International Journal of Mathematical Archive-4(8), 2013, 39-48
Available online through www.ijma.info ISSN 2229 – 5046

PORTFOLIO RANKING EFFICIENCY (II) TRUNCATED LÉVY FLIGHT RETURNS

Werner Hürlimann*

Wolters Kluwer Financial Services, Seefeldstrasse 69, CH-8008 Zürich

(Received on: 27-06-13; Revised & Accepted on: 22-07-13)

ABSTRACT

The truncated Lévy flight (TLF) distribution is viewed as a sub-family of the bilateral tempered stable class of distributions and studied. The domain of variation between skewness and excess kurtosis is derived and a full analytical solution of the moment equations is displayed. Application to portfolio selection with CARA utility is considered. With the TLF as test return distribution, it is analyzed whether a recent approximate ranking function with cubic mean-variance-skewness-kurtosis trade-off should be preferred to the original Gaussian ranking function with linear mean-variance trade-off or not. Based on an appropriate ranking efficiency measure and an empirical data analysis, one notes a systematic efficiency increase of the approximate ranking versus the Gaussian ranking. Comparisons with the normal variance gamma (NVG) distribution as test return distribution are included.

Mathematics Subject Classification: *60E15, 62E15, 62P05, 62P20, 91B16, 91G10.*

Keywords: *bilateral tempered stable, truncated Lévy flight, variance gamma, bilateral gamma, portfolio selection, ranking function, efficiency measure.*

1. INTRODUCTION

In recent years, the topic of non-Gaussian distributions has become very prominent. This is partly due to the sub-prime and Euro crises as well as the new regulations in the finance industry like Basel III and Solvency II, for which an efficient and robust modelling of non-normality plays an increasingly important role. Further financial applications to option pricing, risk management and portfolio optimization are numerous and equally well important. Like its companion paper Hürlimann (2013) the purpose of the present contribution is twofold. In the theoretical part, we aim a brief but comprehensive understanding of the truncated Lévy flight (TLF) distribution with regard to the skewness and (excess) kurtosis parameters. In particular, we display their maximum domain of variation and a simple analytical solution of the moment equations. The application part is directly based on the theoretical results. Due to a recent contribution by Di Pierro and Mosevich (2011), moment methods are particularly suited to analyze the portfolio selection problem within Financial Economics. For this, we use equivalent ranking functions and define an appropriate ranking efficiency measure as explained in Hürlimann (2013), Appendix 1. Its practical use enables taking a decision about whether the recent approximate ranking function with cubic mean-variance-skewness-kurtosis trade-off by Di Pierro and Mosevich (2011) should be preferred to the original Gaussian ranking function with linear mean-variance trade-off by Lévy and Markowitz (1979) or not. A more detailed account of the content follows.

Section 2 starts the theoretical part with a brief taxonomy of the bilateral tempered stable distribution (BTS), which is defined as a shifted convolution of two one-sided tempered stable (TS) distributions. Two important members of the BTS are the five parameter truncated Lévy flight (TLF) and its bilateral gamma (TLF-BG) special case, which are used in the application part. The full analytical solution of the moment equations for the TLF-BG and TLF is presented in Theorem 3.1. The application to portfolio selection is presented in Section 4. The investigation of the ranking efficiency measure for the TLF as test return distribution is illustrated at a case study. Some real-world equity return data sets from the Swiss Market and Standard & Poors 500 indices are fitted to the TLF return distribution and their ranking efficiency measures are calculated and compared. For the convenience of the reader, a comparison with the alternative normal variance gamma (NVG) return distribution, which has been extensively studied in Hürlimann (2013), is included. The empirical data analysis shows that the approximate ranking function with cubic mean-variance-skewness-kurtosis trade-off (4.1) should be preferred to the original linear mean-variance trade-off (4.2), at least for the TLF and NVG test return distributions. The numerical evaluation of the goodness-of-fit statistics encountered in the data analysis are done with the fast Fourier transform (FFT) approximation of a distribution with known characteristic function (see the Appendix for a summary of the method).

Corresponding author: *Werner Hürlimann**
Wolters Kluwer Financial Services, Seefeldstrasse 69, CH-8008 Zürich

2. TAXONOMY OF THE BILATERAL TEMPERED STABLE DISTRIBUTION AND SUB-FAMILIES

The *bilateral tempered stable* (BTS) random variable is defined to be a shifted independent difference of two one-sided tempered stable (TS) random variables with stability parameters restricted to the interval $[0,1)$. If $\gamma_1, \gamma_2 \in (0,1)$ the cumulant generating function (cgf) of a seven parameter BTS random variable

$$X = \xi + TS_1 - TS_2 \sim BTS(\xi, \gamma_1, \lambda_1, c_1, \gamma_2, \lambda_2, c_2)$$

takes the form (e.g. Küchler and Tappe (2012), Remark 2.8)

$$C_X(t) = \xi \cdot t - c_1 \cdot \{(\lambda_1 - t)^{\gamma_1} - \lambda_1^{\gamma_1})\} - c_2 \cdot \{(\lambda_2 + t)^{\gamma_2} - \lambda_2^{\gamma_2})\}, \quad -\lambda_2 < t < \lambda_1. \tag{2.1}$$

