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ON GENERALIZED SKEW-LAPLACE DISTRIBUTION

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ABSTRACT

In this paper, we study the basic properties such as a stochastic representation and moment estimators of the class of the generalized skew-Laplace (GSL) distribution. We consider the general case by inclusion of location and scale parameters. We derive the cumulative distribution function (CDF) and the general properties of the GSL distribution in explicit form, such as: the moment generating function (MGF), characteristic function (CF), Laplace and Fourier transforms. Expressions for the $\mathbf{n^{\underline{th}}}$ *moment, skewness and kurtosis coefficients are obtained. It should be noted that some known results are obtained as special cases. Graphical illustration of the probability density function (pdf) and CDF of the GSL distribution are also given. Skewness- kurtosis graphs for this distribution have been represented. Further, we get a numerical example for skewness-kurtosis coefficients for this distribution.*

Keywords: *Generalized skew-Laplace distribution, Moment generating function, Characteristic function, Laplace and Fourier transforms, Skewness, Kurtosis.*

MSC 2010 Code**:** *60E10*; *62E10*.

1. INTRODUCTION

The fundamental properties and characterizations of symmetric distributions about origin are widely used in engineering applications and have been studied as stated in Johnson *et al.* [13]. In the recent years, there has been quite an intense activity connected to a broad class of continuous probability distributions which are generated starting from a symmetric distributions and applying a suitable form of perturbation of the symmetry. As a general result, Azzalini [9] showed that any symmetric distribution was viewed as a member of more general class of skewed distributions. Many authors have recently studied similar distributions, (see, Arnold and Beaver [4] and [5], Chang *et al.* [10], and Wahed and Ali [21]). The main feature of these models is that a skewness parameter α is introduced to control skewness and kurtosis. These models are useful in many practical situations (see, Arnold et al. [6], and Hill and Dixon [11]), and have also been used in studying robustness and as priors in Bayesian estimation (see, Mukhopadhyay and Vidakovic [16] and O'Hagan and Leonard [19]).

The skew-symmetric models defined by different researchers based on the following general result by Azzalini [9]:

Lemma 1.1: Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two arbitrary absolutely continuous independent random variables symmetric about zero, with probability density functions (pdfs) $\boldsymbol{h}(.)$ and $\boldsymbol{g}(.)$, and cumulative distribution functions (CDFs), $\boldsymbol{H}(.)$ and $\boldsymbol{G}(.)$, respectively. Then for any $\alpha \in \boldsymbol{R}$, the function

$$f(x|\alpha) = 2\, h(x)\, G(\alpha x),\ \alpha \in R \tag{1}$$

is a valid pdf of a random variable, say $\boldsymbol{X}$.

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In fact, when $\boldsymbol{h}(.)$ and $\boldsymbol{G}(.)$ are the pdf and CDF of a Laplace distribution, respectively, then (1) is called a skew-Laplace distribution with parameter $\boldsymbol{\alpha}$, ($SL(\boldsymbol{\alpha})$). Nadarajah and Kotz [17] defined skew-symmetric models with $\boldsymbol{h}(.)$ as the pdf of a Laplace distribution and $\boldsymbol{G}(.)$ being the CDF of: normal, Student's $\boldsymbol{t}$, Cauchy, logistic, and uniform distributions. Ali and Woo [1] defined skew-

symmetric distributions for a number of reflected distributions symmetric about zero and derived their moments. Ali *et al.* [2] discussed skew-symmetric models with reflected gamma kernel and $\boldsymbol{G}(.)$ as the CDF of: Laplace, double Weibull, reflected Pareto, reflected beta prime and reflected generalized uniform distributions. Ali *et al.* [3] constructed some skewed distributions with a Laplace kernel, with $\boldsymbol{h}(.)$ is taken to be a Laplace pdf and $\boldsymbol{G}(.)$ is the one of the following distributions: Laplace, double Weibull, reflected Pareto, reflected beta prime, or reflected generalized uniform distribution. They are studied the properties of the resulting distributions.

Furthermore Inusah and Kozuhowski [12] discussed many properties of $SL\ (\boldsymbol{\alpha})$ on the real line. Kozubowski and Nolan [14] showed that a $SL(\boldsymbol{\alpha})$ is infinity divisible. Arslan [7] produced some fundamental properties of the multivariate $SL(\boldsymbol{\alpha})$ and introduced some examples to demonstrate the modeling strength of $SL(\boldsymbol{\alpha})$. Nekoukhov and Altamatsaz [18] studied more general class of skew-distributions and they generalized the results of Umbach [20], and Aryal and Rao [8] which connection with truncated $SL(\boldsymbol{\alpha})$.

Mazilu [15] presented a new family of skewness distributions, and $SL(\boldsymbol{\alpha})$ including the basic properties. He defined the pdf of GSL random variable X with two parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as follows:

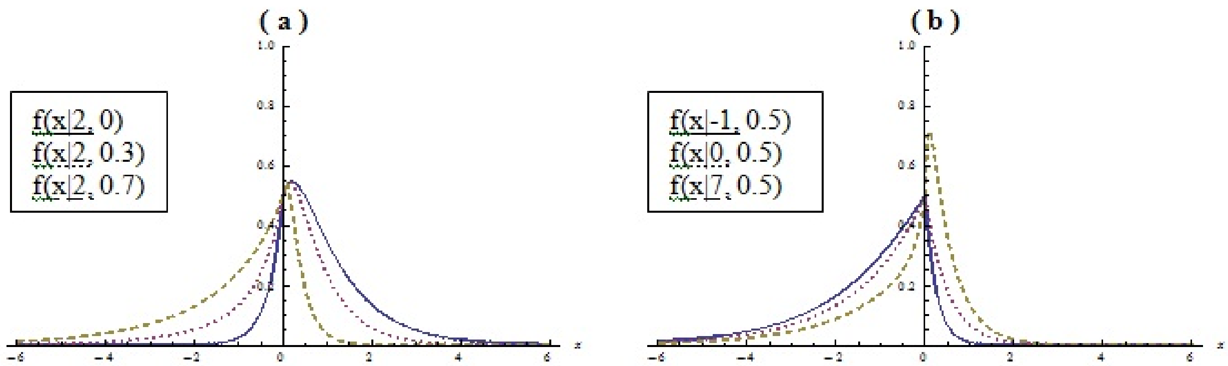
Definition 1.1: The random variable X is distributed according to the GSL distribution with parameters α and β and denote it by $X \sim GSL(\alpha, \beta)$ distribution, if its pdf is given by:

