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# SOME COMMON FIXED POINT THEOREMS IN TWO METRIC SPACES

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***ABSTRACT***

*In this paper we prove some common fixed point theorems for generalized contraction mappings in two complete metric spaces.*

***Keywords and Phrases:*** *fixed point, common fixed point and complete metric space.*

***AMS Mathematics Subject Classification:*** *47H10, 54H25.*

## 1. INTRODUCTION

The study of common fixed point theorems satisfying contractive type mappings and non-expansive mappings has been a very active field of research during the last three decades. In 1922, the polish mathematician, Banach, proved his famous Banach fixed point theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Later many authors proved fixed point theorems in different ways. Some of them ([1]-[6]), [9]) proved fixed point theorems for contractive type mappings and non-expansive mappings. In [7], Fisher proved a related fixed point theorem in two metric spaces. Recently many authors [8], [10], proved common fixed point theorems in various ways. The main purpose of this paper is to present some common fixed point theorems in two complete metric spaces. The following definitions are necessary for present study.

**Definition 1.2:** A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$ if given $\in>0$ there exists a positive integer $n_0$ such that $d(x_n, x) < \in$ for all $n \geq n_0$.

**Definition1.3:** A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence in X if given $\in>0$ there exists a positive integer $n_0$ such that $d(x_m, x_n) < \in$ for all $m, n \geq n_0$ .

**Definition1.4:** A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

**Definition1.5:** Let X be a non-empty set and f: X → X be a map. An element x in X is called a fixed point of X if f(x) = x.

**Definition1.6**. Let X be a non-empty set and f, g: X → X be two maps. An element x in X is called a common fixed point of f and g if f(x) = g(x) = x.

## 2. MAIN RESULTS

**Theorem 2.1:** Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \leq c_1.\max\{ d(x, SAx), d(x', TBx'), e(Ax, Bx'), \frac{d(x,TBx')}{2}, \frac{d(SAx,x')}{2} \} \quad (1)$$

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$e(BSy, ATy') \le c_2.\max\{ e(y, BSy), e(y', ATy'), d(Sy, Ty'), \frac{e(y,ATy')}{2}, \frac{e(BSy,y')}{2} \}$ (2)

for all x, x′ in X and y, y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

**Proof:** Let $x_0$ be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}, Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}; Ty_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .

Now we have

$d(x_{2n+1}, x_{2n}) = d(SAx_{2n}, TBx_{2n-1})$

$\le c_1.\max\{d(x_{2n},SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n},Bx_{2n-1}), \frac{d(x_{2n},TBx_{2n-1})}{2}, \frac{d(SAx_{2n},x_{2n-1})}{2} \}$

$= c_1.\max\{d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \frac{d(x_{2n},x_{2n})}{2}, \frac{d(x_{2n+1},x_{2n-1})}{2} \}$

$\le c_1.\max\{d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), 0, \frac{d(x_{2n+1},x_{2n})+d(x_{2n},x_{2n-1})}{2}\}$

$\le c_1.\max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\}$

Now

$e(y_{2n}, y_{2n+1}) = e(BSy_{2n-1}, ATy_{2n})$

$\le c_2.\max\{e(y_{2n-1}, BSy_{2n-1}), e(y_{2n}, ATy_{2n}), d(Sy_{2n-1}, Ty_{2n}), \frac{e(y_{2n-1},ATy_{2n})}{2}, \frac{e(BSy_{2n-1},y_{2n})}{2} \}$

$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1},y_{2n+1})}{2}, \frac{e(y_{2n},y_{2n})}{2}\}$

$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1},y_{2n})+e(y_{2n},y_{2n+1})}{2}, 0 \}$

$\le c_2.\max\{e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\}$ (3)

Hence

$d(x_{2n+1}, x_{2n}) \le c_1c_2. \max\{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n})\}$ (4)

We have

$d(x_{2n}, x_{2n-1}) = d(x_{2n-1}, x_{2n})$

$= d(SAx_{2n-2}, TBx_{2n-1})$

$\le c_1.\max\{d(x_{2n-2},SAx_{2n-2}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n-2}, Bx_{2n-1}), \frac{d(x_{2n-2},TBx_{2n-1})}{2}, \frac{d(SAx_{2n-2},x_{2n-1})}{2}\}$

