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UNSTEADY FLOW OF VISCOUS FLUID THROUGH POROUS MEDIUM IN A LONG UNIFORM CHANNEL WHOSE CROSS-SECTION IS CURVILINEAR QUADRILATERAL BOUNDED BY THE RADII AND THE ARCS OF TWO CONCENTRIC CIRCLES

Manoj Kumar Arya, Ravindra Kumar* and K. K. Singh

Department of Mathematics, Agra College, Agra (U.P.), India

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ABSTRACT

The aim of the present paper is to discuss the unsteady flow of a viscous incompressible fluid through porous medium in a long uniform channel whose cross-section is curvilinear quadrilateral bounded by the arcs and radii of two concentric circles. Two particular cases when the pressure gradient is (i) constant (ii) an exponentially increasing function of time have been studied in detail. In particular case all the results of case (i) can be determined from the results of case (ii).

INTRODUCTION

The study of viscous fluid flow through channels of various cross-section has been investigated by several research workers using various techniques in different circumstances. In classical viscous fluid, we know that the fluid exerts viscosity effect when there is a tendency of shear flow of the fluid. Various type of basic problems of diversified nature have been solved in this branch.

The study of flow through porous medium is of considerable interest in the field of petroleum engineering concerned with the movement of oil and gas, ground water hydrology, heat transfer in cooling systems and chemical engineering for filtration process etc.

Ahmadi and Manvi (1971) derived a general equation of viscous flow through porous medium and applied them to some basic problems. On account of varied practical applications of the magnetohydrodynamic flow problems in pipes of various cross-section through porous medium several authors: Ram and Mishra (1977), Gupta (1983), Kuiry (2000), Sengupta and Kumar (2001), Avinash and Rao (2006), Kumari and Varshney (2006), Shakya and Johri (2006), Kumar and Singh (2007), Raveendranath and Prasada Rao (2007), Reddy and Verma (2008), Singh *et. al.* (2010), Agarwal and Singh (2011), Pradhan, Dash and Dash (2011), Singh and Tiwari (2011, 2012), Chand, Singh and Kumar (2012) and Sharma, Singh and Chandramouli (2012) etc. have paid their attention in this direction.

In the present paper the laminar flow of viscous fluid through porous medium in a long uniform channel whose cross-section is curvilinear quadrilateral bounded by the arcs and radii of two concentric circles $r=a,\ \ r=b$ and $\theta=-\alpha,\ \theta=+\alpha,$ under the influence of variable time dependent pressure gradient has been studied. The analytical solution for the motion of viscous fluid is obtained by using the finite Fourier cosine transform. Two cases when the pressure gradient is (i) constant, and (ii) an exponentially increasing function of time have been discussed in detail. In particular case all the expressions of case (i) can be deduced from the results of case (ii).

FORMULATION OF THE PROBLEM AND SOLUTION

Let us consider the cylindrical polar coordinates (r,θ,z) of a point in the region of flow. The *z*-axis is taken along the length of the channel in which the porous medium contained. The cross-section of the channel bounded by the arcs of two concentric circles $r=a$ and $r=b\,(a>b)$ and the radii $\theta=\pm\alpha.$ Let u_r, u_θ, u_z be the components of velocity in increasing direction of $r,\ \theta,\ z$ respectively. Since channel is long and motion of fluid taken along *z*-axis, therefore $u_r=0,\ u_\theta=0$.

Corresponding author: *Ravindra Kumar* and K. K. Singh*
Department of Mathematics, Agra College, Agra (U.P.), India

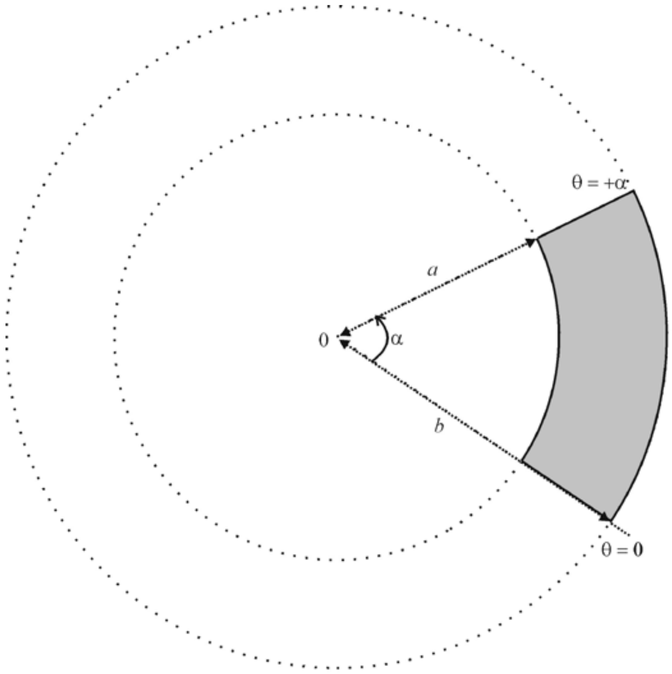

Fig. : *Cross section perpendicular to the axis of the curvilinear quadrilateral channel.*

