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WEAKLY COMPATIBLE MAPPINGS AND FIXED POINTS IN DISLOCATED METRIC SPACES

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ABSTRACT

In this paper we prove a general fixed point theorem for two pairs of weakly compatible mappings, in dislocated metric spaces. We generalize, unify improve fixed point results in these spaces existing in recent literature.

***Keywords**: weakly compatible mappings, dislocated metric, fixed point.*

***Mathematics subject Classification**: 47H10; 54H25.*

1. INTRODUCTION

During the last five decades, study of common fixed point of mappings satisfying contractive type conditions has been a very active field, for many mathematicians. Recently, with the introduction of the concept of dislocated and dislocated quasi-metric spaces by P. Hitzler and A. Seda, F. M. Zeyada *et al*., several author generalized the famous Banach Contraction Principle. They have established a number of important fixed point theorems for a single and a pair of mappings in complete dislocated metric space, by [1, 2, 6, 7, 8, 11, 12] The aims of this paper is to establish a common fixed point theorem involving two pairs of weakly compatible mappings, in context of dislocated metric space. Also our theorem generalizes and improves existing results in such spaces.

2. PRELIMINARIES

We start with base and auxiliary definitions and lemmas, which will be used in this paper.

Definition 2.1: [10] Let X be a non-empty and let $d: X \times X \to \mathbb{R}^+$ be a function, called a distance function if for all $x, y, z \in X$, satisfies:

$$d_1: d(x,x) = 0$$

$$d_2: d(x,y) = d(y,x) = 0 \Rightarrow x = y$$

$$d_3: d(x,y) = d(y,x)$$

$$d_4: d(x,y) \le d(x,z) + d(z,y).$$

If d satisfies the condition $d_1 - d_4$, then d is called a metric on X. If it satisfies the conditions d_1, d_2 and d_4 it is called a quasi-metric. If d satisfies conditions d_2, d_3 and d_4 it is called a dislocated metric (or simply d-metric). If d satisfies only d_2 and d_4 then d is called a dislocated quasi-metric (or simply dq-metric) on X. A nonempty set X with dq-metric d, i. e., (X,d) is called a dislocated quasi-metric space.

Example: Let be $X = R^+$ and the function $d: X \times X \to \mathbb{R}^+$ where $d(x,y) = x + y$. Then (X,d) is a dislocated metric space, but not e metric space.

Definition 2.2: [10] A sequence (x_n) in a d-metric space (X,d) is called a Cauchy sequence if for all $\varepsilon > 0$, $\exists n_0 \in N$ such that $\forall m, n \ge n_0$, we have $d(x_m, x_n) < \varepsilon$.

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Definition 2.3: [10] A sequence in d -metric space converges with respect to d , if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

In this case x is called a d-limit of (x_n) and we write $x_n \to x$.

Definition 2.4: [10] A d -metric space (X, d) is called complete if every Cauchy sequence in it is convergent with respect to d .

Preposition 2.5: Every d -convergent sequence in a d -metric space is a Cauchy sequence.

Proof: Let be a sequence (x_n) in a d -metric space (X, d) which is d -convergent to x , and if we choose an $\varepsilon > o$ (arbitrary) then there exists $n_0 \in N$ that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n > n_0$. Considering

$d(x_m, x_n) \le d(x_m, x) + d(x, x_n) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence (x_n) is a Cauchy sequence.

The following examples validate the preposition.

Example: Let $X = [0,1]$ and $d(x, y) = \max\{x, y\}$. Then the pair (X, d) is a dislocated metric space. We define an arbitrary sequence (x_n) in X by $x_n = \frac{3}{3^n + 2}, n \in \mathrm{N} \cup \{0\}$.

We note that for $n \to \infty$, $x_n \to 0 \in X$. Let $\varepsilon = \sup_{n\in\mathbb{N}} \left\{ \frac{3}{3^n + 2} \right\}$, then for $n, m \in \mathrm{N}$ and $n > m,$ we have $d(x_n, x_m) = d\left(\frac{3}{3^n + 2}, \frac{3}{3^m + 2} \right) = \frac{3}{3^m + 2} \le \varepsilon$. Thus, (x_n) is a Cauchy sequence in X.

Definition 2.6: [10] Let (X, d) be a d-metric space. A mapping $T : X \to X$ is called contraction if there exists a number $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda d(x, y)$ for all $x, y \in X$.

Lemma 2.7: [10] Let (X, d) be a d-metric space. If $T : X \to X$ is a contraction function, then $T^n(x_0)$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 2.8: [10] Limits in a d-metric space are unique.

Definition 2.9: [5] Let F and S be mappings from a metric space (X, d) into itself. Then, F and S are said to be weakly compatible if they commute at their coincidence point; that is $Fx = Sx$ for some $x \in X$ implies $SFx = FSx$.