$= c_1.\max\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2},x_{2n})}{2}, \frac{d(x_{2n-1},x_{2n-1})}{2} \}$

$= c_1.\max\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n})}{2}, 0 \}$

$\le c_1. \max\{d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\}$

Now

$e(y_{2n},y_{2n-1}) = e(BSy_{2n-1}, ATy_{2n-2})$

 

$$\leq c_2.\ \max\{e(y_{2n-1}, BSy_{2n-1}), e(y_{2n-2}, ATy_{2n-2}), d(Sy_{2n-1}, Ty_{2n-2}), \frac{e(y_{2n-1}, ATy_{2n-2})}{2}, \frac{e(BSy_{2n-1}, y_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n-1}, y_{2n-1})}{2}, \frac{e(y_{2n}, y_{2n-2})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0, \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}\}$$

$$\leq c_2.\ \max\{e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \tag{5}$$

Hence

$$d(x_{2n}, x_{2n-1}) \leq c_1c_2.\ \max\{d(x_{2n-1},x_{2n-2}), e(y_{2n-1},y_{2n-2})\} \tag{6}$$

From inequalities (3), (4), (5) and (6), we have

$$d(x_{n+1},x_n) \leq c_1(c_2)^n.\ \max\{d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

$$e(y_{n+1},y_n) \leq c_2^n.\ \max\{d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) and (Y, e) respectively. Since (X, d) and (Y, e) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z.

Suppose SAz $\neq$ z.

We have

$$d(SAz, z) = \lim_{n\to\infty} d(SAz, TBx_{2n-1})$$

$$\leq \lim_{n\to\infty} c_1.\ \max\{ d(z, SAz), d(x_{2n-1},TB\,x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, TBx_{2n-1})}{2}, \frac{d(SAz, x_{2n-1})}{2}\}$$

$$= \lim_{n\to\infty} c_1.\ \max\{ d(z, SAz), d(x_{2n-1}, x_{2n}), e(Az, y_{2n}), \frac{d(z, x_{2n})}{2}, \frac{d(SAz, x_{2n-1})}{2}\}$$

$$= c_1.\max\{ d(z, SAz), d(z,z), e(w, w), \frac{d(z, z)}{2}, \frac{d(SAz, z)}{2}\}$$

$$= c_1.\max\{ d(z, SAz), 0, 0, 0, \frac{d(SAz, z)}{2}\}$$

$$\leq c_1.d(z, SAz)$$

$$< d(z, SAz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose BSw $\neq$ w.

 

We have

$$
\begin{aligned}
e(BSw, w) &= \lim_{n\to\infty} e(BSw, y_{2n+1}) \\
&= \lim_{n\to\infty} e(BSw, ATy_{2n}) \\
&\le \lim_{n\to\infty} c_2 \max\{e(w, BSw), e(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n}), \frac{e(w, ATy_{2n})}{2}, \frac{e(BSw, y_{2n})}{2}\} \\
&= c_2. \max\{e(w, BSw), e(w, w), d(z, z), \frac{e(w,w)}{2}, \frac{e(BSw,w)}{2}\} \\
&< e(w, BSw) \quad (\text{Since } 0 \le c_2 < 1)
\end{aligned}
$$

Which is a contradiction.

Thus $BSw = w$.

Hence $Bz = w$. (Since $Sw = z$)

Now we prove $TBz = z$.

Suppose $TBz \neq z$.

$$
\begin{aligned}
d(z,TBz) &= \lim_{n\to\infty} d(x_{2n+1}, TBz) \\
&= \lim_{n\to\infty} d(SAx_{2n}, TBz) \\
&\le \lim_{n\to\infty} c_1. \max\{ d(x_{2n},SAx_{2n}), d(z, TBz), e(Ax_{2n}, Bz), \frac{d(x_{2n}, TBz)}{2}, \frac{d(SAx_{2n}, z)}{2}\} \\
&= c_1. \max\{d(z, z), d(z, TBz), e(w, Bz), \frac{d(z, TBz)}{2}, \frac{d(z,z)}{2}\} \\
&= c_1. \max\{ 0, d(z, TBz), 0, \frac{d(z, TBz)}{2}, 0\} \\
&< d(z, TBz) \quad (\text{Since } 0 \le c_1 < 1)
\end{aligned}
$$

Which is a contradiction.

