International Journal of Mathematical Archive-4(6), 2013, 207-210
Available online through www.ijma.info ISSN 2229 – 5046

On rg-R_0 and rg-R_1 spaces

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(Received on: 26-04-13; Revised & Accepted on: 06-06-13)

ABSTRACT

The aim of this paper is to introduce rg-R_0 and rg-R_1 spaces. Some existing lower separation axioms are characterized by using these spaces.

Keywords and Phrases: *pre-open, pre-closed sets, pgpr-open sets, pgpr-closed sets.*

MSC 2010: *54A20, 54C10, 54D10.*

1. INTRODUCTION

The separation axioms R_0 and R_1 were introduced and studied by Shanin in [13] and Yang [15]. In 1963, Davis [4] rediscovered them. The notions of semi-R_0 [11]; semi-R_1 [5]; pre-R_0, pre-R_1 [3] and g-R_0, g-R_1 [2] were discussed by Charles Dorsott; Maheswari and Prasad; Caldas *et.al.* and Balasubramanian respectively. Recently the authors studied pgpr-R_0 and pgpr-R_1 spaces in [8]. In this paper, we introduce and investigate rg-R_0 and rg-R_1 spaces.

2. PRELIMINARIES

Throughout this paper (X,τ) denotes a topological space on which no separation axioms are assumed unless explicitly stated. A subset B of (X,τ) is regular open if $B=int(cl(B))$ [14] and is generalized closed(briefly g-closed)[10] if $cl(B)\subseteq U$ whenever $B\subseteq U$ and U is open in X and regular generalized closed (briefly rg-closed)[12] if $cl(B)\subseteq U$ whenever $B\subseteq U$ and U is regular open in X. The complement of a g-closed set is g-open and that of rg-closed set is rg-open. The intersection of all g-closed (resp. rg-closed) sets containing B is called the g-closure (resp. rg-closure) of B and denoted by $cl^*(B)$ [6] (resp. $cl_r^*(B)$ [1]). In a space (X,τ), *Ker* (A) denotes the intersection of all open sets containing A.

Definition 2.1[1]: A topological space (X,τ) is rg-regular if for every regular closed set F and a point $x\notin F$, there exist disjoint rg-open sets U and V such that $x\in U$ and $F\subseteq V$.

Definition 2.2: A space (X,τ) is R_0 [13] (resp.g-R_0 [2]) if for each open (resp. g-open) set U of X, $x\in U$ implies $cl(\{x\})\subseteq U$ (resp. $cl^*(\{x\})\subseteq U$).

Definition 2.3: A topological space (X,τ) is R_1 [15] if for $x, y\in X$ such that $cl(\{x\})\neq cl(\{y\})$, there are disjoint open sets U and V such that $cl(\{x\})\subseteq U$ and $cl(\{y\})\subseteq V$.

3. rg-R_0 spaces

In this section, we introduce rg-R_0 spaces as a generalization of R_0 spaces and obtain some of their basic properties.

Definition 3.1: A topological space (X,τ) is said to be rg-R_0 if every rg-open set contains the closure of each of its points

Proposition 3.2: *If (X,τ) is rg-R_0 then it is R_0 and g-R_0.*

Proof: Suppose (X,τ) is rg-R_0. Let V be an open set in X. Since every open set is rg-open, V is rg-open in X. Since (X,τ) is rg-R_0, by Definition 3.1, $cl(\{x\})\subseteq V$ for every $x\in V$. By using Definition 2.2, (X,τ) is R_0. Let V be a g-open set in X. Since every g-open set is rg-open, V is rg-open in X. Since (X,τ) is rg-R_0, by Definition 3.1, $cl(\{x\})\subseteq V$ for every $x\in V$. Now by using Proposition 5 of [9], $cl^*(\{x\})\subseteq cl(\{x\})\subseteq V$. This proves that (X,τ) is g-R_0.