In general, the parameters can take the following values: $\gamma_1, \gamma_2 \in [0,1)$ (*stability parameters*), $\lambda_1, \lambda_2 > 0$ (*tempering parameters*), $c_1, c_2 > 0$ (*scale parameters*), and $-\infty < \xi < \infty$ (*location parameter*). The *truncated Lévy flight* (TLF) family is identified as the special BTS random variable with identical tempering and stability parameters. Therefore, the cgf of a five parameter TLF random variable $X \sim TLF(\xi, \gamma, \lambda, c_1, c_2)$ is determined by ($\gamma = 0$ identifies with a bilateral gamma special case (TLF-BG) as observed in the Notes 2.1)

$$\begin{aligned} C_X(t) &= \xi \cdot t - c_1 \cdot \{(\lambda - t)^{\gamma} - \lambda^{\gamma})\} - c_2 \cdot \{(\lambda + t)^{\gamma} - \lambda^{\gamma})\}, \quad \gamma \in (0,1), \\ C_X(t) &= \xi \cdot t + c_1 \cdot \ln\{\tfrac{\lambda}{\lambda - t}\} + c_2 \cdot \ln\{\tfrac{\lambda}{\lambda + t}\}, \quad \gamma = 0, \quad -\lambda < t < \lambda. \end{aligned} \tag{2.2}$$

Notes 2.1: The BTS family with cgf (2.1) corresponds to the generalized tempered stable distributions in Rachev et al. (2011). Several sub-families are known from the literature. The case $\gamma_1 = \gamma_2$ defines the so-called Koponen-Boyarchenko-Levendorskii or *KoBoL distribution* first considered in Boyarchenko and Levendorskii (2000). In the special case $\gamma_1 = \gamma_2 = 0$ it identifies with a bilateral gamma distribution, whose general class has been studied in Küchler and Tappe (2008) and Hürlimann (2013). Specializing further to $\gamma_1 = \gamma_2 = 0,\, c_1 = c_2$ one obtains a special instance of the *variance gamma* (VG) distribution, often used in option pricing (e.g. Madan and Seneta (1990), Madan (2001) among many others). The case $\gamma_1 = \gamma_2,\, c_1 = c_2$ is the *CGMY distribution* by Carr et al. (2002), also called classical tempered stable distribution by Rachev et al. (2011). The TLF distibution $\gamma_1 = \gamma_2,\, \lambda_1 = \lambda_2$ defined in (2.2) is another classical BTS sub-family, which has been studied by many authors. According to Imai and Kawai (2011) the class of tempered stable laws was first proposed by Tweedie (1984). The analytical expression for the chf of a TLF distribution was derived in Koponen (1995). The pioneering work of Mantegna and Stanley (1994/2000) established the TLF in econophysics. The origins of TLF are sketched in Figueiredo et al. (2003). Properties of tempered stable laws have been first revealed by Rosinski (2007). Arbitrary truncation of Lévy flights is considered in Vinogradov (2010).

In the present work, the focus is on the TLF. It is viewed as a simple alternative to the NVG studied in Hürlimann (2013). The mean, variance and higher order cumulants of the TLF are obtained from (2.2) and determined by (as usual μ, σ denote the mean and standard deviation)

Case 1: $\gamma = 0$

$$\begin{aligned} &\mu = \xi + (c_1 - c_2) \cdot \lambda^{-1}, \quad \sigma^2 = (c_1 + c_2) \cdot \lambda^{-2}, \\ &\kappa_r = C_X^{(r)}(t=0) = (r-1)! \cdot \left(c_1 + (-1)^r c_2\right) \cdot \lambda^{-r}, \quad r > 2. \end{aligned} \tag{2.3}$$

Case 2: $\gamma \in (0,1)$

$$\begin{aligned} &\mu = \xi + \gamma(c_1 - c_2) \cdot \lambda^{\gamma-1}, \quad \sigma^2 = \gamma(1-\gamma)(c_1 + c_2) \cdot \lambda^{\gamma-2}, \\ &\kappa_r = C_X^{(r)}(t=0) = \gamma \cdot \prod_{i=1}^{r-1}(i-\gamma) \cdot \left(c_1 + (-1)^r c_2\right) \cdot \lambda^{\gamma-r}, \quad r > 2. \end{aligned} \tag{2.4}$$

The BTS and its TLF sub-family satisfy a number of important properties. The BTS is a finite variation process with infinitely many jumps in each interval of positive length (type B in Cont and Tankov (2004), Definition 11.9). Furthermore, the BTS is infinitely divisible, self-decomposable, absolutely continuous, and of class L. The density is smooth (differentiable) of class $C^{\infty}(R)$ and unimodal (e.g. Küchler and Tappe (2012), Theorem 7.8). Explicit analytical expressions for the density exist only for the BG (e.g. Hürlimann (2013), Appendix 4). However, numerical evaluation of BTS and TLF density, distribution function and related VaR and CVaR risk measures, can be performed with the fast Fourier approximation.

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3. SOLVING THE MOMENT EQUATIONS FOR THE TRUNCATED LÉVY FLIGHT DISTRIBUTION

The solution of the moment equations for the truncated Lévy flight depends upon the two cases (2.3) and (2.4). The skewness and (excess) kurtosis parameters are denoted throughout by S, K .