Thus $TBz = z$.

Hence $Tw = z$. (Since $Bz = w$)

Now we prove $ATw = w$.

Suppose $ATw \neq w$.

$$
\begin{aligned}
e(w, ATw) &= \lim_{n\to\infty} e(y_{2n}, ATw) \\
&= \lim_{n\to\infty} e(BSy_{2n-1}, ATw) \\
&\le \lim_{n\to\infty} c_2. \max\{e(y_{2n-1}, BSy_{2n-1}), e(w, ATw), d(Sy_{2n-1}, Tw), \frac{e(y_{2n-1}, ATw)}{2}, \frac{e(BSy_{2n-1}, w)}{2}\} \\
&= c_2. \max\{ e(w, w), e(w, ATw), d(z, z), \frac{e(w,ATw)}{2}, \frac{e(w,w)}{2}\} \\
&= c_2. \max\{ 0, e(w, ATw), 0, \frac{e(w,ATw)}{2}, 0\} \\
&< e(w, ATw) \quad (\text{Since } 0 \le c_2 < 1)
\end{aligned}
$$

Which is a contradiction.

Thus $ATw = w$.

The same results hold if one of the mappings B, S and T is continuous.

**Uniqueness:** Let z′ be another common fixed point of SA and TB in X, w′ be another common fixed point of BS and AT in Y.

We have $d(z, z') = d(SAz, TBz')$

$$\le c_1.\max\{d(z, SAz),\ d(z', TBz'), e(Az, Bz'), \frac{d(z, TBz')}{2}, \frac{d(SAz, z')}{2}\}$$

$$= c_1.\max\{d(z, z),\ d(z', z'), e(w, w'), \frac{d(z, z')}{2}, \frac{d(z, z')}{2}\}$$

$$= c_1.\max\{d(z, z),\ d(z', z'), e(w, w'), \frac{d(z, z')}{2}\}$$

$$= c_1.\max\{0, 0, e(w, w'), \frac{d(z, z')}{2}\}$$

$$\le e(w, w')$$

$$e(w, w') = e(BSw, ATw')$$

$$\le c_2.\max\{e(w, BSw), e(w', ATw'), d(Sw, Tw'), \frac{e(w, ATw')}{2}, \frac{e(BSw, w')}{2}\}$$

$$= c_2.\max\{ e(w, w), e(w', w'), d(z, z'), \frac{e(w, w')}{2}, \frac{e(w, w')}{2}\}$$

$$= c_2.\max\{ 0, 0, d(z, z'), \frac{e(w, w')}{2}\}$$

$$< d(z, z')$$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

**Remark: 2.2:** If we put A = B, S = T in the above theorem 2.1, we get the following corollary.

**Corollary 2.3:** Let (X, d) and (Y, e) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$$d(TAx, TAx') \le c_1.\max\{ d(x, TAx), d(x', TAx'), e(Ax, Ax'), \frac{d(x, TAx')}{2}, \frac{d(TAx, x')}{2}\}$$

$$e(ATy, ATy') \le c_2.\max\{ e(y, ATy), e(y', ATy'), d(Ty, Ty'), \frac{e(y, ATy')}{2}, \frac{e(ATy, y')}{2}\}$$

for all x, x′ in X and y, y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT have a unique fixed point w in Y. Further, Az = w and Tw = z.

**Theorem2.4:** Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \le c_1.\max\{ d(x, x'), d(x, SAx), d(x', TBx'), e(Ax, Bx'), \frac{d(x, TBx') + d(SAx, x')}{2}\} \tag{1}$$

$$e(BSy, ATy') \le c_2.\max\{ e(y, y'), e(y, BSy), e(y', ATy'), d(Sy, Ty'), \frac{e(y, ATy') + e(BSy, y')}{2}\} \tag{2}$$

for all x, x′ in X and y, y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

**Proof:** Let $x_0$ be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$Ax_{2n-2} = y_{2n-1},\ Sy_{2n-1} = x_{2n-1},\ Bx_{2n-1} = y_{2n};\ Ty_{2n} = x_{2n} \text{ for } n = 1, 2, 3 \ldots\ldots$$