$$f(x\,|\alpha,\beta)=\begin{cases} 2\,A_1 h(A_1x)\left[\beta+\dfrac{1}{A_2}G(\alpha A_1x)\right], & if\ x<0 \\ 2h(A_2x)\ G(\alpha A_2 x), & if\ x\geq 0, \end{cases} \tag{2}$$

where $A_1 = 1 / (1 + \beta)$, $A_2 = 1 / (1 - \beta)$, $\alpha \in R$, $\beta \in [0, 1)$, $h(.)$ and $G(.)$ are the pdf and CDF of the standard Laplace distribution.

It is clear that the class of the $GSL\ (\alpha, \beta)$ distribution contains the standard Laplace distribution, (take $\alpha = 0$, and $\beta = 0$) and the $SL(\alpha)$, (take $\beta = 0$).

In this paper, we study the basic properties such as a stochastic representation and moment estimators of the class of the GSL distribution. We consider the general case by inclusion of location and scale parameters. We derive the CDF and the general properties of the $GSL(\boldsymbol{\alpha}, \boldsymbol{\beta})$ density function given in (2) in explicit form, such as: the moment generating function (MGF), characteristic function (CF), Laplace and Fourier transforms. Expressions for the $\boldsymbol{n}^{\underline{th}}$ moment, skewness and kurtosis coefficients are obtained. Graphical illustration of the probability density function (pdf) and CDF of the $GSL\ (\boldsymbol{\alpha},\boldsymbol{\beta})$ distribution are also given. Skewness-kurtosis graphs for this distribution have been represented. Further, we get a numerical example for skewness-kurtosis coefficients for this distribution, and finally conclusions of our results are presented.



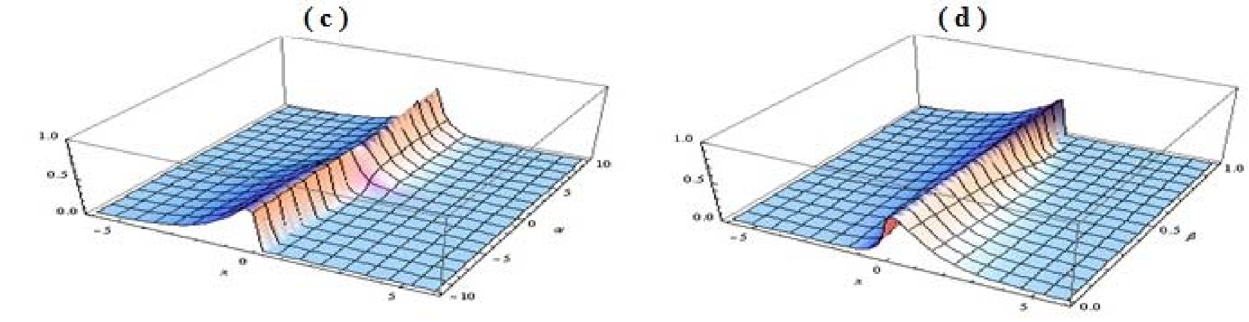


Figure 1: Examples of the $GSL(\alpha, \beta)$ density function for (a) $\alpha = 2$ and $\beta = 0$ (solid line), $\beta = 0.3$ (dotted line) and $\beta = 0.7$ (dashed line), (b) $\beta = 0.5$ and $\alpha = -1$ (solid line), $\alpha = 0$ (dotted line) and $\alpha = 7$ (dashed line), (c) $\beta = 0.5$ and $\alpha \in [-10, 10]$, and (d) $\alpha = 2$ and $\beta \in [0, 1)$.

2. SOME PROPERTIES OF THE *GSL* DISTRIBUTION

In this section, we develop a stochastic representation and some useful properties of $GSL(\alpha, \beta)$ distribution. The main idea is to notice that if the random variable $X \sim GSL\ (\alpha, \beta)$ density function defined by (2), then X can be represented as the product of two dependent random variables.

2.1 CUMULATIVE DISTRIBUTION FUNCTION

The pdf of the random variable $X \sim GSL\ (\alpha, \beta)$ density function given in (2) can be written as follows:

$$f(x \mid \alpha, \beta) = \begin{cases} A_1\left[\exp(A_1 x) - \dfrac{1}{2A_2}\exp\left(\dfrac{A_1}{B_2}x\right)\right], & \text{for } x < 0,\ \alpha < 0 \\ A_1\left[\beta\exp(A_1 x) + \dfrac{1}{2A_2}\exp\left(\dfrac{A_1}{B_1}x\right)\right], & \text{for } x < 0,\ \alpha \ge 0 \\ \dfrac{1}{2}\exp\left(\dfrac{-A_2}{B_2}x\right), & \text{for } x \ge 0,\ \alpha < 0 \\ \exp(-A_2 x) - \dfrac{1}{2}\exp\left(\dfrac{-A_2}{B_1}x\right), & \text{for } x \ge 0,\ \alpha \ge 0 \end{cases}$$

where $\boldsymbol{B_1 = 1 / (1 + \alpha)}$ and $\boldsymbol{B_2 = 1 / (1 - \alpha)}$.

