric spaces and hence proving a fixed point theorem in cone metric spaces having a unique fixed point. Some of the articles dealt with the extension of certain fixed point theorems of cone metric spaces ([1], [2]) and some other with the structure of the spaces themselves ([1], [2], [3]). Recently some results on fixed point theorems have been extended to cone Banach spaces ([3]). Here we will define Pseudo cone metric spaces. Some known results ([4]) are extended to cone Banach spaces where the existence of fixed points for self mappings on cone Banach spaces are investigated.

We will represent a real Banach space by $E = (E, \|.\|)$ in this paper. Let E have a closed non-empty subset $P = P_E$. Then P is called Cone if $ax+by \in P$ for all $x, y \in P$ and non-negative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq \{0\}$.

We define a partial ordering (represented by $\leq$ or $\leq_P$) with respect to a given cone P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$, whereas $x \ll y$ will show that $y - x \in$ int.P where int. P denotes the interior of P . In this paper we assume that int.P $\neq \emptyset$

The cone P is called (N) normal if there exists a number $K \geq 1$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K \|y\| \tag{1.1}$$

and (R) regular if every bounded above increasing sequence is convergent. In other words, if $\{x_n\}$, $n \geq 1$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there exists an element $x \in \mathbf{E}$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$.

In (N), the least positive integer K, satisfying (1.1) is called the normal constant of P. We have the following observation.

Lemma 1: [3] (i) Every regular cone is normal, (ii) There is a normal cone with normal constant $K > l$ for $l > 1$ and (iii) If every bounded below decreasing sequence is convergent then the cone P is regular.

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Now we have the following definitions.

Definition 1: [3] Suppose **X** be a non-empty set and a mapping d:**X**× **X** →**E** satisfies the following four properties:

(A_1) d (x, y) ≥ 0 ∀ x,y ∈ **X**

(A_2) d (x, y) = 0 ⟺ x = y

(A_3) d (x, y) ≤ d (x, z) + d (z, y) for all x, y, z ∈ **X**

(A_4) d (x, y) = d(y, x) for all x, y ∈ **X**

Then the mapping d is called cone metric on **X** and (**X**, d) is said to be a cone metric space (briefly CMS).

Definition 2: [3] Suppose **X** be a vector space over **R**. Let the mapping $\|.\|_P$: **X** → **E** satisfies the four properties as:

(B_1) $\|x\|_P$ >0 for all x ∈ **X**

(B_2) $\|x\|_P$ =0 ⟺ x=0

(B_3) $\|x+y\|_P \leq \|x\|_P + \|y\|_P$ for all x, y ∈ **X**

(B_4) $\|lx\|_P$ = $|l|$ $\|x\|_P$ for all $l \in \boldsymbol{R}$, then the mapping $\|.\|_P$ is called cone norm on **X** and (**X** , $\|.\|_P$) is said to be a cone normed space (briefly CNS). Here we observe that every CNS is CMS. of course d (x, y) = $\|x-y\|_P$

Definition 3: Suppose (**X**, $\|.\|_P$) be a cone normed space and x ∈ **X**. Let $\{x_n\}$, n≥ 1 be a sequence in **.** then

(i) the sequence $\{x_n\}$, n≥ 1 converges to x whenever for every c∈**E** with 0< c there is a natural number N such that $\|x_n - x\|_P <$ c for all n ≥ N. We write it as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as n → ∞.

(ii) the sequence $\{x_n\}$, n≥ 1 is a Cauchy sequence whenever for every c ∈ $\boldsymbol{E}$ with 0< c there is a natural number N such that $\|x_n - x_m\|_P <$ c ∀ n, m ≥ N.

(iii) (**X**, $\|.\|_P$) is a complete cone normed space if every Cauchy sequence is convergent.

Complete cone normed spaces (briefly CCNS) are referred to as cone Banach spaces. We have three well known results given in Lemmas 2, 3 and 4 below.