Now we have

$$d(x_{2n+1}, x_{2n}) = d(SAx_{2n}, TBx_{2n-1})$$

$$\leq c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n},SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n},Bx_{2n-1}), \frac{d(x_{2n}, TBx_{2n-1}) + d(SAx_{2n}, x_{2n-1})}{2}\}$$

$$= c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \frac{d(x_{2n}, x_{2n}) + d(x_{2n+1}, x_{2n-1})}{2}\}$$

$$\leq c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \frac{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})}{2}\}$$

$$\leq c_1.\max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\}$$

Now

$$e(y_{2n}, y_{2n+1}) = e(BSy_{2n-1}, ATy_{2n})$$

$$\leq c_2.\max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n}, ATy_{2n}), d(Sy_{2n-1}, Ty_{2n}), \frac{e(y_{2n-1}, ATy_{2n})+e(BSy_{2n-1}, y_{2n})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n+1}) + e(y_{2n}, y_{2n})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}\}$$

$$\leq c_2. \max\{e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \tag{3}$$

Hence

$$d(x_{2n+1}, x_{2n}) \leq c_1c_2. \max\{ d(x_{2n-1}, x_{2n}) , e(y_{2n-1}, y_{2n})\} \tag{4}$$

we have

$$d(x_{2n}, x_{2n-1}) = d(x_{2n-1}, x_{2n})$$

$$= d(SAx_{2n-2}, TBx_{2n-1})$$

$$\leq c_1.\max\{d(x_{2n-2},x_{2n-1}), d(x_{2n-2},SAx_{2n-2}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n-2}, Bx_{2n-1}), \frac{d(x_{2n-2}, TBx_{2n-1}) + d(SAx_{2n-2}, x_{2n-1})}{2}\}$$

$$= c_1.\max\{d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-2},y_{2n-1}), \frac{d(x_{2n-2}, x_{2n}) + d(x_{2n-1}, x_{2n-1})}{2}\}$$

$$\leq c_1.\max\{d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-2},y_{2n-1}), \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}\}$$

$$\leq c_1.\max\{d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\}$$

Now

$$e(y_{2n},y_{2n-1}) = e(BSy_{2n-1}, ATy_{2n-2})$$

$$\leq c_2. \max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n-2}, ATy_{2n-2}), d(Sy_{2n-1}, Ty_{2n-2}), \frac{e(y_{2n-1}, ATy_{2n-2)} + e(BSy_{2n-1}, y_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n-1}, y_{2n-1}) + e(y_{2n}, y_{2n-2})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \tag{5}$$

Hence

$$d(x_{2n}, x_{2n-1}) \leq c_1c_2. \max \{d(x_{2n-1},x_{2n-2}), e(y_{2n-1},y_{2n-2})\} \tag{6}$$

from inequalities (3), (4), (5) and (6), we have

$$d(x_{n+1},x_n) \leq c_1(c_2)^n. \max\{d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

$$e(y_{n+1},y_n) \leq c_2^n. \max\{d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) and (Y, e) respectively. Since (X, d) and (Y, e) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z. .

Suppose SAz $\neq$ z.

We have

$$\begin{aligned}
d(SAz, z) &= \lim_{n\to\infty} d(SAz, TBx_{2n-1}) \\
&\leq \lim_{n\to\infty} c_1.\max\{ d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1},TB\, x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, TBx_{2n-1}) + d(SAz, x_{2n-1})}{2}\} \\
&= \lim_{n\to\infty} c_1. \max\{ d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1},TB\, x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, x_{2n}) + d(SAz, x_{2n-1})}{2}\} \\
&= c_1. \max\{ d(z, z), d(z, SAz), d(z, z), e(w, w), \frac{d(z, z) + d(SAz, z)}{2}\} \\
&= c_1. \max\{ 0, d(z, SAz), 0, 0, \frac{d(SAz, z)}{2}\} \\
&\leq c_1.d(z, SAz) \\
&< d(z, SAz) \quad (\text{Since } 0 \leq c_1 < 1)
\end{aligned}$$

Which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose BSw $\neq$ w.