Theorem 3.1: (*TLF moment method*). Given is a feasible skewness and kurtosis pair (S, K) satisfying the inequality $S^2 \le \frac{2-\gamma}{3-\gamma} K$. Then, there exists a unique and explicitly given truncated Lévy flight distribution $TLF(\xi, \gamma, \lambda, c_1, c_2)$, which solves the moment equations up to order four. Its parameters are fully analytical and specified as follows.

Case 1: $\gamma = 0$ (TLF-BG special case)

$$\xi = \mu + (c_2 - c_1)\lambda^{-1}, \quad \lambda = \sigma^{-1}\sqrt{p}, \quad c_1 = \frac{p}{1+q^2}, \quad c_2 = \frac{pq^2}{1+q^2},$$
$$p = \frac{6}{K}, \quad q^2 = \frac{1-\frac{1}{2}S\sqrt{p}}{1+\frac{1}{2}S\sqrt{p}}. \tag{3.1}$$

Case 2: $\gamma \in (0,1)$

$$\xi = \mu + (c_2 - c_1)\gamma \cdot \lambda^{\gamma-1}, \quad \lambda = \sigma^{-1}\sqrt{\frac{(2-\gamma)(3-\gamma)}{K}},$$
$$c_1 = \frac{1}{2}\sigma^2\lambda^{2-\gamma}\,\frac{2-\gamma+\lambda\sigma S}{\gamma(1-\gamma)(2-\gamma)}, \quad c_2 = \frac{1}{2}\sigma^2\lambda^{2-\gamma}\,\frac{2-\gamma-\lambda\sigma S}{\gamma(1-\gamma)(2-\gamma)}. \tag{3.2}$$

Proof: According to (2.3) the equations of variance, skewness and kurtosis in Case 1 read

$$\mu = \xi + (c_1 - c_2)\cdot\lambda^{-1}, \quad \sigma^2 = (c_1 + c_2)\cdot\lambda^{-2},$$
$$S\sigma^3 = 2(c_1 - c_2)\cdot\lambda^{-3}, \quad K\sigma^4 = 6(c_1 + c_2)\cdot\lambda^{-4}.$$

It is convenient to use the one-to-one transformation of parameters

$$p = c_1 + c_2, \quad q^2 = c_2 / c_1, \quad c_1 = \frac{p}{1+q^2}, \quad c_2 = \frac{pq^2}{1+q^2}.$$

Comparing the variance and kurtosis equations one obtains that

$$\lambda = \sigma^{-1}\sqrt{p}, \quad p = \frac{6}{K}.$$

Inserting into the skewness equation one sees that

$$\frac{1-q^2}{1+q^2} = \frac{1}{2}S\sqrt{p}, \text{ or equivalently } q^2 = \frac{1-\frac{1}{2}S\sqrt{p}}{1+\frac{1}{2}S\sqrt{p}}.$$

The value of the location parameter follows from the mean equation. The mathematical analysis in Case 2 is similar. Restating the relevant equations in (2.4) one gets

$$\mu = \xi + \gamma(c_1 - c_2)\cdot\lambda^{\gamma-1}, \quad \sigma^2 = \gamma(1-\gamma)(c_1 + c_2)\cdot\lambda^{\gamma-2},$$
$$S\sigma^3 = \gamma(1-\gamma)(2-\gamma)(c_1 - c_2)\cdot\lambda^{\gamma-3}, \quad K\sigma^4 = \gamma(1-\gamma)(2-\gamma)(3-\gamma)(c_1 + c_2)\cdot\lambda^{\gamma-4}.$$

Comparing the variance and kurtosis equation one gets

$$\lambda^2 = \sigma^{-2}\frac{(2-\gamma)(3-\gamma)}{K}.$$

Multiplying the variance equation by $(2-\gamma)$ and the skewness equation by λ one obtains the system of 2 equations in 2 unknowns:

$$\gamma(1-\gamma)(2-\lambda)(c_1 + c_2)\cdot\lambda^{\gamma-2} = (2-\gamma)\sigma^2,$$
$$\gamma(1-\gamma)(2-\gamma)(c_1 - c_2)\cdot\lambda^{\gamma-2} = \lambda S\sigma^3.$$

Elimination of one parameter (addition and subtraction of the 2 equations) shows the validity of the formulas for c_1, c_2 in (3.2). Again, the value of the location parameter follows from the mean equation. It remains to show the validity of the inequality $S^2 \le \frac{2-\gamma}{3-\gamma} K$ between skewness and kurtosis. The BG special case $\gamma = 0$ follows from Hürlimann (2013), Theorem A2.2. If $\gamma \in (0,1)$ one notes that the parameter restriction $c_2 > 0$ is equivalent with the inequality

$$(\lambda\sigma)^2 S^2 < (2-\gamma)^2.$$

Elimination of $\lambda\sigma$ using the second equation in (3.2) yields the desired inequality. ◊

Remarks 3.1: The inequality between skewness and kurtosis in case $\gamma = 0$ holds actually for the whole BG subfamily of the BTS, as shown in Hürlimann (2013), Theorem A2.2. The generic TLF inequality $S^2 \le \frac{2-\gamma}{3-\gamma} K,\ \gamma \in (0,1)$, is more restricted than the BG inequality. However, the worst inequality $S^2 \le \frac{1}{2} K$ in case $\gamma \to 1$ is enough flexible for modelling purposes, as demonstrated in Section 4. A comparison of skewness and kurtosis boundaries between important families of distributions follows along the line of Hürlimann (2013), Appendix 3.