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Theorem 3.3: *A topological space (X,τ) is a rg-R_0 space if and only if for any rg-closed set H, $cl(\{x\})\cap H=\emptyset$ for every $x\in X\backslash H$.*

Proof: Suppose (X,τ) is rg-R_0. Let H be rg-closed in X and $x\in X\backslash H$.Then $X\backslash H$ is rg-open. Since (X,τ) is rg-R_0, by using Definition 3.1, $cl(\{x\})\subseteq X\backslash H$ and so $cl(\{x\})\cap H=\emptyset$. Conversely assume that, for any rg-closed set H of X, $cl(\{x\})\cap H=\emptyset$ for every $x\in X\backslash H$. Let V be any rg-open set in X and $x\in V$. Then $x\in V=X\backslash(X\backslash V)$ and $X\backslash V$ is rg-closed. By our assumption $cl(\{x\})\cap X\backslash V=\emptyset$ which implies that $cl(\{x\})\subseteq V$. This proves that (X,τ) is rg-R_0.

Theorem 3.4: *A topological space X is rg-R_0 if and only if for any points x and y in X, $x\neq y$ implies $cl(\{x\})\cap cl(\{y\})=\emptyset$.*

Proof: Let X be rg-R_0 and $x\neq y\in X$. By Theorem 3.1 of [7], {x} is rg-open. Since $x\in\{x\}$, we have $cl(\{x\})\subseteq\{x\}$.Thus $cl(\{x\})=\{x\}$.Now $cl(\{x\})\cap cl(\{y\})=\{x\}\cap\{y\}=\emptyset$. Conversely suppose for any points x and y in X$\neq$ x y implies $cl(\{x\})\cap cl(\{y\})=\emptyset$. Let V be rg-open and $x\in V$. Let $y\in cl(\{x\})$.Suppose $y\notin V$. Since $x\in V$, $x\neq y$. By assumption, $cl(\{x\})\cap cl(\{y\})=\emptyset$.Then $y\notin cl(\{x\})$ which is a contradiction to $y\in cl(\{x\})$ so we get $y\in V$ and $cl(\{x\})\subseteq V$. Thus X is rg-R_0.

Theorem 3.5: *A topological space (X,τ) is rg-R0 if and only if for any x and y in X, $x\neq y$ implies $Ker(\{x\})\cap Ker(\{y\})=\emptyset$.*

Proof: Assume that (X,τ) is a rg-R_0 space and$\neq$yx $\in$X. By Theorem 3.4, $cl(\{x\})\cap cl(\{y\})=\emptyset$. If $Ker(\{x\})\cap Ker(\{y\})\neq\emptyset$,then there exists $z\in X$ such that $z\in Ker(\{x\})\cap Ker(\{y\})$. Then $z\in Ker(\{x\})$ and $z\in Ker(\{y\})$. Since $z\in Ker(\{x\})$ we have $x\in cl(\{z\})$. Suppose $x\neq z$, $cl(\{x\})\cap cl(\{z\})=\emptyset$ which contradicts $x\in cl(\{x\})\cap cl(\{z\})$ so x=z. Similarly we have y=z. That is x=y=z. This is a contradiction to $x\neq y$. Hence $Ker(\{x\})\cap Ker(\{y\})=\emptyset$.

Conversely, assume that for any $x\neq y$ in X, $Ker(\{x\})\cap Ker(\{y\})=\emptyset$. Suppose $z\in cl(\{x\})\cap cl(\{y\})$.Then $z\in cl(\{x\})$ and $z\in cl(\{y\})$. Now $z\in cl(\{x\})$ implies that $x\in Ker(\{z\})$. Since $x\in Ker(\{x\})$ we have $x\in Ker(\{x\})\cap Ker(\{z\})$ and hence x=z (otherwise $x\neq z\Rightarrow Ker(\{x\})\cap Ker(\{z\})=\emptyset$ which is a contradicts $x\in Ker(\{x\})\cap Ker(\{z\})$). Similarly we have y=z and hence x=y=z. This is a contradiction to $x\neq y$. So $cl(\{x\})\cap cl(\{y\})=\emptyset$. By Theorem 3.4, (X,τ) is rg-R_0.