The relevant equations of motion for viscous incompressible fluid are:

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{1}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{r\partial\theta} \tag{2}$$

$$\frac{\partial u_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\right)-\frac{\nu}{K}u_z \tag{3}$$

and the equation of continuity is

$$\frac{\partial u_z}{\partial z}=0 \tag{4}$$

From (1), (2), (3), (4), we have equation of motion

$$\frac{\partial u_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\right)-\frac{\nu}{K}u_z \tag{5}$$

Introducing the following non-dimensional quantities:

$$u^*=\frac{a}{\nu}u_z,\quad r^*=\frac{r}{a},\quad z^*=\frac{z}{a},\quad p^*=\frac{a^2}{\rho\nu^2}p, t^*=\frac{\nu}{a^2}t,\quad K^*=\frac{1}{a^2}K$$

In equation (5), we get (after dropping the stars):

$$\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial z}+\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)-\lambda^2 u \tag{6}$$

where $\lambda^2=\frac{1}{K}$.

The boundary conditions are:

$$u(r,\theta)=0, \quad \text{at } r=1$$

$$u(r,\theta)=0, \quad \text{at } r=\frac{b}{a}=c<1$$

$$u(r,\theta)=0, \text{ when } \theta=\pm\alpha,\ c<r<1$$

and $$\frac{\partial u}{\partial\theta}=0, \text{ when } \theta=0$$

Equations (1) and (2) show that p is independent to r and θ

$$\therefore \quad p=p(z,t) \tag{8}$$

From (6) and (8), it is clear that $\frac{\partial p}{\partial z}$ is the function of time t or constant, we assume

$$-\frac{\partial p}{\partial z}=P_0 f(t) \tag{9}$$

Now we consider following cases for different pressure gradient:

Case-I: When pressure gradient is constant *i.e.* $-\frac{\partial p}{\partial z}=P_0$

Let $$u=F(r,\theta) \tag{10}$$

The equation (6) becomes

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}-\lambda^2 F=-P_0 \tag{11}$$

since $F=0$ on the plane faces $\theta=\pm\alpha$ for every r, therefore we apply finite Fourier cosine transform defined as

$$\bar{F}=\int_0^{\alpha} F\cos q_m\theta\, d\theta \tag{12}$$

where $q_m=\frac{2m+1}{2\alpha}\pi, \quad m=0,1,2....$

Equation (11) gives

$$\frac{d^2\bar{F}}{dr^2}+\frac{1}{r}\frac{d\bar{F}}{dr}-\frac{q_m^2\bar{F}}{r^2}-\lambda^2\bar{F}=-\frac{(-1)^m}{q_m}P_0$$

or $$\frac{d^2\bar{F}}{dr^2}+\frac{1}{r}\frac{d\bar{F}}{dr}-\left(\lambda^2+\frac{q_m}{r^2}\right)\bar{F}=-\frac{(-1)^m}{q_m}P_0 \tag{13}$$

Assuming that q_m is neither positive integer nor zero.

The general solution of equation (13) is

$$\bar{F}(r,m)=A_1 I_{q_m}(\lambda r)+B_1 I_{-q_m}(\lambda r)+K_0 S_{1,q_m}(\lambda r) \tag{14}$$

where $K_0=\frac{(-1)^m P_0}{q_m\lambda^2}$

$$I_{q_m}^{(\lambda r)} = \sum_{n_1=0}^{\infty} \frac{\left(\frac{1}{2}\lambda r\right)^{q_m+2n_1}}{\underline{|n_1}\,\overline{|1+n_1+q_m}}$$ (modified function of first kind of order q_m)

$$S_{1,q_m}(\lambda r) = \sum_{l_1=0}^{\infty} \frac{(-1)^{l_1}\left(\frac{1}{2}\lambda r\right)^{2(l_1+1)}\overline{\left|\left(1-\frac{q_m}{2}\right)\right.}\,\overline{\left|\left(1+\frac{q_m}{2}\right)\right.}}{\overline{\left|\left(2-\frac{1}{2}q_m+l_1\right)\right.}\,\overline{\left|\left(2+\frac{1}{2}q_m+l_1\right)\right.}}$$ (Lommel function of first kind)

Applying boundary conditions

$$\left.\begin{aligned} \bar{F}(1,m) &= 0 \\ \bar{F}(c,m) &= 0 \end{aligned}\right\} \tag{15}$$

in equation (14), we get

$$A_1 = \frac{K_0\left\{I_{-q_m}(\lambda)S_{1,q_m}(c\lambda) - I_{-q_m}(c\lambda)S_{1},_{q_m}(\lambda)\right\}}{I_{q_m}(\lambda)I_{-q_m}(c\lambda) - I_{q_m}(c\lambda)I_{-q_m}(\lambda)}$$

$$B_1 = \frac{K_0\left\{I_{-q_m}(c\lambda)S_{1,q_m}(\lambda) - I_{q_m}(\lambda)S_{1},_{q_m}(c\lambda)\right\}}{I_{q_m}(\lambda)I_{-q_m}(c\lambda) - I_{q_m}(c\lambda)I_{-q_m}(\lambda)}$$