4. APPLICATION TO THE RANKING EFFICIENCY IN PORTFOLIO SELECTION

Our application to portfolio selection is based on the financial economics ranking efficiency measure defined and motivated in Hürlimann (2013), Appendix 1, Proposition A1.1. The investigation of this ranking efficiency measure for the TLF as test return distribution is illustrated with a case study. Several real-world equity return data sets from the Swiss Market and Standard & Poors 500 indices are fitted to the TLF return distribution and their ranking efficiency measures are calculated and compared. We note a systematic efficiency increase of the approximate ranking versus the Gaussian ranking, which is comparable in size with the observed efficiency increase for the normal variance-gamma (NVG) return distribution used in Hürlimann (2013) (see Table 4.3). This means that the approximate ranking function with cubic mean-variance-skewness-kurtosis trade-off (4.1) should be preferred to the original linear mean-variance trade-off (4.2), at least for the TLF and NVG test return distributions.

Let us recall briefly the definition and aim of the used ranking efficiency measure. Given is a finite set of portfolios, each with its own return distribution $p = p(x)$, and a rational investor with utility function $U(x)$. The *portfolio selection problem* consists to rank portfolios using the expected utility ranking function $R_U(p) = \int_{-\infty}^{\infty} U(x)p(x)dx$, or a function equivalent to it. Two ranking functions R_1 and R_2 are *equivalent*, written $R_1 \sim R_2$, if, and only if, there exists a monotone increasing function $h(x)$ such that $R_2(p) = h(R_1(p))$ for all p. To fix ideas assume a CARA utility function $U_{CARA}(x) = -\exp(-mx)$, also called *exponential utility*. For portfolio selection without risk-free asset, and assuming *finite moments*, Di Pierro and Mosevich (2011) derive through a simple Taylor series expansion the *approximate* ranking equivalence such that

$$R_{U_{CARA}}(p) \sim R_*(p) = -\ln(-R_{U_{CARA}}(p))/m \approx R_*^A(p) = \mu - \frac{m\sigma^2}{2} + \frac{m^2\sigma^3 S}{6} - \frac{m^3\sigma^4 K}{720}, \quad (4.1)$$

where the parameters μ, σ, S, K represent the mean, standard deviation, skewness and excess kurtosis of the portfolio return p, and the approximation error is of order $O(m^4\sigma^5)$. For Gaussian distributed return p^G equation (4.1) reduces to the *exact* ranking function

$$R_*(p^G) = \mu - \frac{m\sigma^2}{2}, \quad (4.2)$$

due to Lévy and Markowitz (1979). It is important to ask whether the approximate ranking function (4.1) with *cubic* mean-variance-skewness-kurtosis trade-off should be preferred to the original ranking function with *linear* mean-variance trade-off (4.2) or not.

To answer this question one examines the efficiency increase/decrease obtained using $R_*^A(p)$ instead of $R_*(p^G)$. For this, let S be an appropriate set of *test* return distributions, whose ranking functions $R_*(p) = -\log(-R_U(p))/m$ can be determined exactly or to an arbitrary level of accuracy for all $p \in S$. A *naive* *approach* to efficiency consists to measure the distance between two portfolio returns p_1 and p_2 through the *ranking distance* $D^*(p_1, p_2) = \left| R_*(p_1) - R_*(p_2) \right|$. A meaningful *ranking efficiency measure*, given a test return $p^S \in S$, is described by the deviation of the distance measures $D^*(p^S, p)$ and $D^*(p^G, p)$, $\forall p \in L$, relative to the distance $D^*(p^G, p)$, in formula

$$E_S^*(p^G, p) = \frac{D^*(p^S, p) - D^*(p^G, p)}{D^*(p^G, p)} = \frac{\left| R_*(p^S) - R_*^A(p) \right|}{\left| R_*(p^G) - R_*^A(p) \right|} - 1, \ \forall p \in L, \ p^S \in S, \tag{4.3}$$

which quantifies the *efficiency increase* (if positive) respectively *decrease* (if negative) of the approximate ranking versus the Gaussian ranking. The efficiency measure (4.3) has been shown to be consistent with a certainty equivalent return methodology that must be considered in financial economics (see Hürlimann (2013), Proposition A1.1).