Then the CDF of the random variable X is given by

$$F(x \mid \alpha, \beta) = P(X \le x) = \int_{-\infty}^{x} f(t \mid \alpha, \beta)\, dt$$

$$= \begin{cases} \exp(A_1 x) - \dfrac{B_2}{2A_2}\exp\left(\dfrac{A_1}{B_2}x\right), & \text{for } x < 0,\ \alpha < 0 \\ \beta\exp(A_1 x) + \dfrac{B_1}{2A_2}\exp\left(\dfrac{A_1}{B_1}x\right), & \text{for } x < 0,\ \alpha \ge 0 \\ 1 - \dfrac{B_2}{2A_2}\exp\left(-\dfrac{A_2}{B_2}x\right), & \text{for } x \ge 0,\ \alpha < 0 \\ 1 - \dfrac{1}{A_2}\left[\exp(-A_2 x) - \dfrac{B_1}{2}\exp\left(-\dfrac{A_2}{B_1}x\right)\right], & \text{for } x \ge 0,\ \alpha \ge 0 \end{cases}$$

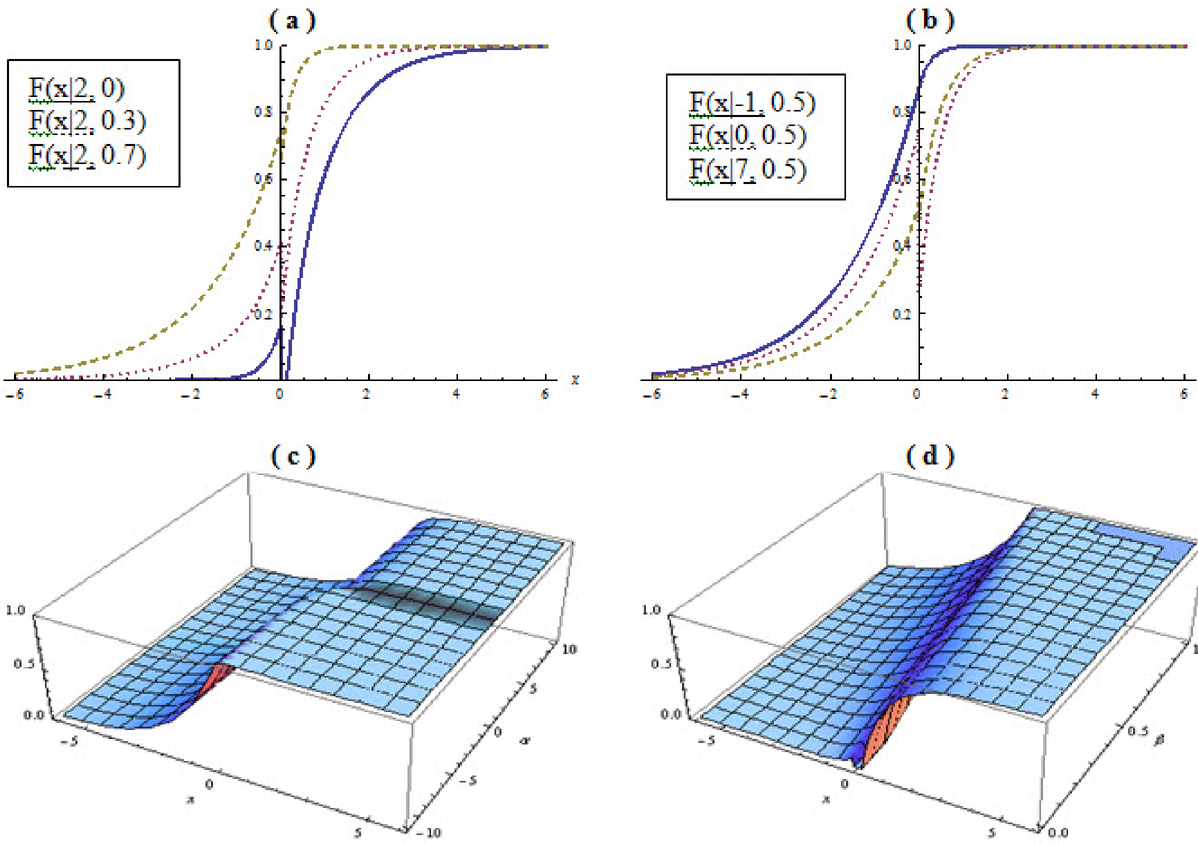


Figure 2: Examples of the $GSL(\alpha, \beta)$ distribution function for (a) $\alpha = 2$ and $\beta = 0$ (solid line), $\beta = 0.3$ (dotted line) and $\beta = 0.7$ (dashed line), (b) $\beta = 0.5$ and $\alpha = -1$ (solid line), $\alpha = 0$ (dotted line) and $\alpha = 7$ (dashed line), (c) $\beta = 0.5$ and $\alpha \in [-10, 10]$, and (d) $\alpha = 2$ and $\beta \in [0, 1)$.

2.2 Moments and Moment Generating Function

The MGF of the random variable $\boldsymbol{X} \sim GSL\,(\boldsymbol{\alpha}, \boldsymbol{\beta})$ distribution is given by the following theorem.