Theorem 3.6: *A topological space (X,τ) is rg-R_0 if and only if it is T_1.*

Proof: Let (X,τ) be rg-R_0 and let $x\in X$. By using Theorem 3.1 of [7], {x} is rg-open. Since (X,τ) is rg-R_0, by using Definition 3.1, $cl(\{x\})\subseteq\{x\}$ and hence $cl(\{x\})=\{x\}$. That is {x} is closed. It follows that every singleton set is closed. Therefore (X,τ) is T_1.Conversely suppose (X,τ) is T_1. Let V be rg-open and let $x\in V$. Then $cl(\{x\})=\{x\}\subseteq V$. Thus (X,τ) is rg-R_0.

Theorem 3.7: *For a topological space (X,τ), the following are equivalent:*
(a) (X,τ) is rg-R_0.
(b) If H is rg-closed, then H=Ker (H).
(c) If H is rg-closed and $x\in H$, then $Ker(\{x\})\subseteq H$.

Proof:
(a)$\Rightarrow$(b): Assume that (X,τ) is rg-R_0. Let H be any rg-closed set in X and $x\in Ker(H)$. Suppose $x\notin H$. Then $X\backslash H$ is rg-open and $x\in X\backslash H$. Since (X,τ) is rg-R_0, $cl(\{x\})\subseteq X\backslash H$ which implies $H\subseteq X\backslash cl(\{x\})$. Since $X\backslash cl(\{x\})$ is open, we have $Ker(H)\subseteq X\backslash cl(\{x\})$. Since $x\notin X\backslash cl(\{x\})$, we have $x\notin Ker(H)$. This is a contradiction to $x\in Ker(H)$ so we get $x\in H$. That is $Ker(\{x\})\subseteq H$. But always $H\subseteq Ker(H)$. This proves that $H=Ker(H)$.

(b)$\Rightarrow$(c): Let H be a rg-closed set and $x\in H$. Then $Ker(\{x\})\subseteq Ker(H)=H$, by (b).

(c)$\Rightarrow$(a): Let V be any rg-open set and $x\in V$. Let $y\in cl(\{x\})$.Then $x\in Ker(\{y\})$. Suppose $y\notin V$.

Then $y\in X\backslash V$ and $X\backslash V$ is rg-closed. By (c), $Ker(\{y\})\subseteq X\backslash V$. This implies that $x\in X\backslash V$ and hence $x\notin V$. This is a contradiction to $x\in V$ and we get $y\in V$. That is $cl(\{x\})\subseteq V$. Thus (X,τ) is rg-R_0.

Theorem 3.8: *For a topological space (X,τ), the following are equivalent:*
(i) (X,τ) is a rg-R_0 space.
(ii) For any $A\neq\emptyset$ and $G\in RGO(X,\tau)$ such that $A\cap G\neq\emptyset$, there exists a closed set F such that $A\cap F\neq\emptyset$, and $F\subseteq G$.
(iii) Any $G\in RGO(X,\tau)$, $G=\bigcup\{F: F\subseteq G$ and F is closed$\}$.
(iv) Any $F\in RGC(X,\tau)$, $F=\bigcap\{G: G\subseteq F$ and G is open$\}$.

Proof: (i)⇒(ii): Let A be a nonempty subset of X and G∈RGO(X,τ) such that A∩G≠∅. Then there exists x∈A∩G. Since X is rg-R_0 and x∈G, *cl*({x})⊆G. Take F=*cl*({x}). Then F is closed and F⊆G. Now x∈*cl*({x})=F and x∈A implies that x∈A∩F and A∩F≠∅.