Computational evaluation of (4.3) requires formulas for the test ranking function $R_*(p^{TLF})$ and the approximate ranking function $R_*^A(p)$ defined in (4.1). For a CARA utility function one has $R_*(p^{TLF}) = -\frac{1}{m} \cdot \ln(-R_U(p^{TLF})) = -\frac{1}{m} C_X(-m)$, which implies by (2.2) the formulas:

Case 1: $\gamma = 0$

$$R_*(p^{TLF-BG}) = \xi - \tfrac{1}{m} \cdot \left(c_1 \cdot \ln\{\tfrac{\lambda}{\lambda+m}\} + c_2 \cdot \ln\{\tfrac{\lambda}{\lambda-m}\} \right) \tag{4.4}$$

Case 2: $\gamma \in (0,1)$

$$R_*(p^{TLF}) = \xi + \tfrac{1}{m} \cdot \left(c_1 \cdot \{(\lambda+m)^\gamma - \lambda^\gamma)\} + c_2 \cdot \{(\lambda-m)^\gamma - \lambda^\gamma)\} \right) \tag{4.5}$$

To illustrate with a case study, we consider some stock market indices. Return observations stem from seven different Swiss Market (SMI) and Standard & Poors 500 (SP500) data sets:

SMI 3Y/1D: 758 historic daily closing prices over 3 years from 04.01.2010 to 28.12.2012
SMI 24Y/1D: 6030 historic daily closing prices over 12 years from 03.01.1989 to 28.12.2012
SMI 24Y/1M: 288 historic end of month prices over 24 years from Jan. 1989 to Dec. 2012
SP500 3Y/1D: 754 historic daily closing prices over 3 years from 04.01.2010 to 31.12.2012
SP500 24Y/1D: 6049 historic daily closing prices over 12 years from 03.01.1989 to 31.12.2012
SP500 24Y/1M: 288 historic end of month prices over 24 years from Jan. 1989 to Dec. 2012
SP500 63Y/1M: 756 historic end of month prices over 63 years from Jan. 1950 to Dec. 2012

These data sets are typical as they contain short to medium high volatile periods (recent 3 years), long term periods (24 years) as well as very long term periods (63 years). The SMI exists only for 24.5 years. Hence, the SMI cannot be compared with the SP500 for longer periods.

The observed sample logarithmic returns of stock-market indices are negatively skewed and have a much higher excess kurtosis than is allowed by a normal distribution, at least over shorter daily and even monthly periods. One observes that the Bera-Jarque test statistic of normality is far beyond the critical value except for the monthly returns over 24 years (see Table 4.2 in Hürlimann (2013)). Therefore, the normal distribution is retained for comparison for the 3 monthly return data sets only. The TLF distribution is fitted to the data following the moment method described in Theorem 3.1. If the empirical counterparts of the domains of variation of the skewness and kurtosis are big enough, a unique solution is obtained, which is the case here.

To do so, the mean, variance, skewness and kurtosis, which are used in the moment method, must be estimated. We use the well-known *k-statistics* of Fisher (1928), which provide unbiased estimates of the cumulants as follows (assume $n > 3$):

$$\begin{gathered} \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} r_i, \quad \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\mu})^2, \quad \hat{\kappa}_3 = \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}(r_i - \hat{\mu})^3, \quad \hat{S} = \hat{\kappa}_3 / \hat{\sigma}^3, \\ \hat{\kappa}_4 = \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{i=1}^{n}(x_i - \hat{\mu})^4 - 3 \cdot \frac{1}{(n-2)(n-3)}\left(\sum_{i=1}^{n}(x_i - \hat{\mu})^2\right)^2, \quad \hat{K} = \hat{\kappa}_4 / \hat{\sigma}^4, \end{gathered} \tag{4.6}$$

where r_i, $i = 1,\ldots,n$, are the sample logarithmic returns (Table 4.3 lists the obtained values).

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The *goodness-of-fit* (GoF) of the chosen estimation method is based on statistics, which measure the difference between the empirical distribution function $F_n(x)$ and the fitted distribution function $F(x)$. We use the *Cramér-von Mises* statistic W^2 and the *Anderson-Darling* statistic A^2. Given the order statistics of the return data such that $r_1 \le r_2 \le ... \le r_n$, the fitted values of the distribution function are $\hat{F}(r_i), i = 1,...,n,$. Then, one has the formulas

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\hat{F}(r_i) - \frac{2i-1}{n} \right)^2, \quad A^2 = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \cdot \ln\left\{ \hat{F}(r_i) \cdot \hat{\overline{F}}(r_{n-i+1}) \right\}. \tag{4.7}$$

The fitted values $\hat{F}(r_i)$ are obtained numerically through application of the FFT approximation formulas in the Appendix with $N = 2^{12}$ disjoint subintervals.

Fitting results are summarized and compared in the Table 4.1 below. Some comments are in order. Except for the SMI 24Y/1M data set, which fits "best" the TLF with $\gamma = 0$, the TLF with $\gamma \in (0,1)$ always provides the smallest GoF statistics. In four cases the "best" fitted TLF is a TLF with $\gamma \to 1$ (SMI 24Y/1D, SP500 24Y/1D, SP500 24Y/1M and SP500 63Y/1M). Even if the normal distribution is not rejected by the Bera-Jarque test, its fit is rather poor compared to the "best" TLF fit. The fitting results are compared with those of the NVG in Hürlimann (2013) and summarized in Table 4.2, where the GoF statistics are calculated with the FFT approximation method. We also list the mode of the TLF. The mode of the NVG is also computed numerically using an analytical convolution formula for the density similar to (A4.28) in Hürlimann (2013). Up to some VG's (SMI 24Y/1D and SP500 24Y/1D) the two calculated modes almost coincide. With the exception of the SP500 24Y/1D and the SP500 3Y/1D (W^2 statistic) the NVG fits better the data than the TLF in terms of the GoF statistics.