Theorem 2.2.1: The MGF of the random variable $\boldsymbol{X}$ with pdf $f(x|\alpha, \beta)$ defined by (2) is given by:

$$M(t) = \begin{cases} \dfrac{(1-\alpha+\beta t)^2 + t(1-\beta-\beta^2 t)}{[1+t(1+\beta)][(1-\alpha+\beta t)^2 - t^2]}, & \text{for } \alpha < \mathbf{0} \\[2ex] \dfrac{[1+t(1-\beta)][(1+\alpha+\beta t)^2 - t^2] - t(1-\beta)[(1+\beta t)^2 - t^2]}{[(1+\beta t)^2 - t^2][(1+\alpha+\beta t)^2 - t^2]}, & \text{for } \alpha \geq \mathbf{0}. \end{cases} \tag{3}$$

Proof: By using $f(x|\alpha, \beta)$ given by (2), it is then necessary to consider two cases separately, when $\alpha < 0$ and $\alpha \geq 0$. Firstly, when $\alpha < 0$, we have

$$M(t) = E\left(\exp(tx)\right) = \int_{-\infty}^{\infty} \exp(tx)\, f(x|\alpha, \beta)\,\mathrm{d}x = \frac{(1-\alpha+\beta t)^2 + t(1-\beta-\beta^2 t)}{[1+t(1+\beta)][(1-\alpha+\beta t)^2 - t^2]},$$

and, when $\alpha \geq 0$, we can obtain

$$M(t) = \frac{[1+t(1-\beta)][(1+\alpha+\beta t)^2 - t^2] - t(1-\beta)[(1+\beta t)^2 - t^2]}{[(1+\beta t)^2 - t^2][(1+\alpha+\beta t)^2 - t^2]}.$$

Consequently, the n$^{\text{th}}$ moment of the random variable $\boldsymbol{X} \sim GSL\,(\boldsymbol{\alpha}, \boldsymbol{\beta})$ distribution is given by:

$$E(X^n) = \left.\frac{d^n M(t)}{dt^n}\right|_{t=0} = \frac{n!}{2}\begin{cases} (2(-1)^n(1+\beta)^n + B_2^{n+1}[(-1)^{n+1}(1-\beta)(1+\beta)^n + (1-\beta)^{n+1}]), & \text{for } \alpha<0 \\ (2(-1)^n\beta(1+\beta)^n + 2(1-\beta)^{n+1} + B_1^{n+1}[(-1)^{n+1}(1-\beta)(1+\beta)^n - (1-\beta)^{n+1}]), & \text{for } \alpha\geq 0 \end{cases} \tag{4}$$

The first and second moments are given as follows:

$$E(X) = \begin{cases} (1-\beta)B_2^2 - (1+\beta), & \text{for } \alpha < 0 \\ (1-3\beta) - (1-\beta)B_1^2, & \text{for } \alpha \geq 0 \end{cases} \tag{5}$$

and

$$E(X^2) = \begin{cases} 2\left[(1+\beta)^2 - 2\beta(1-\beta)B_2^3\right], & \text{for } \alpha < 0 \\ 2\left[\beta(1+\beta)^2 + (1-\beta)^3 + 2\beta(1-\beta)B_1^3\right] & \text{for } \alpha \geq 0. \end{cases} \tag{6}$$

Therefore, the variance is given by

$$Var(X) = \sigma^2 = \begin{cases} \left[(1+\beta)^2 + 2\,(1-\beta^2)B_2^2 - 4\beta(1-\beta)B_2^3 - (1-\beta)^2B_2^4\right], & \text{for } \alpha < 0 \\ \left[(1+\beta)^2 + 2\,(1-\beta)(1-3\beta)B_1^2 + 4\beta(1-\beta)B_1^3 - (1-\beta)^2B_1^4\right], & \text{for } \alpha \geq 0 \end{cases} \tag{7}$$

In Equations (4) to (7), if $\boldsymbol{\beta} = 0$, the results of Mazilu [15] are obtained as special cases and if $\boldsymbol{\beta} = 1$, the results of Ali et al. [3] are obtained as special cases.

The characteristic function (CF) is the general case of the MGF which is given by the following theorem.

Theorem 2.2.2: Let $\boldsymbol{X}$ be a random variable with $GSL\,(\boldsymbol{\alpha}, \boldsymbol{\beta})$ distribution**.** Then its CF is given by

$$\Psi(t) = \begin{cases} \dfrac{\left(1-\alpha+i\beta t\right)^2 + \beta^2t^2 + it\left(1-\beta\right)}{\left[1+it(1+\beta)\right]\left[(1-\alpha+i\beta t)^2 + t^2\right]}, & \text{for } \alpha < \mathbf{0} \\ \dfrac{\left[1+it(1-\beta)\right]\left[(1+\alpha+i\beta t)^2+t^2\right] - it(1-\beta)\left[(1+i\beta t)^2+t^2\right]}{\left[(1+i\beta t)^2+t^2\right]\left[(1+\alpha+i\beta t)^2+t^2\right]}, & \text{for } \alpha \geq \mathbf{0}. \end{cases}$$

Proof: By using the formula**,** $\boldsymbol{\Psi(t) = E(\exp[itx])}$**,** $\boldsymbol{i = \sqrt{-1}}$ we can prove Theorem 2.2.2.

The results given by Equations (4) to (7) can be obtained by using the formula $E(X^n) = \left.\dfrac{1}{i^n}\dfrac{d^n\Psi(t)}{dt^n}\right|_{t=0}$.

2.3. THE SKEWNESS AND KURTOSIS COEFFICIENTS

The skewness coefficient is measured by, $\boldsymbol{\gamma_1 = M_3 / \sigma^3}$**,** and the kurtosis coefficient is measured by, $\boldsymbol{\gamma_2 = M_4 / \sigma^4}$**,** where $\boldsymbol{M_3}$ and $\boldsymbol{M_4}$ are the third and fourth moments about the mean and $\boldsymbol{\sigma^2}$ is the variance of the random variable $\boldsymbol{X}$ given by (7).

