International Journal of Mathematical Archive-4(6), 2013, 93-100
Available online through www.ijma.info ISSN 2229 – 5046

NEW SEPARATION AXIOMS BY M-OPEN SETS

A. I. EL-Maghrabi*- & M. A. AL-Juhani***

****Department of Mathematics, Faculty of Science, Taibah University P. O. Box, 344, AL-Madinah AL-Munawarah, K.S.A***

***Department of Mathematics, Faculty of Science, Kafr El-Sheikh University, Kafr El-Sheikh, Egypt*

(Received on: 18-05-13; Revised & Accepted on: 04-06-13)

ABSTRACT

The aim of this paper is to introduce and study new forms of separation axioms by M-open sets. Moreover; basic properties and preservation theorems of these separation axioms are investigated. Also; the relationships between them and other forms are discussed.

(2000) Mathematical Subject Classifications: *54C05; 54C08.*

Key Words and Phrases: *M-T_i-space; for i = 0, 1, 2. M-regular, M-normal spaces.*

1. INTRODUCTION

A.I. EL-Maghrabi and M.A. AL-Juhani [5] introduced and investigated the notions of M-open sets. By this concept we define and investigate many topological properties. In recent literature, we find many topologists had focused their research in the direction of investigating different types of separation axioms. This paper is devoted to introduce and investigate a new class of separation axioms called M-T_i-spaces, i = 0, 1, 2. Also, the M-regularity and the M-normality are examined in the context of these new concepts. Further, some of fundamental properties of them are studied.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) (Simply, X and Y) represent topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure of subset A of X, the interior of A and the complement of A is denoted by cl(A), int(A) and A^c or X\A, respectively. A subset A of a space (X, τ) is called regular open [15] if A= int(cl(A)). A point $x \in X$ is said to be a θ-interior of A [11] if there exists an open set U containing x such that $U \subseteq cl(U) \subseteq A$. The set of all θ-interior points of A is said to be the θ-interior set and denoted by $int_\theta(A)$. A subset A of X is called θ-open (resp. δ-open [11] if $A = int_\theta(A)$ (resp. It is the union of regular open sets). The complement of δ-open set is called δ-closed. The δ-interior of a subset A of X is the union of all δ-open sets of X contained in A and denoted by $int_\delta(A)$. A subset A of a space (X,τ) is called preopen [12] or locally dense [2] (resp. δ-preopen [14], semi-open [10], δ-semi-open [13], θ-semi-open [3] , e-open [3], if $A \subseteq int(cl(A))$ (resp. $A \subseteq int(cl_\delta(A))$, $A \subseteq cl(int(A))$, $A \subseteq cl(int_\delta(A))$, $A \subseteq cl(int_\theta(A))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$,

A subset A of a space (X,τ) is called M-open [5] if $A \subseteq cl(int_\theta(A)) \cup int(cl_\delta(A))$.

The complement of preopen (resp. δ-preopen, semi-open, e-open, δ-semi-open, θ-semi-open, θ-open, M-open) set is called preclosed (resp. δ-preclosed, semiclosed, e-closed, δ-semiclosed, θ-semiclosed, θ-closed, M-closed). The family of all preopen (resp. δ-preopen, semi-open, θ-semi-open, e-open, δ-semi-open , θ-open, M-open) is denoted by PO(X) (resp. δ-PO(X), SO(X), θ-SO(X), e-O(X), δ-SO(X), θ-O(X), MO(X)). The union of all M-open (resp. θ-open, θ-semi-open, δ-preopen, e-open) sets contained in A is called the M-interior [5] (resp. θ-interior [11], θ-semi-interior [1], δ-pre-interior [14], e-interior [3]) of A and it is denoted by M-int(A) (resp. $int_\theta(A)$, $sint_\theta(A)$, $pint_\delta(A)$, e-int(A)). The intersection of all M-closed (resp.θ-semi-closed, δ-preclosed, e-closed) sets containing A is called the M-closure [5] (resp. θ-semi-closure [1], δ-preclosure [14], e-closure [3]) of A and it is denoted by M-cl(A) (resp. $scl_\theta(A)$, $pcl_\delta(A)$, e-cl(A)). A point $x \in X$ is called a θ-cluster [11] (resp. δ-cluster [16]) point of A if $cl(U) \cap A \neq \phi$ (resp. $int(cl(U)) \cap A \neq \phi$) for every open set U of X containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the θ-closure (resp. δ-closure) of A and is denoted by $cl_\theta(A)$ (resp. $cl_\delta(A)$).