Table 4.1: Parameter estimates, mode, and GoF statistics for the TLF family

data set	parameter estimates					mode	FFT GoF statistics	
	γ	ξ	λ	c1	c2	FFT	A²	W²
SMI 3Y/1D	0	0.00227	128.68	0.70219	0.98954	79.98	1.20887	0.21035
	0.68	0.00543	91.931	0.07484	0.10857	55.21	0.33983	0.04511
	0.69	**0.00558**	**91.384**	**0.07218**	**0.10477**	**55.06**	0.33883	**0.04508**
	0.7	0.00574	90.838	0.06970	0.10124	54.92	0.33816	0.04511
	0.71	**0.00592**	**90.291**	**0.06740**	**0.09796**	**54.78**	**0.33782**	0.04520
	0.72	0.00610	89.743	0.06528	0.09493	54.64	0.33783	0.04536
SP500 3Y/1D	0	0.00432	108.54	0.58644	1.02245	69.76	1.85807	0.42951
	0.38	0.00596	91.290	0.31226	0.55681	52.72	1.49247	0.33659
	0.39	**0.00603**	**90.834**	**0.29308**	**0.52300**	**52.52**	**1.49216**	0.33619
	0.40	0.00611	90.378	0.27535	0.49172	52.33	1.49227	0.33589
	0.42	0.00626	89.465	0.24373	0.43591	51.96	1.49378	0.33555
	0.43	**0.00634**	**89.009**	**0.22963**	**0.41100**	**51.77**	1.49516	**0.33552**
	0.44	0.00643	88.552	0.21654	0.38787	51.60	1.49695	0.33558
SMI 24Y/1D	0	0.00174	77.0290	0.36209	0.47621	252.39	84.434	16.396
	0.99999	98.766	44.4731	264.79	363.56	50.606	1.82255	0.32345
	0.999999	987.66	44.4728	2647.8	3635.4	50.606	1.82239	0.32342
	0.9999999	**9876.5**	**44.4727**	**26478**	**36354**	**50.606**	**1.82238**	**0.32342**
SP500 24Y/1D	0	0.00131	71.8447	0.31002	0.38443	392.87	96.689	19.013
	0.99999	69.048	41.4799	244.52	313.58	54.478	0.87274	0.15408
	0.999999	690.47	41.4796	2445.1	3135.6	54.478	0.87272	0.15408
	0.9999999	**6904.7**	**41.4796**	**24451**	**31356**	**54.478**	**0.87272**	**0.15408**
SMI 24Y/1M	**0**	**0.06608**	**38.3146**	**0.52690**	**2.85564**	**9.9918**	**0.43016**	**0.05798**
	0.01	0.06649	38.1549	50.798	275.914	9.9874	0.43129	0.05805
	0.02	0.06691	37.9952	24.490	133.316	9.9829	0.43244	0.05813
	0.05	0.06821	37.5160	8.789	48.168	9.9696	0.43591	0.05839
	normal distribution					0.50	2.60679	0.43223
SP500 24Y/1M	0	0.06727	44.472	0.48854	3.23731	10.775	0.71028	0.12646
	0.999	41.232	25.6953	3.6052	45.007	10.341	0.57221	0.10460
	0.9999	412.08	25.6780	35.843	448.10	10.340	0.57217	0.10459
	0.99999	**4120.6**	**25.6762**	**358.23**	**4479.0**	**10.340**	**0.57217**	**0.10459**
	normal distribution					0.50	2.19806	0.37436
SP500 63Y/1M	0	0.04012	37.304	0.59984	1.87778	12.802	1.94245	0.36893
	0.999	22.856	21.5537	7.7941	30.737	11.323	0.46985	0.09274
	0.9999	228.40	21.5392	77.578	306.07	11.322	0.46914	0.09259
	0.99999	**2283.8**	**21.5377**	**775.42**	**3059.4**	**11.300**	**0.46907**	**0.09258**
	normal distribution					0.50	3.26036	0.51397