Lemma 2: [3] Suppose (**X**, $\|.\|_P$) be a cone normed space (CNS) and P be a normal cone with normal constant K and let $\{x_n\}$ be a sequence in **X**, then

(i) $\{x_n\}$ converges to x iff $\|x_n - x\|_P \to 0$ as n → ∞

(ii) $\{x_n\}$ is a Cauchy sequence iff $\|x_n - x_m\|_P \to 0$ as n, m → ∞

(iii) if $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then $\|x_n - y_n\|_P \to \|x-y\|_P$

Lemma 3: [3] Suppose (**X**, $\|.\|_P$) be a cone normed space (CNS) over a cone P in **E**, then

(i) Int.(P)+ Int.(P) ⊆ Int.(P) and λ Int.(P) ⊆ Int.(P), $\lambda > 0$

(ii) If c > 0 then ∃ a $\delta > 0$ such that $\|b\| < \delta$ implies that b $<$ c

(iii) For any given c > 0 and $c_0 > 0$, ∃ $n_o \in N$ such that ${c_0}/{n_o} < c$

(iv) $\{a_n\}$, $\{b_n\}$ are two sequences in **E** s.t. $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$ ∀ n then a ≤ b.

Before mentioning Lemma 4, we have the following well known definition.

Definition 4: [3] P is said to be a minihedral cone if sup{x, y}exists for all x, y∈ **E** and it is said to be strongly minihedral if every subset of **E** which is bounded above has a supremum.

Lemma 4: [3] Every strongly minihedral normal cone is regular.

Finally we define a Pseudo cone metric space as follows.

Definition 5: Consider a non-empty set **X** and a mapping d: **X** × **X** →**E** which satisfies four properties as

(C_1) d (x, y) ≥ **0** for all x, y ∈ **X**

(C_2) d (x, y) = **0** if x = y, for x, y ∈ **X**

(C_3) d (x, y) = d(y, x) for all x, y ∈ **X**

(C_4) d (x, y) ≤ d (x, z) + d (z, y) for all x, y, z ∈ **X**, then d is said to be a Pseudo cone metric on **X** and (**X**, d) is called a Pseudo cone metric space.

2. MAIN RESULTS

Our main purpose in this paper is to prove three Theorems on fixed points.

Theorem 1: Every cone metric space is a Pseudo cone metric space but converse need not be true.

Proof: It is clear from the definitions of a cone metric space and a Pseudo cone metric space that every cone metric space is a Pseudo cone metric space.

To prove that the converse need no t be true, we consider the following example.

Suppose **E**=R^3, P = {(x, y, z) ∈ $\boldsymbol{E}$: x, y, z ≥ 0} and **X** = $\boldsymbol{R}$**.** We define a mapping d: **X** × **X** →**E** by d(x,$\tilde{x}$) = ($\alpha|x^2 - \tilde{x}^2|$,β$|x^2 - \tilde{x}^2|$, $\gamma|x^2 - \tilde{x}^2|$) where α, β, γ are non-zero positive constants. It is very easy to verify that d is a Pseudo cone metric space on **X**.

Now d(x,$\tilde{x}$) = **0** implies that ($\alpha|x^2 - \tilde{x}^2|$, β$|x^2 - \tilde{x}^2|$, $\gamma|x^2 - \tilde{x}^2|$) = **0** which implies $|x^2 - \tilde{x}^2|$ = **0** ⇒ x = $\pm\,\tilde{x}$.

Thus d(x, $\tilde{x}$) = **0** does not necessarily imply that x =$\tilde{x}$. Hence (**X**, d) is a Pseudo cone metric space but not cone metric space. Therefore, the result follows.

Theorem 2: Suppose C be a closed convex subset of a cone Banach space **X** with the norm$\|x\|_P$ = d(x,0) and T: C → C be a mapping which satisfies the condition

d(Tx, Ty)+ l [d(x,Tx)+ d(y,Ty)] ≤ a d(x,Ty)+ b d(y,Tx)+ c d(x,y) (2.1)

$for\ all\ x, y \in$ C, where a, b, c, $l \geq 0$ and 3a+b+c < 1+ 4 l , a+b+c < 1, and 1+ l > b.

Then T has a unique fixed point.