Corresponding author: *M. A. AL-Juhani**
Department of Mathematics, Faculty of Science, Taibah University P. O. Box, 344, AL-Madinah AL-Munawarah, K.S.A

Definition 2.1: For a space (X, τ), a point $p \in X$ is called an M-limit point of A [5] if for every M-open set G containing p contains a point of A other than p . The set of all M-limit points of A is called M-derived set of A and is denoted by M-d(A).

Lemma 1.1: [5] For a space (X, τ), a subset A of X is called M-closed if and only if M-cl(A) = A∪M-d(A).

Definition 2.2: A function f: $(X, \tau) \to (Y, \sigma)$ is called
(i) M-continuous [6] if, $f^{-1}(U) \in MO(X)$, for each $U \in \sigma$,
(ii) M-irresolute [6] if, $f^{-1}(U) \in MO(X)$, for each $U \in MO(Y)$,
(iii) M-open [7] if, $f(U) \in MO(Y)$, for each $U \in \tau$,
(iv) pre-M-open [7] if, $f(U) \in MO(Y)$, for each $U \in MO(X)$,
(v) super M-open if, $f(U) \in \sigma$, for each $U \in MO(X)$,
(vi) M-homeomrphism, if , f is bijective, M-irresolute and pre-M-open.

Definition 2.3: A topological space (X, τ) is said to be:

(i) δ-regular if for every closed set F of X and each point $x \in X$ and $x \in X \setminus F$, there exist two disjoint δ-open sets U,V such that $F \subseteq U$ and $x \in V$,

(ii) δp-regular [9] if for every closed set F of X and each point $x \in X$ and $x \in X \setminus F$, there exist two disjoint δp-open sets U,V such that $F \subseteq U$ and $x \in V$,

(iii) θs-regular if for every closed set F of X and each point $x \in X$ and $x \in X \setminus F$, there exist two disjoint θ-semi-open sets U, V such that $F \subseteq U$ and $x \in V$,

(iv) e-regular [17] if for each closed set F and each point $x \notin F$, there exist two disjoint e-open sets U and V such that $F \subseteq U$ and $x \in V$,

(v) θ-regular if for each closed set F and each point $x \notin F$, there exist two disjoint θ-open sets U and V such that $F \subseteq U$ and $x \in V$,

(vi) θs-normal if for any pair of disjoint closed sets F_1 and F_2 of X, there exist two disjoint θ-semi-open sets U, V such that $F_1 \subseteq U$ and $F_2 \subseteq V$,

(vii) e-normal [8] if for every pair of disjoint closed sets F_1 and F_2 of X, there exist two disjoint e-open sets U, V such that $F_1 \subseteq U$ and $F_2 \subseteq V$,

(viii) θ-normal if for every pair of disjoint closed sets F_1 and F_2 of X, there exist two disjoint θ-open sets U, V such that $F_1 \subseteq U$ and $F_2 \subseteq V$,

(ix) δ-normal if for every pair of disjoint closed sets F_1, F_2 of X, there exist two disjoint δ-open sets U,V such that $F_1 \subseteq U$ and $F_2 \subseteq V$,

(x) δp-normal[4] if for every pair of disjoint closed sets F_1, F_2 of X, there exist two disjoint δ-preopen sets U,V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Lemma 1.2: [6] For a space (X, τ). If $A \in \theta\text{-}O(X, \tau)$ and $B \in MO(X, \tau)$, then $A \cap B \in MO(X, \tau_A)$.

3. M-T_i-SPACES

Definition 3.1: A space (X, τ) is called

1) M-T_0 if for every two distinct points x, y of X, there exists an M-open set U such that either $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$,

2) M-T_1 if for every two distinct points x, y of X, there exist two M-open sets U, V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$,

3) M-T_2 or M-Hausdorff if for every two distinct points x, y of X, there exist two disjoint M-open sets U, V such that $x \in U$, $y \in V$.