Putting the values of A_1 and B_1 in (14), we get solution of equation (13)

$$\begin{aligned} \bar{F}(r,m) = {} & \frac{K_0}{\{I_{q_m}(\lambda)I_{-q_m}(c\lambda) - I_{q_m}(c\lambda)I_{-q_m}(\lambda)\}}\Big[\{I_{-q_m}(\lambda)S_{1,q_m}(c\lambda) - I_{-q_m}(c\lambda) \\ & S_{1},_{q_m}(\lambda)\}\,I_{q_m}(\lambda r) + \{I_{q_m}(c\lambda)S_{1,q_m}(\lambda) - I_{q_m}(\lambda)S_{1,q_m}(c\lambda)\}I_{-q_m}(\lambda r) + \\ & \{I_{q_m}(\lambda)I_{-q_m}(c\lambda) - I_{q_m}(c\lambda)I_{-q_m}(\lambda)\}S_{1,q_m}(\lambda r)\Big] \end{aligned} \tag{16}$$

Now taking inverse finite Fourier cosine transform of (16) and putting in (10), we get fluid velocity

$$u = F(r,\theta) = \frac{2}{\alpha}\sum_{m=0}^{\infty}\bar{F}(r,m)\cos q_m\theta \tag{17}$$

Case-II: When pressure gradient is exponentially increasing function of time *i.e.* $-\frac{\partial p}{\partial z} = P_0 e^{\omega^2 t}$

Let $u(r,\theta,t) = F(r,\theta)e^{\omega^2 t}$ (18)

The from equation (6), we get

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \left(\omega^2 + \lambda^2\right)F = -P_0 \tag{19}$$

Taking finite Fourier cosine transform, we get

$$\frac{\partial^2 \bar{F}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{F}}{\partial r} - \left(\xi^2 + \frac{q_m}{r^2}\right)\bar{F} = -\frac{(-1)^m P_0}{q_m}$$

$$r^2\frac{\partial^2 \bar{F}}{\partial r^2} + r\frac{\partial \bar{F}}{\partial r} - \left(\xi^2 r^2 + q_m^2\right)\bar{F} = -\frac{(-1)^m P_0}{q_m} \tag{20}$$

where $\xi^2 = \omega^2 + \lambda^2$,

General solution of equation (19) is

$$\bar{F} = (r,m) = A_2 I_{q_m}(\xi r) + B_2 I_{-q_m}(\xi r) + K_1 S_{1,q_m}(\xi r) \tag{21}$$

where $K_1 = \dfrac{(-1)^m P_0}{q_m \xi^2}$

Applying boundary conditions (15) in equation (21) and solving for arbitrary constants A_2, B_2, we get

$$A_2 = \frac{K_1 \left\{ I_{-q_m}(\xi) S_{1,q_m}(c\xi) - I_{-q_m}(c\xi) S_{1,q_m}(\xi) \right\}}{I_{q_m}(\xi) I_{-q_m}(c\xi) - I_{q_m}(c\xi) I_{-q_m}(\xi)}$$

$$B_2 = \frac{K_1 \left\{ I_{q_m}(c\xi) S_{1,q_m}(\xi) - I_{q_m}(\xi) S_{1,q_m}(c\xi) \right\}}{I_{q_m}(\xi) I_{-q_m}(c\xi) - I_{q_m}(c\xi) I_{-q_m}(\xi)}$$

Putting the values of A_2 and B_2 in (21)

$$\begin{aligned} \bar{F}(r,m) = \frac{K_1}{\{I_{q_m}(\xi) I_{-q_m}(c\xi) - I_{q_m}(c\xi) I_{-q_m}(\xi)\}} \Big[&\{I_{-q_m}(\xi) S_{1,q_m}(c\xi) - I_{-q_m} \\ &(c\xi) S_{1,q_m}(\xi)\} I_{q_m}(\xi r) + \{I_{q_m}(c\xi) S_{1,q_m}(\xi) - I_{q_m}(\xi) S_{1,q_m}(c\xi)\} \cdot \\ &I_{-q_m}(\xi r) + \{I_{q_m}(\xi) I_{-q_m}(c\xi) - I_{q_m}(c\xi) I_{-q_m}\} S_{1,q_m}(\xi r) \Big] \end{aligned} \tag{22}$$

Taking inverse finite Fourier cosine transform of (22) and then putting in (18), we get fluid velocity

$$u = F(r,\theta) e^{\omega^2 t} = \frac{2e^{\omega^2 t}}{\alpha} \sum_{m=0}^{\infty} \bar{F}(r,m) \cos q_m \theta \tag{23}$$

If we put $\omega^2 = 0$, in case II, all the results of case I i.e. for constant pressure gradient can be obtained with the slight change of symbols.

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