Theorem 2.3.1: The skewness coefficient of the random variable $\boldsymbol{X}$ with $GSL(\boldsymbol{\alpha}, \boldsymbol{\beta})$ distribution is given as follows

$$\gamma_1(X) = \begin{cases} \dfrac{K_1}{\sigma^3}, & \text{for } \alpha < 0 \\ \dfrac{K_2}{\sigma^3}, & \text{for } \alpha \geq 0, \end{cases} \tag{8}$$

where,

$$K_1 = -2(1+\beta)^3 + 2(1-\beta)B_2^3\left[12\beta^2B_2 + 6\beta(1-\beta)B_2^2 + (1-\beta)^2B_2^3 - 6\beta(1+\beta)\right],$$

and

$$K_2 = 2(1-9\beta+3\beta^2-3\beta^3) + 2(1-\beta)\,B_1^2\left[12\beta(1-\beta) - 6\beta(1-3\beta)B_1 - 12\beta\,B_1^2 + 6\beta(1-\beta)B_1^3 - (1-\beta)^2B_1^4\right]$$

If $\boldsymbol{\beta}$ = 0, in (8), we get the result of Mazilu [15] as a special case.

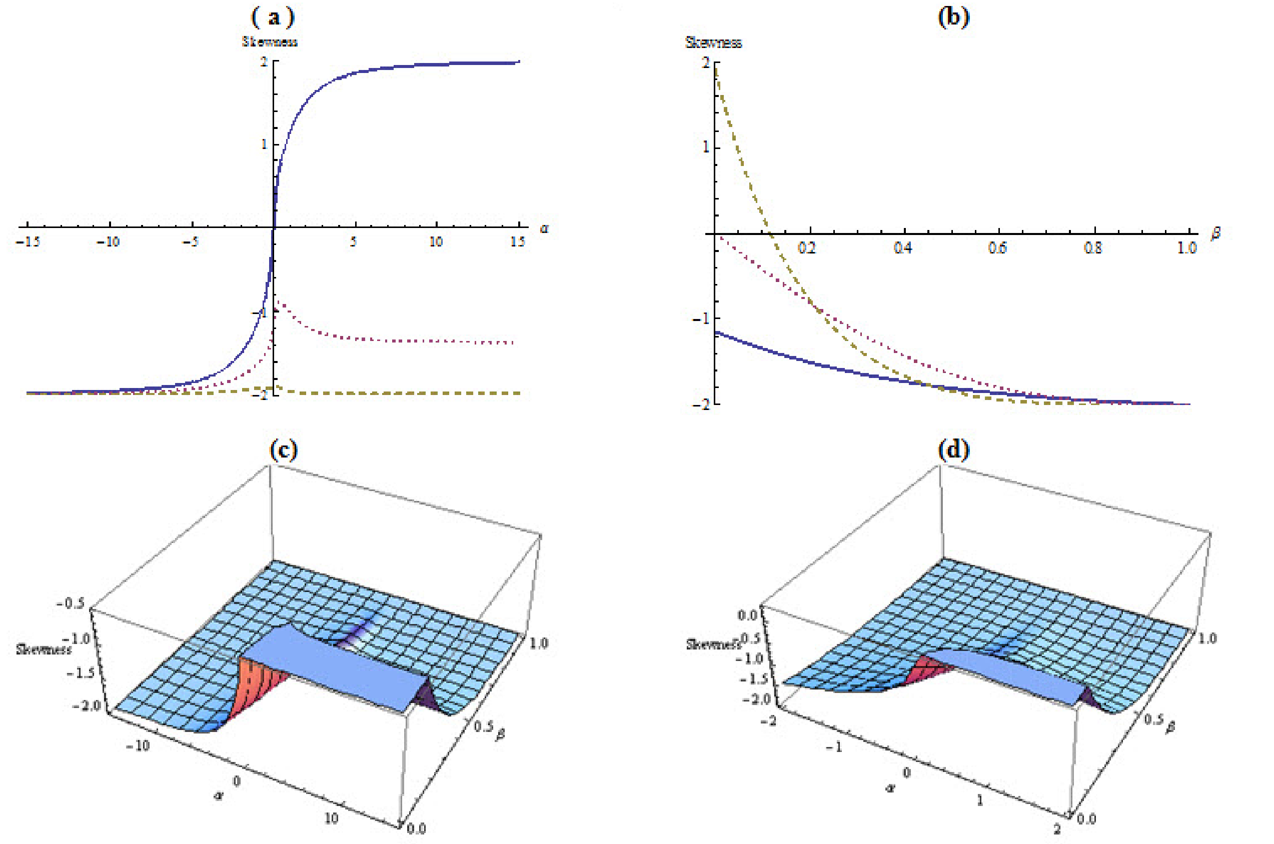


Figure 3: Examples of the skewness coefficient of $GSL(\alpha, \beta)$ distribution for (a) $\beta = 0$ (solid line), $\beta = 0.3$ (dotted line), $\beta = 0.7$ (dashed line) and $\alpha \in [-15, 15]$, (b) $\alpha = -1$ (solid line), $\alpha = 0$ (dotted line), $\alpha = 7$ (dashed line) and $\beta \in [0, 1)$, (c) $\alpha \in [-15, 15]$ and $\beta \in [0, 1)$ and (d) $\alpha \in [-2, 2]$ and $\beta \in [0, 1)$.

From equation (8), we see that the admissible intervals for the skewness if $\beta = 0$ is $-2 < \gamma_1(x) < 2$.

Theorem 2.3.2: The kurtosis coefficient of the random variable X with GSL (α, β) distribution, is given as follows

$$\gamma_2(X) = \begin{cases} \dfrac{K_3}{\sigma^4}, & \text{for } \alpha < 0 \\[2ex] \dfrac{K_4}{\sigma^4}, & \text{for } \alpha \geq 0, \end{cases} \tag{9}$$

where

$$\begin{aligned} K_3 = {} & 9(1+\beta)^4 + 6(1-\beta^2)B_2^2[2(1+\beta)^2 - 4\beta(1+\beta)B_2 + (3+13\beta^2)B_2^2] - 3(1-\beta)B_2^5[16\beta(1+3\beta^2) \\ & + 4(1-\beta)(1+7\beta^2)B_2 + 8\beta(1-\beta)^2 B_2^2 + (1-\beta)^3 B_2^3], \end{aligned}$$

and

$$\begin{aligned} K_4 = {} & 3(3+12\beta+18\beta^2-44\beta^3-37\beta^4) + 6(1-\beta)B_1^2[2(1-9\beta+11\beta^2-11\beta^3) + 4\beta(1-3\beta)^2 B_1 \\ & + (3+3\beta-21\beta^2-19\beta^3)B_1^2 + 8\beta(1+4\beta-\beta^2)B_1^3] \\ & - 3(1-\beta)^2 B_1^6\, [4(1+4\beta+3\beta^2) - 8\beta(1-\beta)B_1 + (1-\beta)^2 B_1^2]. \end{aligned}$$

The result of Mazilu [15] is obtained as a special case by putting $\boldsymbol{\beta} = 0$, in (9).