Remark 3.1: The implication between some types of separation axioms are given by the following diagram.

$$\begin{tikzcd}
T_2\text{-space} \arrow[r] \arrow[d] & T_1\text{-space} \arrow[r] \arrow[d] & T_0\text{-space.} \arrow[d] \\
M\text{-}T_2\text{-space} \arrow[r] & M\text{-}T_1\text{-space} \arrow[r] & M\text{-}T_0\text{-space.}
\end{tikzcd}$$

The converse of these implications are not true in general and the following examples.

Example 3.1: A Sierpinski space X = {0, 1} with τ = {X,ϕ, {0}} is M-T_0 ,but not M-T_1-space.

Example 3.2: Let X = {a, b, c, d} and τ ={X,ϕ, {a},{b},{a, b},{a, b, c},{a, b, d}}. Then X is M-T_1, but not M-T_2-space.

Example 3.3: Let X = {a, b, c, d} and τ ={X,ϕ, {a, c},{b, d},{d},{a, c, d}}.Then X is an M-T_2-space**.**

Theorem 3.1: For a space (X, τ), the following statements are equivalent:
(i) X is an M-T_0-space,
(ii) for every two distinct points x , y $\in$X , M-cl({x}) $\neq$ M-cl({y}).

Proof: (i) → (ii). For every x, y of X and $x \neq$ y. Since X is an M-T_0-space, then there exists an M-open set U such that $x\in$U , y$\notin$U. So, y$\in$ X\U. Hence, {y}$\subseteq$ X\U which is an M-closed set, M-cl{y} $\subseteq$ X\U and hence $x\notin$ M-cl{y}. Therefore, M-cl({x}) $\neq$ M-cl({y}).

(ii)→ (i). Suppose that for every x, y $\in$ X, $x \neq$ y and M-cl({x}) $\neq$ M-cl({y}). Let z $\in$ X such that z $\in$ M-cl({x}), hence z $\notin$ M-cl({y}). If, x $\in$ M-cl({y}), then {x} $\subseteq$ M-cl({y}) which implies that M-cl({x}) $\subseteq$ M-cl({y}) and hence z $\in$ M-cl({y}) which is a contradiction, thus x $\notin$ M-cl({y}) which implies x $\in$ (M-cl({y}))^c and hence (M-cl({y}))^c is an M-open set containing x but not y. Therefore, X is M-T_0.

Definition 3.2: A function f: (X, τ) →(Y, σ) is said to be injective point M-closure if and only if for every x, y$\in$ X such that M-cl({x}) $\neq$ M-cl({y}), then M-cl(f(x)) $\neq$ M-cl(f(y)).

Theorem 3.2: If, f: (X, τ) →(Y, σ) is an injective point M-closure and X is an M-T_0-space, then f is injective.

Proof; Let x, y $\in$X and $x \neq$ y. Since X is an M-T_0-space, then M-cl({x}) $\neq$ M-cl({y}). Where f is injective point M-closure, then M-cl(f(x)) $\neq$ M-cl(f(y)), hence f(x) $\neq$ f(y). Therefore, f is injective.

Remark 3.2; An M-T_0-space is not hereditary property.

Example 3.4: Suppose that X = {a, b, c, d} with topology τ ={X, ϕ,{a},{c},{a, c}} and Y = {a, b, c} $\subseteq$ X. with topology τ_Y = {Y, ϕ,{a}, {c},{a, c}}.Then X is an M-T_0-space but (Y, τ_Y) is not M-T_0. Since b, c $\in$ Y and b $\neq$ c but no exists U$\in$MO(Y) such that b$\in$U, c $\notin$U or b$\notin$U, c$\in$ U.

Corollary 3.1: An M-T_0-space is a topological property.

Proof: Let f: (X, τ)→(Y, σ) be M-homeomorphism and x, y $\in$ X such that $x \neq$ y. Since f is injective, then f(x) $\neq$ f(y). Since X is M-T_0, then there exists an M-open set G such that x$\in$G, y$\notin$G. Since f is pre-M-open, then f(G) is M-open in Y such that f(x) $\in$ f(G), f(y) $\notin$ G. Hence, Y is M-T_0.

Theorem 3.3: Let X be a topological space. Then the following statements are equivalent:
(i) X is an M-T_1-space,
(ii) for every point $x\in$X the singleton set {x} is M-closed,
(iii) for every point $x\in$X, M-d({x}) = ϕ.