Proof: Suppose x_0 be an arbitrary element in C. We define a sequence $\{x_n\}$ as

$x_{n+1} = \frac{x_n + Tx_n}{2}$, n=0,1,2,3,… (2.2)

Here we notice that

($x_n - Tx_n$) = 2 { $x_n - (\frac{x_n + Tx_n}{2})$} =2$(x_n - x_{n+1})$. (2.3)

This yields that

d(x_n, Tx_n)= $\|x_n - Tx_n\|_P$ = 2 $\|x_n - x_{n+1}\|_P$ = 2 d(x_n, x_{n+1}) (2.4)

and ($x_n - Tx_{n-1}$) = $\frac{x_{n-1} + Tx_{n-1}}{2} - Tx_{n-1} = \frac{x_{n-1} - Tx_{n-1}}{2}$ which implies that

d(x_n, Tx_{n-1})= $\|x_n - Tx_{n-1}\|_P = \frac{1}{2}\|x_{n-1} - Tx_{n-1}\|_P = \frac{1}{2}$ d(x_{n-1}, Tx_{n-1}) (2.5)

Therefore the triangle inequality implies that

d(x_n, Tx_n) ≤ d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n)

or d(x_n, Tx_n) − d(x_n, Tx_{n-1}) ≤ d(Tx_{n-1}, Tx_n)

Hence by (2.4) and (2.5) we get

2 d(x_n, x_{n+1}) − d(x_{n-1}, x_n) ≤ d(Tx_{n-1}, Tx_n) (2.6)

Now replacing x by x_{n-1} and y by x_n in (2.1) and regarding (2.4) and (2.6) we get,

2 d(x_n, x_{n+1}) − d(x_n, x_{n-1}) + l { d(x_{n-1}, Tx_{n-1})+ d(x_n, Tx_n) } ≤ a d(x_{n-1}, Tx_n)+ b d(x_n, Tx_{n-1}) + c d(x_{n-1}, x_n) (2.7)

which further gives

2 d(x_n, x_{n+1}) − d(x_n, x_{n-1}) + l {2 d(x_{n-1}, x_n) +2 d(x_n, x_{n+1})}

≤ a { d(x_{n-1}, x_n)+ d(x_n, Tx_n)} + $\frac{b}{2}$ d(x_{n-1}, Tx_{n-1}) + c d(x_{n-1}, x_n)

which further gives

2 d(x_n, x_{n+1}) − d(x_n, x_{n-1}) + 2l { d(x_{n-1}, x_n) + d(x_n, x_{n+1})} ≤ (a+b+c) d(x_{n-1}, x_n)+2a d(x_n, x_{n+1})

$\Longrightarrow (2 + 2l - 2a)$ d(x_n, x_{n+1}) ≤ $(1 - 2l + a + b + c)$ d(x_{n-1}, x_n)

$\Longrightarrow$ d(x_n, x_{n+1}) ≤ $\frac{(1-2l+a+b+c)}{2(1+l-a)}$ d(x_{n-1}, x_n) .

$\Longrightarrow$ d(x_n, x_{n+1}) ≤ K_1 d(x_{n-1}, x_n) where $K_1 = \frac{(1-2l+a+b+c)}{2(1+l-a)} < 1$ as 3a+b+c $< 1 + 4l$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in C and thus converges to some point z ∈ C.

Now we have by triangle inequality

d(z, Tx_n) ≤ d(z, x_n)+ d(x_n, Tx_n) = d(z, z)+2 d(x_n, x_{n+1})

and by using lemma 2, (iii) we have $Tx_n \to z$ (2.8)

Now considering (2.4) and (2.1), putting x = z and y =x_n, we get

d(Tz, Tx_n) + l {d(z, Tz)+ d(x_n, Tx_n)} ≤ a d(z, Tx_n)+ b d(x_n, Tz) + c d(z, x_n)

which on further simplification gives

$(1 + l - b)$ d$(Tz, z) \leq 0$ which implies that d(Tz, z) = 0, since $(1 + l) > b$.

Therefore, we have $Tz = z$ (2.9)

which shows that the mapping T: C → C has a fixed point z.

Now we prove the uniqueness part of the Theorem.

Uniqueness of point z

If possible, suppose p be another fixed point of T so that Tp = p, then by (2.1) we get

d(Tz, Tp) + l {d(z, Tz) + d(p, Tp) } ≤ a d(z, Tp) + b d(p, Tz) + c d(z, p)

⟹ d(z, p) ≤ a d(z, p) + b d(z, p) + c d(z, p)

⟹ $(1 - a - b - c)$ d$(z, p) \leq 0$

⟹ d(z, p) = 0 (since a + b + c < 1) ⟹ $z = p$ (2.10)

Hence T has a unique fixed point, therefore, the proof of Theorem 2 follows completely.

Now we prove one more result where a mapping T has a unique fixed point.