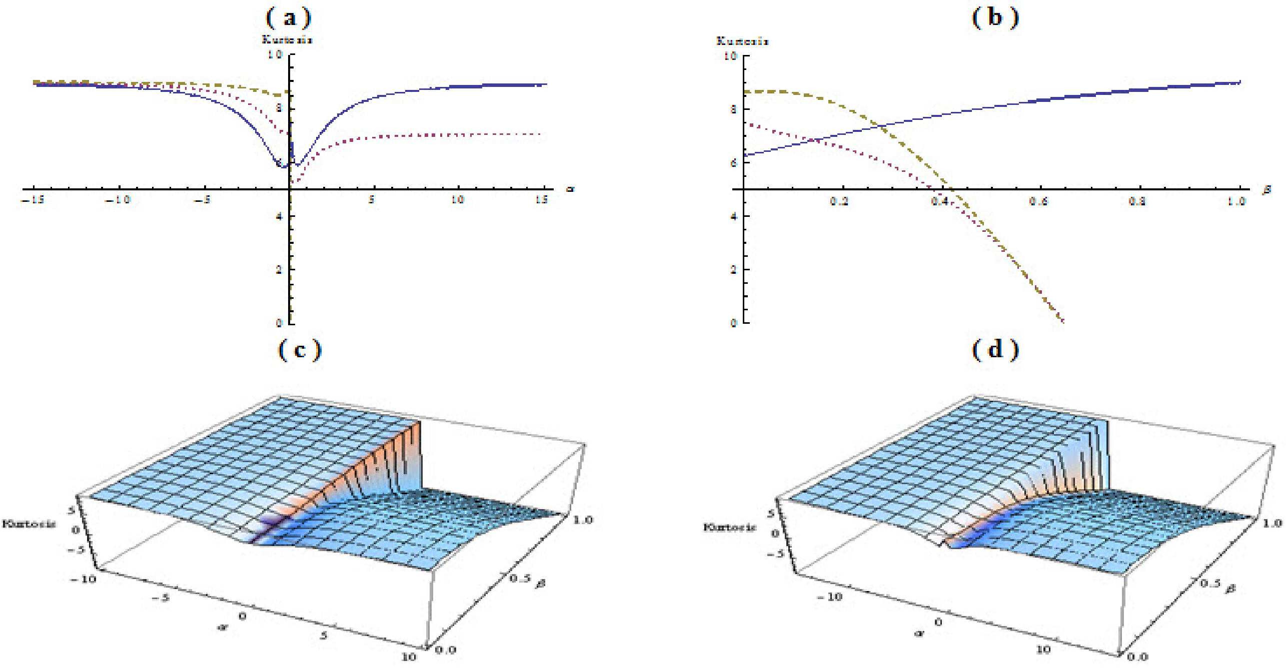


Figure 4: Examples of the kurtosis coefficient of $GSL(\alpha, \beta)$ distribution for (a) $\beta = 0$ (solid line), $\beta = 0.3$ (dotted line), $\beta = 0.7$ (dashed line) and $\alpha \in [-15, 15]$, (b) $\alpha = -1$ (solid line), $\alpha = 0$ (dotted line), $\alpha = 7$ (dashed line) and $\beta \in [0, 1)$, (c)
$\alpha \in [-10, 10]$ and $\beta \in [0, 1)$ and (d) $\alpha \in [-15, 15]$ and $\beta \in [0, 1)$.

From equation (9), we see that the admissible intervals for the kurtosis if $\boldsymbol{\beta} = 0$ is $5.8105 < \gamma_2(\boldsymbol{x}) < 8.8080$.

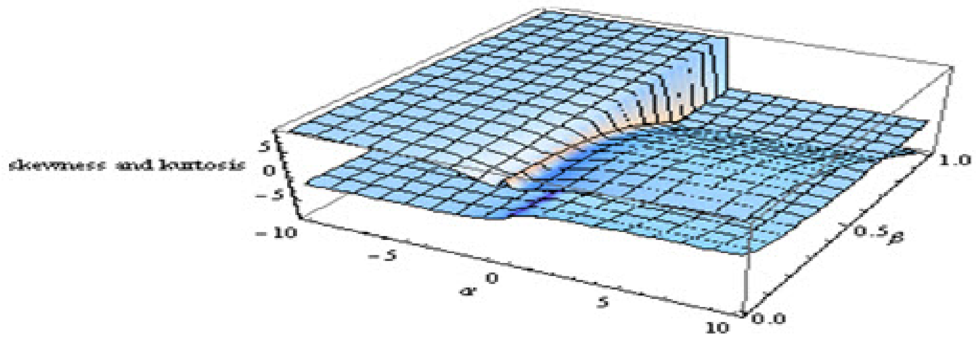


Figure 5: The skewness and kurtosis of $GSL(\alpha, \beta)$ distribution, for $\alpha \in [-10, 10]$ and $\beta \in [0, 1)$.

Other useful properties of $GSL(\boldsymbol{\alpha}, \boldsymbol{\beta})$ distribution are the Laplace and Fourier transforms, which are given by the following theorem.