Theorem 2.10: [10] Let (X, d) be complete d-metric space and let $T : X \to X$ be a contraction mapping then, T has a unique fixed point.

3. MAIN RESULTS

We consider the set G_3 of all continuous functions $g : [0, \infty)^3 \to [0, \infty)$ with the following properties:

(a) g is non-decreasing in respect to each variable

(b) $g(t, t, t) \le t, t \in [0, \infty)$

Some examples of these functions are as follows:

$g_1 : g(t_1,t_2,t_3) = \max\{t_1,t_2,t_3\}$

$g_2 : g(t_1,t_2,t_3) = \max\{t_1+t_2, t_1+t_3, t_2+t_3\}$

$g_3 : g(t_1,t_2,t_3) = \left[\max\{t_1t_2, t_2t_3, t_3t_1\}\right]^{\frac{1}{2}}$

$g_4 : g(t_1,t_2,t_3) = \left[\max\{t_1^{\,p}, t_2^{\,p}, t_3^{\,p}\}\right]^{\frac{1}{p}}, p>0$

$g_5 : g(t_1,t_2,t_3) = c_1t_1 + c_2t_2 + c_3t_3 \quad with\ 0 \le c_1+c_2+c_3 < 1$

Theorem 3.1: Let (X,d) be a complete dislocated metric space and $S,T,F,G : X \to X$ are continuous mappings, satisfying the conditions:

(3.1.1) $S(X) \subset G(X), T(X) \subset F(X)$

(3.1.2) The pairs (S,F) and (T,G) are weakly compatible and

(3.1.3) $d(Sx,Ty) \le c\, g\left[d(Fx,Ty), d(Gy,Sx), d(Fx,Gy)\right]$ for all $x,y \in X$,

where $g \in G_3$ and $0 \le c < \frac{1}{2}$.Then $F,G,S,$ and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X .Define the sequence (x_n) and (y_n) in X as follows:

$$y_{2n} = Sx_{2n} = Gx_{2n+1};\ y_{2n+1} = Tx_{2n+1} = Fx_{2n+2} \text{ for } n = 0,1,2,...$$

Let consider these cases:

Case: 1 If $y_{2n} = y_{2n+1}$ for some n, then $Gx_{2n+1} = Tx_{2n+1}$. Hence x_{2n+1} is a coincidence point of G and T.

If $y_{2n+1} = y_{2n+2}$ for some n, then $Fx_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of F and T.

Case: 2 If $y_n \ne y_{n+1}$. for all n, by condition (3.1.3) we consider:

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\le cg\left[d(Fx_{2n}, Tx_{2n+1}), d(Gx_{2n+1}, Sx_{2n}), d(Fx_{2n}, Gx_{2n+1})\right] \\ &= cg\left[d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n})\right] \\ &\le 2cd(y_{2n-1}, y_{2n}) \end{aligned}$$

Thus by property of g get:

$$d(y_{2n}, y_{2n+1}) \le 2cd(y_{2n-1}, y_{2n}) \tag{1}$$

Similarly by condition (3.1.3) have:

$$\begin{aligned} d(y_{2n-1}, y_{2n}) &= d(Tx_{2n-1}, Sx_{2n}) \\ &= d(Sx_{2n}, Tx_{2n-1}) \\ &\le cg\left[d(Fx_{2n}, Tx_{2n-1}), d(Gx_{2n-1}, Sx_{2n}), d(Fx_{2n}, Gx_{2n-1})\right] \\ &= cg\left[d(y_{2n-1}, y_{2n-1}), d(y_{2n-2}, y_{2n}), d(y_{2n-1}, y_{2n-2})\right] \\ &\le 2cd(y_{2n-2}, y_{2n-1}) \end{aligned}$$

Thus $d(y_{2n-1}, y_{2n}) \le 2cd(y_{2n-2}, y_{2n-1})$ (2)

Generally by conditions (1) and (2) show that,

$$d\left(y_n, y_{n+1}\right) \le 2cd\left(y_{n-1}, y_n\right) \le \ldots \le \left(2c\right)^n d\left(y_0, y_1\right) \text{ for } n \in \mathbb{N}$$

We put $2c = \lambda$.