Proof: (i) → (ii). For every x, y$\in$X, $x \neq$ y. Since X is an M-T_1-space, then there exists an M-open set U containing y but not x. Hence, y$\in$ U $\subseteq$ X\{x}. Thus X\{x} =$\cup${U: U is M-open, y$\in$ X\{x}} which is the union of an M-open sets. Then X\{x} is M-open. Therefore, {x} is M-closed.

(ii) → (i). For every x, y$\in$X, x $\neq$ y. By hypothesis {x}, {y} are M-closed sets. Therefore, X\{x}, X\{y} are M-open sets such that x$\notin$ X\{x}, y$\in$ X\{x} and x$\in$ X\{y}, y$\notin$ X\{y}. Therefore, X is M-T_1.

(ii) → (iii). For every $x\in$X, {x} is an M-closed set, hence {x} = M-cl{x} = {x} $\cup$ M-d({x}). Therefore, M-d({x}) = ϕ.

(iii) → (ii). For every $x\in$X, M-d($\{x\}$) = ϕ. But M-cl($\{x\}$) = $\{x\} \cup$ M-d($\{x\}$), hence, M-cl($\{x\}$) = $\{x\}$ if and only if $\{x\}$ is M-closed.

Corollary 3.2; An M-T_1-space is a topological property.

Proof: Let f: (X, τ) →(Y, σ) be M-homeomorphism and x, y $\in$ X such that $x \neq$ y. Since f is injective, then f(x) $\neq$ f(y). Since X is M-T_1, then there exist two M-open sets U, V such that $x\in$U, y$\notin$U and $x \notin$V, y$\in$V. Since f is pre-M-open, then f(U), f(V) are M-open in Y such that f(x)$\in$f(U), f(y) $\notin$ f(U) and f(x)$\notin$f(V), f(y)$\in$f(V). Hence, Y is M-T_1.

Theorem 3.4: Let X be a topological space. Then the following statements are equivalent:
(i) X is an M-T_2-space**,**
(ii) If $x\in$X, then for each $x \neq$ y, there exist an M-open set U containing x such that y$\notin$M-cl(U).

Proof: (i) → (ii). Let $x\in$X. Then for each $x \neq$ y, there exist an M-open sets U and V such that $x\in$U, y$\in$V and U$\cap$V=ϕ. Hence, $x\in$U$\subseteq$ X\V. Put X\V= F. Then F is M-closed, U $\subseteq$ F and y$\notin$ F. That implies y$\notin\cap$ {F: F is M-closed and U $\subseteq$ F} = M-cl(U).

(ii)→ (i). Let x, y$\in$X and $x \neq$ y. By hypothesis, there exist M-open set U containing x such that y$\notin$M-cl(U). Thus y$\in$X\(M-cl(U)) which is M-open and $x\notin$ X\(M-cl(U)). Also, U$\cap$(X\M-cl(U)) = ϕ. Therefore, X is M-T_2.

Remark 3.3: An M-T_2-space is not hereditary property.

Example 3.5: Let X = {a, b, c, d} with topology τ ={X, ϕ,{a},{b},{a, b}, {a, d}, {a, b, d}, {a, c, d}} and Y = {a, b, c} $\subseteq$ X. with topology τ_Y = {Y, ϕ,{a},{b},{a, b}, {a, c}}.Then (X, τ) is M-T_2-space but (Y, τ_Y) is not M-T_2. Since a,c $\in$ Y and a $\neq$ c but no exists U,V $\in$ MO(Y) such that a $\in$ U, c $\in$V and U $\cap$ V = ϕ .

Theorem 3.5: Every θ-open subspace of an M-T_i-space is M-T_i, where i = 0, 1, 2.

Proof: We prove that the theorem for M-T_2-space.

Let Y be a θ-open subspace of an M-T_2-space (X, τ) and x, y $\in$Y such that $x \neq$y. Then x, y be two distinct points of X. But, X is M-T_2, then there exist two disjoint M-open sets U, V with $x\in$U and y$\in$V. Suppose that U_1 = Y $\cap$ U and V_1 = Y $\cap$ V. Hence by Lemma 1.2., U_1, V_1 are M-open sets of Y containing x, y respectively and $U_1 \cap V_1$ = Y $\cap$ (U $\cap$ V) = ϕ. Therefore, (Y, τ_Y) is M-T_2.