(ii)⇒(iii): Let G∈RGO(X,τ), then G⊇ ⋃{F/F⊆G and F is closed}. Let x∈G. Then {x}∩G≠∅. Now by using (ii), there exists a closed set H such that {x}∩H≠∅ and H⊆G. That is x∈H ⊆⋃{F: F⊆G and F is closed} and G⊆⋃{F/F⊆G and F is closed}. It follows that G=⋃{F:F⊆G and F is closed}.

(iii)⇒(iv): Let F∈RGC(X,τ). Then X\F∈RGO(X,τ) and by (iii), X\F= ⋃{H: H⊆X\F and H is closed}. Since H is closed, X\H is open. Now H⊆X\F implies X\H⊆F and X\F=⋃H implies that F=X\ (∪H) = ∩(X\H) where X\H is open and X\H⊆F. So F= ∩{G: G⊆F and G is open}.

(iv)⇒(i): Let F∈RGC(X,τ) and x∈F. By (iv), F= ∩ {G/ G⊆F and G is open}. Then x∈G, for all open set G containing F. Since G is open, *Ker* ({x}) ⊆G, for all open set G containing F. That is Ker ({x}) ⊆ ∩ {G/ G⊆F and G is open} =F. By using Theorem 3.7(c), (X,τ) is rg-R_0.

Theorem 3.9: *If (X,τ) is rg-R_0 if and only if for any rg-closed set U and x∉U, there exists an open set G such that U⊆G and x∉G.*

Proof: Suppose (X,τ) is rg-R_0. Let U be any rg-closed set and x∉U. Then x∈X\U and X\U is rg-open. Since (X,τ) is a rg-R_0 space, by Definition 3.1, *cl*({x})⊆X\U. Put G=X\ *cl*({x}). Then x∉G and U⊆ X*cl*({x})=G. Since *cl*({x}) is closed, we have G=X\ *cl*({x}) is open.

Conversely, suppose for any rg-closed set U and x∉U, there exists an open set G such that U⊆G and x∉G. Let U be any rg-closed set and x∉U. Then by our assumption, there exists an open set G such that U⊆G and x∉G. That is x∈X\G and X\G is closed. Also *cl*({x})⊆X\G and *cl*({x})∩G=∅. Thus *cl*({x})∩U⊆*cl*({x})∩G =∅. By Theorem 3.4, (X,τ) is rg-R_0

Corollary 3.10: *If (X,τ) is rg-R_0, then it is rg-regular.*

Proof: Suppose (X,τ) is rg-R_0. Let H be regular closed and x∉H. Since every regular closed set is rg-closed, H is rg-closed. By using Theorem 3.9, there exists an open set V such that H⊆V and x∉V. Since every open set is rg-open, V is rg-open. Put U={x}. By Theorem 3.1 of [7], U is rg-open. Also U∩V={x}∩V=∅. Thus, U and V are disjointing rg-open sets containing x and H respectively. Then by Definition 2.1, (X,τ) is rg-regular.

4. rg-R_1 spaces

In this section, we introduce and investigate rg-R_1 spaces using the notion of rg-open sets.

Definition 4.1: A space (X,τ) is said to be rg-R_1, if for x, y in X with *cl*({x})≠ *cl*({y}), there exist disjoint rg-open sets U and V such that *cl*({x})⊆U and *cl*({y}) ⊆V.

Proposition 4.2:
(i) *Every rg-R_0 space is rg-R_1.*
(ii) Every R_1 space is rg-R_1.
(iii) Every T_2 space is rg-R_1.

Proof:
(i) Suppose (X,τ) is rg-R_0. Let x, y∈X with *cl*({x})≠ *cl*({y}). Then by Theorem 3.6, *cl*({x})={x} and *cl*({y})={y}. By using Theorem 3.1[7], {x}, {y} are rg-open sets and {x}∩{y}=∅. This shows that (X,τ) is rg-R_1.