Theorem 2.3.3: The Laplace and Fourier transforms of the random variable $\boldsymbol{X}$ having $GSL(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are:

$$L(t) = E(\exp(-tx)) = \begin{cases} \dfrac{(1-\alpha-\beta t)^2 - t(1-\beta+\beta^2 t)}{\left[1-t(1+\beta)\right]\left[(1-\alpha-\beta t)^2 - t^2\right]}, & \text{for } \alpha < \mathbf{0} \\[2ex] \dfrac{\left[1-t(1-\beta)\right]\left[(1+\alpha-\beta t)^2 - t^2\right] + t\,(1-\beta)[(1-\beta t)^2 - t^2]}{[(1-\beta t)^2 - t^2]\left[(1+\alpha-\beta t)^2 - t^2\right]}, & \text{for } \alpha \geq \mathbf{0} \end{cases}$$

and

$$Fo(t) = E(\exp(-itx)) = \begin{cases} \dfrac{(1-\alpha-i\beta t)^2 - it(1-\beta+i\beta^2 t)}{\left[1-it(1+\beta)\right]\left[(1-\alpha-i\beta t)^2+t^2\right]}, & for\ \alpha < \mathbf{0} \\ \\ \dfrac{\left[1-it(1-\beta)\right]\left[(1+\alpha-i\beta t)^2+t^2\right]+it\ (1-\beta)[(1-i\beta t)^2+t^2]}{[(1-i\beta t)^2+t^2]\left[(1+\alpha-i\beta t)^2+t^2\right]}, & for\ \alpha \geq \mathbf{0}. \end{cases}$$

3. TRANSFORMATION OF VARIABLES

In practice, one often works with the family of distribution generated by linear transformation $\boldsymbol{Y = \mu + \eta X}$, where $\boldsymbol{X}$ has *GSL* ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) distribution. The random variable $\boldsymbol{Y}$ gives the general class of the *GSL* ($\boldsymbol{\alpha,\beta}$) distributions by inclusion of the location parameter $\boldsymbol{\mu}$ and the scale parameter $\boldsymbol{\eta}$. It is easy to show that the random variable $\boldsymbol{Y}$ having also *GSL*($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\eta}$) distribution.

Theorem 3.1: Let $\boldsymbol{X}$ be a random variable having *GSL* ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) distribution, and $\boldsymbol{Y = \mu + \eta X}$. Then, the n<u>th</u> moment of the random variable $\boldsymbol{Y}$ is given by $E(Y^n) = \sum_{j=0}^{n} C_j^n\ \mu^{n-j}\eta^j E(X^j)$.

By elementary calculations, we can prove the theorem. Consequently, we obtain

i. $E(Y) = \begin{cases} \mu - \eta(1+\beta) + \eta(1-\beta)B_2^2 \quad , & \text{for } \alpha < 0 \\ \mu + \eta(1-3\beta) - \eta(1-\beta)B_1^2, & \text{for } \alpha \geq 0 \end{cases}$

ii. $\mathrm{Var}(Y) = \eta^2\,\mathrm{Var}(X)$, where $\mathrm{Var}(X)$ is given by (7).

iii. $\gamma_1(Y) = \eta^3\gamma_1(X)$, where $\boldsymbol{\gamma_1(X)}$ is given by (8).

iv. $\gamma_2(Y) = \eta^4\gamma_2(X)$, where $\boldsymbol{\gamma_2(X)}$ is given by (9).

4. NUMERICAL EXAMPLE

In this section, we express the flexibility of the distribution to account for a wide range for the skewness and the kurtosis.

Table (1): The range of the skewness and the kurtosis of *GSL* (α, β) distribution, for some particular values of β while $\alpha \in [-10, 10]$.

		Skewness	Kurtosis
$\boldsymbol{\beta} = 0$	$\boldsymbol{\alpha} \in [-10, 10]$	$-2 < \boldsymbol{\gamma_1(x)} < 2$	$5.8105 \leq \boldsymbol{\gamma_2(x)} \leq 8.0808$
$\boldsymbol{\beta} = 0.1$		$-2 < \boldsymbol{\gamma_1(x)} < 0.3036$	$6.158 \leq \boldsymbol{\gamma_2(x)} \leq 8.692$
$\boldsymbol{\beta} = 0.3$		$-2 < \boldsymbol{\gamma_1(x)} < -1.393$	$6.924 \leq \boldsymbol{\gamma_2(x)} \leq 8.928$
$\boldsymbol{\beta} = 0.5$		$-2 < \boldsymbol{\gamma_1(x)} < -1.826$	$3.389 \leq \boldsymbol{\gamma_2(x)} \leq 8.928$
$\boldsymbol{\beta} = 0.7$		$-2 < \boldsymbol{\gamma_1(x)} < -1.978$	$8.692 \leq \boldsymbol{\gamma_2(x)} \leq 8.928$
$\boldsymbol{\beta} = 0.9$		$-2 < \boldsymbol{\gamma_1(x)} < -1.979$	$8.928 \leq \boldsymbol{\gamma_2(x)} \leq 8.986$

For the standard Laplace distribution, $\boldsymbol{\gamma_1(x)} = 0$ and $\boldsymbol{\gamma_2(x)} = 6$, which means it is a symmetric platykurtic distribution. In order to examine how flexible the *GSL* distribution defined is, in the sense of skewness and peakedness, we draw the skewness-kurtosis graphs. The graphs have been drawn by computing ($\boldsymbol{\gamma_1(x)}$, $\boldsymbol{\gamma_2(x)}$) for the set of parameter values as follows: $\boldsymbol{\alpha} \in [-10, 10]$ and $\boldsymbol{\beta} = 0, 0.1, 0.3, 0.5, 0.7, 0.9$.