Corollary 3.3; An M-T_2-space is a topological property.

Proof; Let f: (X, τ) →(Y, σ) be M-homeomorphism and x, y $\in$ X such that $x \neq$ y. Since f is injective, then f(x) $\neq$ f(y). Since X is M-T_2, then there exist two disjoint M-open sets U, V such that $x\in$U, y$\in$V. Since f is pre-M-open, then f(U), f(V) are two disjoint M-open sets in Y such that f(x)$\in$f(U) and f(y)$\in$f(V). Hence, Y is M-T_2.

Theorem 3.6: If, f:(X, τ)→(Y, σ) is an injective M-continuous function and Y is T_i-space, then X is M-T_i, where i = 0, 1, 2.

Proof; We prove that the theorem for M-T_0-space.

Let x_1, x_2 $\in$ X and $x_1 \neq x_2$. Since f is injective, then f(x_1) $\neq$ f(x_2) in Y. But, Y is T_0, then there exist an open set U such that f(x_1) $\in$ U, f(x_2) $\notin$ U or f(x_2) $\in$ U , f(x_1) $\notin$ U. Since f is M-continuous, then f^{-1}(U) is an M-open set of X such that $x_1\in f^{-1}$(U), $x_2 \notin f^{-1}$(U) or $x_2\in f^{-1}$(U), $x_1 \notin f^{-1}$(U). Therefore, X is M-T_0.

Theorem 3.7: If, f: (X, τ) →(Y, σ) is an injective M-irresolute function and Y is an M-T_i-space, then X is M-T_i, where i = 0, 1, 2

Proof: We prove that the theorem for M-T_2-space.

Let x, y $\in$X and $x \neq$ y. Since f is injective, then f(x) $\neq$ f(y). But, Y is M-T_2, then there exists two disjoint M-open sets U, V of Y such that f(x)$\in$U, f(y)$\in$V . By using M-irresoluteness of f, then f^{-1}(U), f^{-1}(V) are M-open set of X such that $x\in f^{-1}$(U), y $\in f^{-1}$(V) and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Therefore, X is M-$T_2$.

Theorem 3.8; If, f:(X, τ)→(Y, σ) is a bijective M-open function and X is a T_i-space, then Y is M-T_i, where i = 0,1,2.

Proof: We prove that the theorem for M-T_2-space.

Let y_1, y_2 be two distinct points in Y. Since f is bijective, then there exist $x_1, x_2 \in$ X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is T_2, then there exist two disjoint M-open sets U, V in X such that $x_1 \in U$, $x_2 \in V$. Since f is M-open, then f(U), f(V) are M-open in Y with $y_1 \in f(U)$, $y_2 \in f(V)$. Therefore, Y is M-T_2.

Theorem 3.9: If, f: $(X, \tau) \to (Y, \sigma)$ is a bijective super M-open function and X is an M-T_0-space, then Y is T_0, where i = 0, 1, 2.

Proof; We prove that the theorem for T_0-space.

Let y_1, y_2 be two distinct points in Y. Since f is bijective, then there exist $x_1, x_2 \in$ X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is M-T_0 then there exist an M-open set U in X such that $x_1 \in U$, $x_2 \notin U$ or $x_2 \in U$, $x_1 \notin U$. Since f is super M-open, then f(U) is open in Y with $y_1 \in f(U)$, $y_2 \notin f(U)$ or $y_2 \in f(U)$, $y_1 \notin f(U)$. Therefore, Y is T_0.

Theorem 3.10: If, f:$(X, \tau) \to (Y, \sigma)$ is a bijective pre-M-open function and X is an M-T_i-space, then Y is M-T_i, where i = 0, 1, 2.

Proof: We prove that the theorem for M-T_0-space.

Let y_1, y_2 be two distinct points in Y. Since f is bijective, then there exist $x_1, x_2 \in$ X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is M-T_0 then there exist an M-open set U in X such that $x_1 \in U$, $x_2 \notin U$ or $x_2 \in U$, $x_1 \notin U$. Since f is pre-M-open, then f(U) is M-open in Y with $y_1 \in f(U)$, $y_2 \notin f(U)$ or $y_2 \in f(U)$, $y_1 \notin f(U)$. Therefore, Y is M-T_0.