Theorem 3: Suppose C be a closed convex subset of a cone Banach space **X** with the norm $\|x\|_P$ = d(x, 0) and T: C → C be a mapping which satisfies the condition

d(Tx, Ty) ≤ q_1 d(x, Tx)+ q_2 d(y,Ty)+ q_3 d(x,Ty)+ q_4 d(y,Tx)+ q_5 d(x,y) (2.11)

for all x, y ∈ C where $q_1, q_2, q_3, q_4, q_5 \geq 0$ and $2q_1$+2 q_2+3 q_3+ q_4+ $q_5 < 1$, then T has a unique fixed point.

Proof: Let us again define a sequence $\{x_n\}$ by $x_{n+1} = \frac{x_n + Tx_n}{2}$, n=0,1,2,3,….. Then we have (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) as in the proof of Theorem 2 above. We now claim the inequality (2.11) for x= x_{n-1} , y=x_n implies that

d(Tx_{n-1}, Tx_n) ≤ q_1 d(x_{n-1},Tx_{n-1})+ q_2 d(x_n,Tx_n)+ q_3d(x_{n-1},Tx_n)+ q_4d(x_n,Tx_{n-1}) + q_5 d(x_{n-1},x_n)

which on simplification gives

$(2-2\, q_2 - 2\, q_3)$ d(x_n, x_{n+1})≤ $(2q_1 + + q_3 + q_4 + q_5 + 1)$ d(x_n, x_{n-1}) which further implies that

d(x_n, x_{n+1})≤ $\frac{(2q_1 + + q_3 + q_4 + q_5 + 1)}{(2-2\, q_2 - 2\, q_3)}$ d(x_n, x_{n-1})

⟹ d(x_n, x_{n+1}) ≤ K_2 d(x_n, x_{n-1}) where $K_2 = \frac{(2q_1 + + q_3 + q_4 + q_5 + 1)}{(2-2\, q_2 - 2\, q_3)} < 1$

(Since $2q_1$+2 q_2+3 q_3+ q_4+ $q_5 < 1$).

Therefore $\{x_n\}$ is a Cauchy sequence in C and hence converges to some point z∈ C.

Now we have by triangle inequality that

d$(z, Tx_n) \leq$ d(z, x_n) + d(x_n, Tx_n) = d(z, x_n) +2 d(x_n, x_{n+1}) and by using Lemma 2, (iii)

We have $Tx_n \to z$ (2.12)

Now considering (2.4) and (2.11) putting x = z and y = x_n, we get

d$(Tz, Tx_n) \leq q_1$ d(z,Tz) + q_2 d(x_n,Tx_n) + q_3d(z,Tx_n) + q_4d(x_n,Tz)+ q_5 d(z,x_n)

$\Longrightarrow$d$(Tz, Tx_n) \leq q_1$ d(z,Tz) + 2q_2 d(x_n,x_{n+1}) + q_3d(z,Tx_n) + q_4d(x_n,Tz) + q_5 d(z,x_n)

which further implies that

$(1 - q_1 - q_4)$ d$(Tz, z) \leq 0$ (as $n \to \infty) \Longrightarrow$ d$(Tz, z) = 0$ $(since\ q_1 + q_4 < 1)$

Hence $Tz == z$ (2.13)

which shows that the mapping T: C → C has a fixed point z ∈ C.

Now we show the uniqueness of z in C.

Uniqueness of point z

If possible, suppose q be another fixed point of the mapping T, i.e, Tq = q, then by (2.11) we obtain

d(Tz, Tq) ≤ q_1 d(z, Tz) + q_2 d(q, Tq)+ q_3 d(z, Tq) + q_4 d(q, Tz) + q_5 d(z,q) which further implies that

d(Tz, Tq) ≤ q_1 d(z, z) + q_2 d(q,q)+ q_3 d(z,q) + q_4 d(q,z) + q_5 d(z,q)

$\Longrightarrow$ d(z, q) ≤ (q_3+ q_4+ q_5) d(z,q) $\Longrightarrow$ $1 -$ (q_3+ q_4+ q_5) d(z,q) ≤ 0

(Since (q_3+ q_4+ q_5) < 1). Therefore, we have d(z, q) = 0 and hence z = q .

Thus the mapping T: C → C has a unique fixed point which proves Theorem 3 completely.

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