Table 4.2: Parameter estimates, mode, and GoF statistics for the NVG family

data set	parameter estimates					mode	mode	FFT GoF statistics	
	s	υ	ρ	α	β	anal.	FFT	A²	W²
SMI 3Y/1D	1	0.00079	0.85686	137.59	122.70	81.82	81.80	1.00069	0.16215
	0.93	0.00069	0.64228	128.73	113.79	55.82	55.82	0.35449	0.04051
	0.915	**0.00067**	**0.60214**	**126.83**	**111.88**	**54.65**	**54.65**	**0.33591**	**0.03947**
	0.90	0.00065	0.56391	124.94	109.97	53.65	53.65	0.33907	0.04235
SP500 3Y/1D	1	0.00170	0.83167	122.17	101.43	73.74	73.70	0.44795	0.10181
	0.995	0.00168	0.81544	121.65	100.90	61.16	61.16	0.38528	0.08617
	0.985	**0.00166**	**0.78370**	**120.61**	**99.83**	**56.52**	**56.52**	**0.36132**	**0.07944**
	0.98	0.00164	0.76818	120.10	99.30	55.15	55.15	0.37241	0.08147
SMI 24Y/1D	1	0.00075	0.42263	81.135	74.047	233.7	253.3	83.150	16.275
	0.83	0.00060	0.20133	68.181	61.048	46.79	46.79	1.050	0.142
	0.82	**0.00059**	**0.19186**	**67.422**	**60.285**	**46.24**	**46.24**	**0.958**	**0.136**
	0.81	0.00058	0.18273	66.662	59.522	45.72	45.72	0.999	0.153
SP500 24Y/1D	1	0.00062	0.34901	74.756	69.580	316.2	386.6	94.534	18.612
	0.88	0.00054	0.20961	66.186	60.997	53.16	53.16	2.997	0.355
	0.87	**0.00054**	**0.20028**	**65.473**	**60.282**	**52.19**	**52.19**	**2.742**	**0.348**
	0.85	0.00052	0.18254	64.046	58.853	50.50	50.50	2.917	0.454
SMI 24Y/1M	**1**	**0.03248**	**2.15686**	**62.759**	**35.041**	**10.25**	**10.25**	**0.33470**	**0.04218**
	0.99	0.03214	2.08393	62.778	34.712	10.20	10.20	0.34726	0.04402
	0.98	0.03180	2.01310	62.819	34.384	10.15	10.15	0.36013	0.04595
	0.95	0.03083	1.81285	63.102	33.408	10.00	10.00	0.39936	0.05212
	normal distribution					0.50		2.60679	0.43223
SP500 24Y/1M	1	0.03468	2.48138	78.840	40.882	10.99	10.99	0.43351	0.08095
	0.93	0.03267	1.96388	81.799	38.342	10.75	10.75	0.40177	0.07393
	0.92	**0.03243**	**1.89885**	**82.520**	**37.991**	**10.72**	**10.72**	**0.40121**	**0.07381**
	0.91	0.03221	1.83596	83.348	37.642	10.69	10.69	0.40127	0.07382
	normal distribution					0.50		2.19806	0.37436
SP500 63Y/1M	1	0.01915	1.40715	50.036	33.985	13.52	13.52	1.40332	0.26798
	0.77	0.01478	0.54708	45.817	26.233	11.08	11.08	0.25981	0.03687
	0.76	**0.01462**	**0.52287**	**45.785**	**25.909**	**11.03**	**11.03**	**0.25742**	0.03578
	0.745	0.01440	0.48825	45.784	25.426	10.95	10.95	0.25775	0.03484
	0.73	**0.01419**	**0.45558**	**45.848**	**24.947**	**10.91**	**10.91**	0.26252	**0.03468**
	0.72	0.01405	0.43485	45.931	24.630	10.87	10.87	0.26802	0.03498
	normal distribution					0.50		3.26036	0.51397

Let us now return to the main application, which is the evaluation of the efficiency measure (4.3). Since the chosen estimation method is the moment method, the approximate ranking function $R_*^A(p)$ follows from (4.1) through direct insertion of the sample values. Moreover, to each solution of the TLF moment problem, the corresponding test ranking function $R_*(p^{TLF})$ is evaluated using the formulas (4.4)-(4.5). In this way, the ranking efficiency measure is obtained. The numerical results of our case study are summarized in Table 4.3. We note a systematic efficiency increase of the approximate ranking over the Lévy-Markowitz benchmark. For each feasible value $\gamma \in [0,1)$ the efficiency increase is limited to a small range of variation. The maximum efficiency increase is here attained for the TLF with $\gamma \to 1$ and the minimum for the TLF with $\gamma = 0$. The latter assertion has been verified for a finite number of values $\gamma \in \{0, 0.01, 0.1 \cdot k, k = 1,...,9, 0.99\}$. The maximum efficiency increase is a bit higher for the TLF than for the NVG.

Table 4.3: NVG vs. TLF efficiency measures for SMI and SP500 data sets

data set	unbiased estimates				NVG efficiency		TLF efficiency	
	μ	σ	S	K	min	max	min	max
SMI 3Y/1D	0.00004	0.01011	-0.26118	3.54668	92.83657	93.35457	93.35602	93.36398
SP500 3Y/1D	0.00031	0.01169	-0.42731	3.72928	94.78351	95.04308	95.04488	95.05577
SMI 24Y/1D	0.00026	0.01189	-0.29736	7.15740	83.64588	86.15996	86.16163	86.18644
SP500 24Y/1D	0.00027	0.01160	-0.25716	8.63988	76.13769	81.19789	81.19789	81.22844
SMI 24Y/1M	0.00530	0.04800	-0.74866	1.77381	94.14608	94.27808	94.25105	94.33978
SP500 24Y/1M	0.00546	0.04340	-0.76442	1.61037	95.35316	95.42011	95.39562	95.46018
SP500 63Y/1M	0.00586	0.04220	-0.65537	2.42167	91.77448	92.20746	92.19859	92.29538

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Appendix: Numerical approximations of the TLF distribution and related risk functions