4. M-REGULAR SPACE

Definition 4.1: A space X is said to be M-regular if for every closed set F of X and each point $x \in X$ such that $x \in X \backslash F$, there exist two disjoint M-open sets U, V such that $F \subseteq U$ and $x \in V$.

Remark 4.1: The implication between some types of topological spaces are given by the following diagram.

$$\begin{tikzcd}
\theta\text{-regular} \arrow[r] \arrow[d] & \delta\text{-regular} \arrow[r] & \delta p\text{-regular} \arrow[d] & \\
\theta s\text{-regular} \arrow[rr] & & \textbf{M-regular} \arrow[r] & e\text{-regular}
\end{tikzcd}$$

None of these implications is reversible by [9, 17] and the following examples.

Example 4.1: Let X = {a, b, c, d} and τ ={X,ϕ, {a},{b, c},{a, b, c}}. Then X is an M-regular space but not δp-regular and it is not θs-regular.

Example 4.2: Let X = {a, b, c, d} and τ ={X,ϕ, {a},{b},{a, b},{a, b, c},{a, b, d}}. Then X is an e-regular space but it is not M-regular.

Theorem 4.1: Let X be a space. Then the following statements are equivalent:
(i) X is M-regular,
(ii) For each closed set $F \subseteq X$ and $x \in X \backslash F$, there exists an M-open set U such that $x \in U \subseteq$ M-cl(U) $\subseteq X \backslash F$.

Proof; (i) → (ii). Let X be an M-regular space, $F \subseteq X$ and $x \notin F$. Then there exist two disjoint M-open sets U, V such that $x \in U$ and $F \subseteq V = X \backslash$M-cl(U). Since $F \subseteq X \backslash$M-cl(U), then M-cl(U) $\subseteq X \backslash F$. Therefore, $x \in U \subseteq$ M-cl(U) $\subseteq X \backslash F$.

(ii) → (i). Let $x \in X$ and $F \subseteq X \backslash \{x\}$ be a closed set such that $x \in U \subseteq$ M-cl(U) $\subseteq X \backslash F$. Then $F \subseteq X \backslash$M-cl(U) which is an M-open set and disjoint with U. Therefore, X is M-regular.

Theorem 4.2: In an M-regular space, for any two points x, y of X then either M-cl($\{x\}$) = M-cl({y}) or M-cl($\{x\}$) $\cap$ M-cl({y}) = ϕ .

Proof: Let be M-cl($\{x\}$) $\neq$ M-cl({y}) then either $x \notin$ M-cl({y}) or y $\notin$ M-cl($\{x\}$). Suppose that y $\notin$ M-cl($\{x\}$). Since X is M-regular, then there exists an M-open set G such that M-cl($\{x\}$) $\subseteq$ G and y $\in X \backslash G$. Where $X \backslash G$ is M-closed and M-cl({y}) $\subseteq X \backslash G$. Hence M-cl($\{x\}$) $\cap$ M-cl({y}) $\subseteq G \cap (X \backslash G) = \phi$.

Theorem 4.3: If, f: $(X, \tau) \to (Y, \sigma)$ is a bijective continuous and pre-M-open functions and X is an M-regular space, then Y is M-regular.

Proof: Let $F \subseteq Y$ be a closed set and $y \in Y \backslash F$. Since f is bijective continuous, then $f^{-1}(F)$ is closed of X. Put $f(x) = y$, then $x \in X \backslash f^{-1}(F)$. Since X is M-regular, then there exist two disjoint M-open sets U, V such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since f is bijective and pre-M-open, then $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore, Y is M-regular.

Theorem 4.4: If, f: $(X, \tau) \rightarrow (Y, \sigma)$ is an injective M-irresolute and closed functions and Y is an M-regular space, then X is M-regular.

Proof; Let $F \subseteq X$ be a closed set and $x \notin F$. Since f is injective closed, then f(F) is closed of Y and $f(x) \notin f(F)$, hence, $f(x) \in Y \backslash F$. By M-regularity of Y, there exist two disjoint M-open sets U, V such that $f(F) \subseteq U$ and $f(x) \in V$. Since f is M-irresolute, then $F \subseteq f^{-1}(U)$ and $x \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence, X is M-regular.

Remark 4.2: An M-regular-space is not hereditary property.