For $n, m \in N$ with $n < m$, we have

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots + d(y_{m-1}, y_m) \\ &\le \lambda^n d(y_0, y_1) + \lambda^{n+1} d(y_0, y_1) + \ldots\ldots + \lambda^{m-1} d(y_0, y_1) \\ &= \left(\lambda^n + \lambda^{n+1} + \ldots.. + \lambda^{m-1}\right) d(y_0, y_1) \\ &\le \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \end{aligned}$$

Since $0 \le \lambda < 1$, for $n, m \to \infty$ we have $d\left(y_n, y_m\right) \to 0$. Hence $\left(y_n\right)$ is a Cauchy sequence in complete dislocated metric space $\left(X, d\right)$. So there exists $z \in X$ such that $\left(y_n\right)$ dislocated converges to z. Therefore, the subsequences:

$$\left(Sx_{2n}\right) \to z, \left(Gx_{2n+1}\right) \to z, \left(Tx_{2n+1}\right) \to z \textit{ and } \left(Fx_{2n+2}\right) \to z.$$

Since $T\left(x\right) \subset F\left(x\right)$, there exists a point $u \in X$ such that $Fu = z$. Again using (3.1.3), have:

$$\begin{aligned} d\left(Su, z\right) &= d\left(Su, Tx_{2n+1}\right) \\ &\le cg\left[d\left(Fu, Tx_{2n+1}\right), d\left(Gx_{2n+1}, Su\right), d\left(Fu, Gx_{2n+1}\right)\right] \\ &= cg\left[d\left(z, Tx_{2n+1}\right), d\left(Gx_{2n+1}, Su\right), d\left(z, Gx_{2n+1}\right)\right] \end{aligned}$$

By property of g, and taking limit as $n \to \infty$, we get, $d\left(Su, z\right) \le 2cd\left(Su, z\right)$. Since $0 \le 2c < 1$, then $d\left(Su, z\right) = 0$. Since $\left(X, d\right)$ is a dislocated metric space, have $Su = z$.

Hence, $Su = Fu = z$.

Again, since $S(X) \subset G\left(X\right)$, there exists a point $v \in X$ such that $Gv = z$.

Similarly using condition (3.1.3) and the above argument we show that $Tv = z$.

Thus we have $Su = Fu = Tv = Gv = z$.

Since the pair $\left(S, F\right)$ are weakly compatible and by definition, $SFu = FSu$ implies $Sz = Fz$.

Let prove that z is the fixed point of S.

From theorem have:

$$\begin{aligned} d\left(Sz, z\right) &= d\left(Sz, Tv\right) \\ &\le cg\left[d\left(Fz, Tv\right), d\left(Gv, Sz\right), d\left(Fz, Gv\right)\right] \\ &= cg\left[d\left(Sz, z\right), d\left(z, Sz\right), d\left(Sz, z\right)\right] \\ &\le cd\left(Sz, z\right) \end{aligned}$$

Since $0 \le c < \frac{1}{2}$ we get $d\left(Sz, z\right) = 0$ and, from being $\left(X, d\right)$ a dislocated metric space have $Sz = z$.

So, this implies $Fz = Sz = z$.

Similarly, since the pair (T,G) are weakly compatible and, $TGv = GTv \Rightarrow Tz = Gz$, we show that z is the fixed point of T. Using the same argument:

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\le cg\left[d(Fz,Tz), d(Gz,Sz), d(Fz,Gz)\right] \\ &= cg\left[d(z,Tz), d(Tz,z), d(z,Tz)\right] \\ &\le 2cd(z,Tz) \end{aligned}$$

So $d(z,Tz) = 0$ since $0 \le 2c < 1$ Thus $Tz = z$ since (X,d) is a dislocated metric space.

Hence, we proved $Sz = Fz = Tz = Gz = z$. This shows that z is the common fixed point of the mappings $S,T,F,G: X \to X$.

Uniqueness: Let suppose that u and v are two fixed points of the mappings T,S,F *and* G

From condition (3.1.3) we have:

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\le cg\left[d(Fu,Tv), d(Gv,Su), d(Fu,Gv)\right] \\ &= cg\left[d(u,v), d(v,u), d(u,v)\right] \\ &\le cd(u,v) \end{aligned} \tag{4}$$

From (4) since $0 \le 2c < 1$ get $d(u,v) = 0$, which implies $u = v$ because (X,d) is a dislocated metric space. Thus fixed point is unique.

The following example illustrate theorem 3.1.

Example 3.2: Let $X = [0,1]$ and $d(x,y) = \max\{x,y\}$. Then (X,d) is a dislocated metric space. Define, $Sx = 0, Tx = \frac{x}{8}, Fx = x \text{ and } Gx = \frac{x}{4}$. Clearly T,S,F *and* G are continuous and (3.1.1) and (3.1.2) are satisfied.

We take the function $g(t_1,t_2,t_3) = \max\{t_1,t_2,t_3\}$ and also the condition (3.1.3) is satisfied for all $x,y \in X$, where $c = \frac{3}{8}$.

Clearly 0 is the unique common fixed point of T,S,F *and* G.