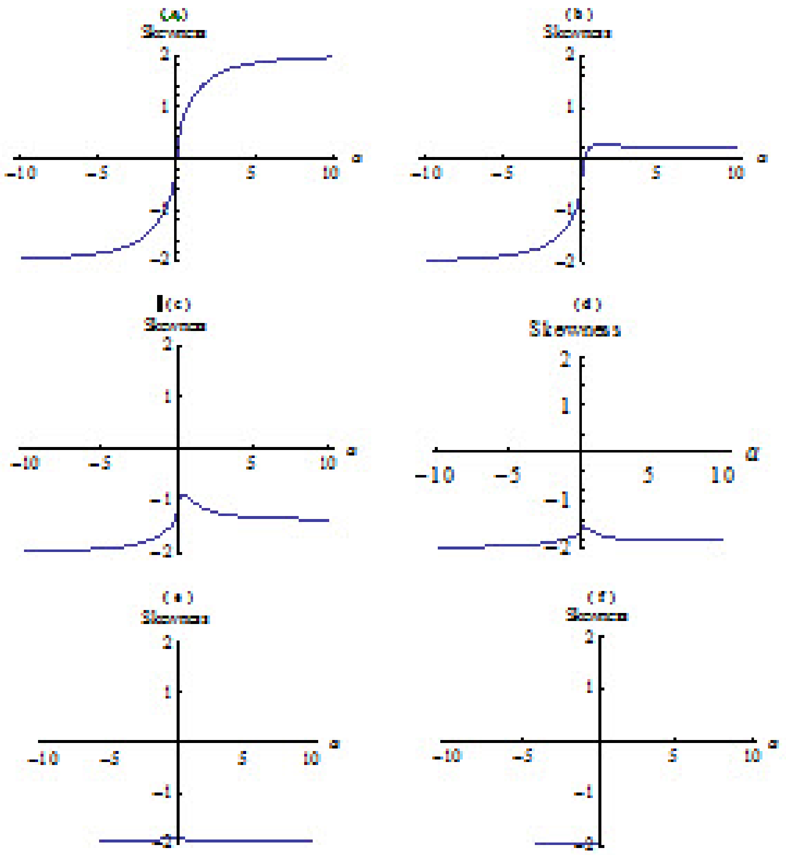


Figure 6: Examples of the skewness of *GSL* (α, β) distribution, for $\alpha \in$ [-10, 10] and (a) β = 0, (b) β = 0.1, (c) β = 0.3, (d) β = 0.5, (e) β = 0.7, and β = 0.9.

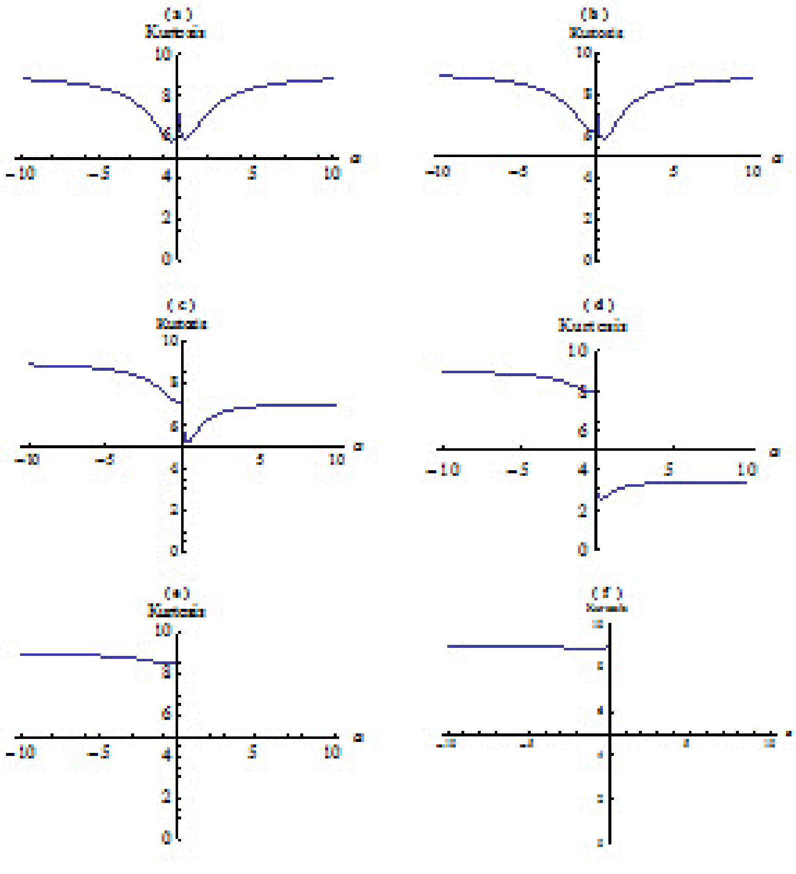


Figure 7: Examples of the kurtosis of *GSL*(α, β) distribution, for $\alpha \in$ [-10, 10] and (a) β = 0, (b) β = 0.1, (c) β = 0.3, (d) β = 0.5, (e) β = 0.7, and β = 0.9.

It is quite evident from the graphs that the *GSL* distribution investigated is very flexible in terms of exhibiting both positive and negative skewness, as well as high and low degrees of peakedness.

5. CONCLUSIONS

I. The skew parameter of the *GSL* distribution is more flexible in the sense that it can take values in a wide range than the skew parameter of the ordinary *SL* distribution.

II. The family of *GSL* distributions has higher degrees of skewness than of ordinary *SL* distribution.

III. The family of *GSL* distributions is more flexibility in its shape by changing the skewness and kurtosis of the model.

IV. The flexibility of the class of *GSL* distributions in terms of accommodating more general types of skewness than the ordinary *SL* distribution is illustrated by computing moments and, in particular, skewness and kurtosis coefficients.

V. The *SL* distribution is frequently used to fit the logarithm of particle sizes and it is also used in Economics, Engineering (reliability), Finance and Biology.

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