Example 4.3: In Example 3.4, X is an M-regular space but (Y, τ_Y) is not M-regular. Since {b, c} is a closed set of Y and $a \notin \{b, c\}$ but no exists $U, V \in MO(Y)$ such that $\{b, c\} \subseteq U$, $a \in V$ and $U \cap V = \phi$.

Theorem 4.5: Every θ-open subspace of an M-regular space is M-regular.

Proof: Let Y be a θ-open subspace of an M-regular space (X, τ) and F be a closed set of Y with $x \notin F$. Then $F \subseteq X$ be a closed set with $x \notin F$. By M-regularity of X, there exist two disjoint M-open sets U, V such that $F \subseteq U$ and $x \in V$. Let $U_1 = Y \cap U$ and $V_1 = Y \cap V$. Hence by Lemma 1.2., U_1, V_1 are M-open of Y containing F, x respectively and $U_1 \cap V_1 = Y \cap (U \cap V) = \phi$. Therefore, (Y, τ_Y) is M-regular.

Corollary 4.1: An M-regular-space is a topological property.

Proof: Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be an M-homeomorphism. Then f is a bijective pre-M-open continuous function. Assume that $F \subseteq Y$ is a closed set and $y \in Y \backslash F$, then $f^{-1}(F)$ is closed set of X and $x \in X \backslash f^{-1}(F)$. Since X is M-regular, then there exist two disjoint M-open sets U, V such that $x \in U$ and $f^{-1}(F) \subseteq V$. By using pre-M-open, $y \in f(U)$ and $F \subseteq f(V)$ such that $f(U) \cap f(V) = \phi$. Therefore, Y is M-regular.

5. M-NORMAL SPACE

Definition 5.1: A space X is called M-normal if for every pair of disjoint closed sets F_1 and F_2, there exist two disjoint M-open sets U, V such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Remark 5.1: The implication between some types of topological spaces are given by the following diagram.

$$\begin{tikzcd} \theta\text{-normal} \arrow[r] \arrow[d] & \delta\text{-normal} \arrow[r] & \delta p\text{-normal} \arrow[d] & \\ \theta s\text{-normal} \arrow[rr] & & M\text{-normal} \arrow[r] & e\text{-normal} \end{tikzcd}$$

None of these implications is reversible by [4, 8] and the following examples.

Example 5.1: Let X = {a, b, c, d} and τ = {X,ϕ, {a},{b}, {a, b},{a, b, c},{a, b, d}}. Then X is e-normal space but it is not M-normal.

Example 5.2: Let X = {a, b, c, d} and τ = {X,ϕ, {a},{c}, {a, b},{a, c},{a, b, c}, {a, c, d}}. Then X is an M-normal space but it is not θs-normal.

Theorem 5.1: Every M-regular space finite is M-normal.

Proof: Let F_1, F_2, be two disjoint closed sets of X and $x \in F_1$, hence $x \notin F_2$. Since X is M-regular space, then there exist two disjoint M-open sets U, V such that $x \in U_x$, $F_2 \subseteq V_x$. Suppose that $U = \cup U_x$, $V = \cap V_x$. Since X is finite, then F_1 is finit and hence $U = \cup U_x$, $V = \cap V_x$ are M-open sets such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \cap V = \phi$. Therefore, X is M-normal.

Remark 5.2: The converse of above theorem is not true in general. Suppose that X = {a, b, c} with topology τ ={X, ϕ, {a},{b}, {a, b}, {a, c}}. Hence, X is M-normal space but it is not M-regular. Since, {b, c} is closed set and $a \notin \{b, c\}$ but no exists two disjoint M-open sets containing them.

Theorem 5.2: Let X be a topological space. Then the following statements are equivalent:

(i) X is an M-normal space,

(ii) For every pair of open sets U and V whose union is X, there exist M-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$,

(iii) For every closed set F and every open set G containing F, there exists an M-open set U such that $F \subseteq U \subseteq$ M-cl(U) $\subseteq G$.