Corollary 3.3: [6] Let (X,d) be a complete dislocated metric space and $S,T: X \to X$ two continuous mappings satisfying the following condition:

$d(Sx,Ty) \le c\, g\left[d(x,Ty), d(y,Sx), d(x,y)\right]$ for all $x,y \in X$, where $0 \le c < \frac{1}{2}$.

Then S and T have a unique common fixed point.

Proof: Taking $F = G = I$ (identity mapping) in above theorem 3.1 and using the similar proof as in that we establish this corollary 3.3

If we put $S = T$ in corollary 3.3 we obtain the following,

Corollary 3.4: [6] Let (X,d) be a complete dislocated metric space and $T:X\to X$ a continuous mapping satisfying the following condition:

$d(Tx,Ty)\le c\,g\left[d(x,Ty),d(y,Tx),d(x,y)\right]$ for all $x,y\in X$, where $0\le c<\frac{1}{2}$.

Then T has a unique common fixed point.

Theorem 3.5: Let (X,d) be a complete dislocated metric space and $S,T,F,G:X\to X$ are continuous mappings, satisfying the conditions:

(3.5.1) $S(X)\subset G(X), T(X)\subset F(X)$

(3.5.2) the pairs (S,F) and (T,G) are weakly compatible

(3.5.3) $d(Sx,Ty)\le c\,g\left[d(Fx,Gy),d(Fx,Sx),d(Gy,Ty)\right]$ for all $x,y\in X$, and $0\le c<\frac{1}{2}$.

Then $F,G,S,$ and T have a unique common fixed point in X.

Proof: The proof of this theorem follows in the same way as in theorem 3.1 replacing the condition (3.1.3) with (3.5.3)

Corollary 3.6: Let (X,d) be a complete dislocated metric space and $S,T,F,G:X\to X$ are continuous mappings, satisfying the conditions:

(3.6.1) $S(X)\subset G(X), T(X)\subset F(X)$

(3.6.2) the pairs (S,F) and (T,G) are weakly compatible

(3.6.3) $d(Sx,Ty)\le c\max\begin{Bmatrix} d(Fx,Ty)+d(Gy,Sx) \\ d(Fx,Ty)+d(Fx,Gy) \\ d(Gy,Sx)+d(Fx,Gy)\end{Bmatrix}$ for all $x,y\in X$, and $0\le c<\frac{1}{2}$.

Then $F,G,S,$ and T have a unique common fixed point in X.

Proof: This theorem is taken as corollary of theorem 3.1 if we use the function $g_2\in G_3$.

Corollary 3.7: Let (X,d) be a complete dislocated metric space and $S,T,F,G:X\to X$ are continuous mappings, satisfying the conditions:

(3.7.1) $T(X)\subset G(X), T(X)\subset F(X)$

(3.7.2) the pairs (T,F) and (T,G) are weakly compatible

(3.7.3) $d(Tx,Ty)\le c\,g\left[d(Fx,Ty),d(Gy,Tx),d(Fx,Gy)\right]$ for all $x,y\in X$, and $0\le c<\frac{1}{2}$.

Then F,G and T have a unique common fixed point in X.

Proof: This theorem is taken as corollary of theorem 3.1 if we put $S=T$ in it.

4. CONCLUDING REMARKS

1. If we use the function $g_5\in G_3$, in theorem 3.1 we obtain as a corollary theorem 3.1 of K. Jha and D. Panthi [8]
2. If we use the function $g_5\in G_3$ and taking $F=G=I$ (identity mapping) or $F=T$ *and* $G=S$ in theorem 3.1, we generalizes the result of A. Isufati [1], improves and extend the results of K. Zoto and E. Hoxha [6], C. T. Aage and J. N. Salunke [2, 3], R. Shrivastava, Z. K. Ansari and M. Sharma [13], K. P. R. Rao and P. Rangaswamy [9] and other result in the literature.
3. Putting $F=G=I$ (identity mapping) in above corollary 3.6 we obtain the result for two continuous mappings.

4. Putting $F = G = I$ (identity mapping) and $S = T$ in above corollary 3.6 we obtain result for a continuous mapping in context of dislocated quasi-metric space.
5. Using different types of functions $g \in G_3$, such as g_3, g_4 etc in the main theorem we obtain other corollaries.
6. If we use the function $g_1 \in G_3$ in theorem 3.5 we obtain theorem 3.1 in [12].
7. If we take $S = T$ in theorem3.5 and corollary 3.6, they reduced to other corollaries, which can be considered or called generalized (F, G)-contraction for a mapping T. It is obvious that the generalized (F, G)-contraction contains the (F, G)-contraction. Furthermore, the contraction is its main subclass also (when $F = G = I$ in (F, G)-contraction).

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