(ii) Suppose (X,τ) is R_1. Let x, y∈X with *cl*({x})≠ *cl*({y}). Since (X,τ) is R_1, by Definition 2.3, there exist disjoint open sets U and V such that *cl*({x})⊆U and *cl*({y})⊆V. Since every open set is rg-open, U and V are rg-open. This proves that (X,τ) is rg-R_1.

(iii) Let (X,τ) be a T_2 space. Since every T_2 space is T_1, by using Theorem 3.6, (X,τ) is rg-R_0. Now by using (i), (X,τ) is rg-R_1.

Theorem 4.3: *If a topological space (X,τ) is rg-R_1 then either cl({x})=X for each x∈X or cl({x})≠X for each x∈X.*

Proof: Assume that (X,τ) is rg-R_1. If $cl(\{x\})=X$ for all $x\in X$, then the theorem is proved. If not, then there exists $y\in X$ such that $cl(\{y\})\neq X$. To prove $cl(\{x\})\neq X$ for all $x\in X$. Suppose not, then there exists $z\in X$ such that $cl(\{z\})=X$. Now $cl(\{y\})\neq X= cl(\{z\})$. Since (X,τ) is rg-R_1, there exist disjoint rg-open sets U and V containing $cl(\{y\})$ and $cl(\{z\})$ respectively. Since $cl(\{z\})=X$, we have V=X. This implies that $U\cap V=U\cap X=U\neq\emptyset$, because $y\in U$. This is a contradiction to $U\cap V=\emptyset$. Therefore $cl(\{z\})\neq X$. Thus $cl(\{x\})\neq X$ for all $x\in X$.

REFERENCES

[1] I.Arockia Rani, Studies on generalizations of generalized closed sets and maps in topological spaces, Ph.D Thesis, Bharathiyar University, Coimbatore, (1997).

[2] S. Balasubramanian, Generalized separation axioms, Scientia Magna, 6(4) (2010), 1-14.

[3] M.Caldas, S.Jafari and T.Noiri, Characterizations of pre-R_0 and pre-R_1 topological spaces, Topology Proceedings. 25(2000), 17-30.

[4] A.S.Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly, 68 (1961), 886-893.

[5] C.Dorsott, Semi-T_2, Semi-R_1 and Semi-R_0 topological spaces, Annales de la Socete Scientifigue de Bruxelles, T. 92, III (1978), 143-150.

[6] W. Dunham and N. Levine, Further results on generalized closed sets, kyungpook Math. J.20 (1980), 169-175.

[7] P.Gnanachandra and P. Thangavelu, Remarks on rg-closure, gpr-closure operators and gpr-separation axioms, *Ultra Scientist*, 24(1) A (2012), 185-191.

[8] P. Gnanachandra and P. Thangavelu, Characterizations of pgpr-R_0 and pgpr-R_1, American Journal of Mathematics and Mathematical sciences, 1(1) (2012), 55-60.

[12] Jin Han Park, Jin Keun Park and Seong Jun Park, Further results on rg-continuity, Far East J. Math.Soc. special volume part II(2000), 237-245.

[13] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo 19(2) (1970), 89-96.

[14] S.N. Maheswari and R. Prasad, On $(R0)_s$- spaces, Portugaliae Math. 34(1975), 213-217.

[16] N. Palaniappan and K. Chandrasekhara Rao, Regular generalized closed sets, Kyungpook Mathematical Journal, 33(2)(1993), 211-219.

[17] N. A. Shanin, On separation in topological spaces, C.R. (Doklady) Acad. Sci. URSS (N.S.), 38(1943), 110-113.

[19] M. H. Stone, Applications of the theory of Boolean rings to the general topology, Trans.A. M.S. 41(1937), 375-481.

[20] C. T. Yang, On paracompact spaces, Proc. Amer. Math. Soc. (5) 1954, 185-189.

Source of support: Nil, Conflict of interest: None Declared