Proof: (i) → (ii). Let U and V be two open sets in M-normal space X such that $X = U \cup V$. Then X\U, X\V are disjoint closed sets. Since X is M-normal, then there exist two disjoint M-open sets U_1, V_1 such that $X\setminus U \subseteq U_1$ and $X\setminus V \subseteq V_1$. Let $A = X\setminus U_1$, $B = X\setminus V_1$. Then A and B are M-closed sets such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(ii) → (iii). Let $F \subseteq X$ be a closed set and G be an open set containing F. Then X\F and G are open sets such that $X = X\setminus F \cup G$. Then by (ii), there exist two M-closed sets W_1, W_2 such that $W_1 \subseteq X\setminus F$, $W_2 \subseteq G$ and $W_1 \cup W_2 = X$. Then $F \subseteq X\setminus W_1$, $X\setminus G \subseteq X\setminus W_2$ and $(X\setminus W_1) \cap (X\setminus W_2) = \phi$. Let $U = X\setminus W_1$ and $V = X\setminus W_2$. Then U and V are disjoint M-open sets such that $F \subseteq U \subseteq X\setminus V \subseteq G$. Therefore, $F \subseteq U \subseteq$ M-cl(U) $\subseteq G$.

(iii)→ (i). Let F_1, $F_2 \subseteq X$ be two disjoint closed sets. Put $G = X\setminus F_2$, then $F_1 \subseteq G$ where G is an open set. By (iii) there exists an M-open set U of X such that $F_1 \subseteq U \subseteq$ M-cl(U) $\subseteq G$. It follows that $F_2 \subseteq X\setminus G \subseteq X\setminus$M-cl(U) = V. Then there exist two M-open sets U, V such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \cap V = \phi$. Therefore, X is M-normal.

Theorem 5.3: Every θ-open subspace of an M-normal space is M-normal.

Proof: Let Y be a θ-open subspace of an M-normal space (X, τ) and F_1, F_2 are two disjoint closed sets of Y. Then F_1, F_2 are two disjoint closed sets of X. By M-normality of X, there exist two disjoint M-open sets U, V such that $F_1 \subseteq U$, $F_2 \subseteq V$. Let $U_1 = Y \cap U$ and $V_1 = Y \cap V$. Hence by Lemma 1. 2., U_1, V_1 are M-open of Y containing F_1, F_2 respectively and $U_1 \cap V_1 = Y \cap (U \cap V) = \phi$. Therefore, (Y, τ_Y) is M-normal.

Theorem 5.4: If, f: (X, τ) →(Y, σ) is a surjective pre-M-open continuous and M-irresolute functions from an M-normal space X onto a space Y, then Y is M-normal.

Proof: Let F be a closed subset of Y and B be an open set containing F. Then by continuity of f, we have $f^{-1}(F)$ is closed and $f^{-1}(B)$ is open of X such that $f^{-1}(F) \subseteq f^{-1}(B)$. By M-normality of X and by Theorem 5.2, there exists an M-open set U in X such that $f^{-1}(F) \subseteq U \subseteq$ M-cl(U) $\subseteq f^{-1}(B)$.Then $f(f^{-1}(F)) \subseteq f(U) \subseteq f(\text{M-cl}(U)) \subseteq f(f^{-1}(B))$. Since f is surjective pre-M-open M-irresolute, we have $F \subseteq f(U) \subseteq$ M-cl(f(U)) $\subseteq B$. Therefore, Y is M-normal.

Theorem 5.5: If, f: (X, τ) →(Y, σ) is a bijective continuous and pre-M-open functions from an M-normal space X onto a space Y, then Y is M-normal.

Proof: Let F_1, F_2 be two disjoint closed sets of Y. Since f is continuous, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets of X. By M-normality of X, there exist two disjoint M-open sets U, V such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. By bijective and pre-M-open of f we have $F_1 \subseteq f(U)$, $F_2 \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore, Y is M-normal.

Theorem 5.6: If, f: (X, τ) →(Y, σ) is an injective closed and M-irresolute functions and Y is an M-normal space, then X is M-normal.

Proof: Let F_1, F_2 be two disjoint closed sets of X. Since f is closed function, then $f(F_1)$, $f(F_2)$ are disjoint closed sets of Y. By M-normality of Y, there exist two disjoint M-open sets U, V such that $f(F_1) \subseteq U$ and $f(F_2) \subseteq V$. By using injective and M-irresolute, we have $F_1 \subseteq f^{-1}(U)$, $F_2 \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Therefore, X is M-normal.

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Source of support: Nil, Conflict of interest: None Declared