Analytically, the pdf $f_X(x)$ of a one-sided TS random variable $X \sim TS(\gamma,\lambda,c)$ with cgf

$$C_X(t) = -c\cdot\{(\lambda-t)^{\gamma}-\lambda^{\gamma})\},\ \gamma\in(0,1),\ \lambda,c>0,$$

can be represented as product of a tempering function and a stable Paretian pdf (e.g. Bauemer and Meerschaert (2010), Janczura and Wylomanska (2012)):

$$f_X(x) = \exp\{-\lambda x + c\lambda^{\gamma}\}\cdot f_S(x),\quad S\sim S_{\gamma}(\delta,1,0),\quad \delta=\{c\cdot\cos(\tfrac{1}{2}\pi\gamma)\}^{1/\gamma},$$

where $S_{\gamma}(\delta,\beta,\mu)$ is a *stable Paretian* random variable with stability index γ, scale parameter δ, skewness parameter β, and location parameter μ. Therefore, the TLF pdf can be represented as convolution of two such products. Though algorithmic calculation of the stable Paretian pdf is possible (e.g. Nolan (1997/2005)), it is usually not easy. On the other hand, the fast Fourier transform (FFT) approximation of the stable Paretian density cannot approach extreme heavy tails with $\gamma<1$, as noted by Menn and Rachev (2006), Section 3. Therefore, it is preferable to apply the FFT to directly approximate the integral expression of the density in terms of the characteristic function as in Scherer et al. (2012) for example. However, we prefer the alternative interpolation scheme by Jelonek (2012), Appendix B, which has been adapted here to the mid-point rule (MPR) (instead of the left-point rule) for a higher accuracy.

Consider a finite interval $[a,b]$ that is divided into N disjoints subintervals of equal length $h=(b-a)N^{-1}$ and assume that the random variable X with pdf $f_X(x)$ has a known chf $\phi_X(z),\ z\in C$. For $k=0,...,N-1$ set $x_k=a+hk$. For N sufficiently large the constant $c=\pi\cdot h^{-1}$ is also large and one has the pdf approximation

$$f_X(x_k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-izx_k}\cdot\phi_X(z)dz\approx\frac{1}{2\pi}\int_{-c}^{c}e^{-izx_k}\cdot\phi_X(z)dz=\int_{-N/2(b-a)}^{N/2(b-a)}e^{-2\pi i\cdot ux_k}\cdot\phi_X(2\pi\cdot u)du.$$

For $j=0,...,N$ set $u_j=(j-\frac{N}{2})(b-a)^{-1}$ and consider the mid-points

$$m_j=\tfrac{1}{2}(u_j+u_{j+1})=(j-\tfrac{N-1}{2})(b-a)^{-1},\quad j=0,...,N-1.$$

Applying the MPR to the right-hand side integral one obtains the finite sum approximation

$$f_X(x_k)\approx(b-a)^{-1}\cdot\sum_{j=0}^{N-1}e^{-2\pi i\cdot m_j x_k}\cdot\phi_X(2\pi\cdot m_j)=(b-a)^{-1}\cdot\sum_{j=0}^{N-1}e^{-2\pi i\cdot(\frac{a}{h}+k)(j-\frac{N-1}{2})\frac{1}{N}}\cdot\phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2})).$$

Since $e^{\pi i}=-1$ one has further $e^{-2\pi i\cdot(\frac{a}{h}+k)(j-\frac{N-1}{2})\frac{1}{N}}=(-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)}\cdot(-1)^{(\frac{2a}{b-a})\cdot j}\cdot e^{-2\pi i\cdot k\frac{j}{N}}$. Inserted into the above sum, one gets the desired representation

$$f_X(x_k)\approx(b-a)^{-1}\cdot(-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)}\cdot\sum_{j=0}^{N-1}(-1)^{(\frac{2a}{b-a})\cdot j}\cdot\phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2}))\cdot e^{-2\pi i\cdot k\frac{j}{N}},$$

which one interprets as k-th component of a Discrete Fourier Transform (DFT)

$$f_X(x_k)\approx C_k\cdot DFT(y)_k,\quad C_k=(b-a)^{-1}\cdot(-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)},$$

$$y=(y_0,...,y_{N-1}),\quad y_j=(-1)^{(\frac{2a}{b-a})\cdot j}\cdot\phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2})),\quad j=0,...,N-1.$$

An efficient software implementation of the DFT is based on the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey (1965) (see also Schwartz (1977/78), Heideman et al. (1985), Duhamel and Vetterli (1990), Batenkov (2005), among others). For numerical approximation of the distribution function $F_X(x)=\int_{-\infty}^{x}f_x(t)dt$ one derives a similar DFT approximation in terms of the chf (e.g. Kim et al. (2010), Proposition 1) or one uses the recursive formula

$$F_X(x_k)=F_X(x_{k-1})+hf_X(x_{k-1}),\quad k=1,...,N-1,\quad F_X(x_0)=0,$$

and a simple piecewise linear interpolation for intermediate values:

$$F_X(x)=F_X(x_{k-1})+h^{-1}(x-x_{k-1})\{F_X(x_k)-F_X(x_{k-1})\},\quad x\in[x_{k-1},x_k],\quad k=1,...,N-1.$$

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Finally, we note that similar approximations can be obtained for the value-at-risk measure (VaR), the stop-loss transform and the related conditional value-at-risk measure (CVaR) (see Kim et al. (2010) for formulas in terms of the chf). They can be used for further important financial applications of the TLF in option pricing and risk management.

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Source of support: Nil, Conflict of interest: None Declared