We have

$$\begin{aligned}
e(BSw, w) &= \lim_{n\to\infty} e(BSw , y_{2n+1}) \\
&= \lim_{n\to\infty} e(BSw, ATy_{2n}) \\
&\leq \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, BSw), e(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n}), \frac{e(w, ATy_{2n}) + e(BSw, y_{2n})}{2}\} \\
&= c_2.\max \{e(w, w), e(w, BSw), e(w, w), d(z, z), \frac{e(w, w) + e(BSw, w)}{2}\} \\
&< e(w, BSw) \quad (\text{Since } 0 \leq c_2 < 1)
\end{aligned}$$

Which is a contradiction.

Thus BSw = w.

 

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

Suppose TBz $\neq$ z.

$$
\begin{aligned}
d(z, TBz) &= \lim_{n\to\infty} d(x_{2n+1}, TBz) \\
&= \lim_{n\to\infty} d(SAx_{2n}, TBz) \\
&\leq \lim_{n\to\infty} c_1.\max\{d(x_{2n}, z), d(x_{2n}, SAx_{2n}), d(z, TBz), e(Ax_{2n}, Bz), \frac{d(x_{2n}, TBz) + d(SAx_{2n}, z)}{2}\} \\
&= c_1.\max\{d(z, z), d(z, z), d(z, TBz), e(w, Bz), \frac{d(z, TBz) + d(z, z)}{2}\} \\
&= c_1.\max\{0, 0, d(z, TBz), 0, \frac{d(z, TBz)}{2}\} \\
&< d(z, TBz) \quad (\text{Since } 0 \leq c_1 < 1)
\end{aligned}
$$

Which is a contradiction.

Thus TBz = z.

Hence Tw = z. (Since Bz = w)

Now we prove ATw = w.

Suppose ATw $\neq$ w.

$$
\begin{aligned}
e(w, ATw) &= \lim_{n\to\infty} e(y_{2n}, ATw) \\
&= \lim_{n\to\infty} e(BSy_{2n-1}, ATw) \\
&\leq \lim_{n\to\infty} c_2.\max\{ e(y_{2n-1}, w), e(y_{2n-1}, BSy_{2n-1}), e(w, ATw), d(Sy_{2n-1}, Tw), \frac{e(y_{2n-1}, ATw) + e(BSy_{2n-1}, w)}{2}\} \\
&= c_2.\max\{ e(w, w), e(w, w), e(w, ATw), d(z, z), \frac{e(w, ATw) + e(w, w)}{2}\} \\
&< e(w, ATw) \quad (\text{Since } 0 \leq c_2 < 1)
\end{aligned}
$$

Which is a contradiction.

Thus ATw = w.

The same results hold if one of the mappings B, S and T is continuous.

**Uniqueness:** Let z′ be another common fixed point of SA and TB in X, w′ be another common fixed point of BS and AT in Y.

We have d(z, z′) = d(SAz, TBz′)

$$
\begin{aligned}
&\leq c_1.\max\{ d(z, z'), d(z, SAz), d(z', TBz'), e(Az, Bz'), \frac{d(z, TBz') + d(SAz, z')}{2}\} \\
&\leq c_1.\max\{ d(z, z'), d(z, SAz), d(z', z'), e(Az, Bz'), \frac{d(z, z') + d(z, z')}{2}\} \\
&= c_1.\max\{d(z, z'), d(z, z), d(z', z'), e(w, w'), d(z, z')\} \\
&= c_1.\max\{d(z, z'), 0, 0, e(w, w'), d(z, z')\} \\
&\leq e(w, w')
\end{aligned}
$$

$e(w, w') = e(BSw, ATw')$

$\leq c_2.\max\{e(w, w'), e(w, BSw), e(w', ATw'), d(Sw, Tw'), \dfrac{e(w, ATw') + e(BSw, w')}{2}\}$

$= c_2.\max\{e(w, w'), e(w, w), e(w', w'), d(z, z'), \dfrac{e(w, w') + e(w, w')}{2}\}$

$= c_2.\max\{e(w, w'), 0, 0, d(z, z'), e(w, w')\}$

$< d(z, z')$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

**Remark 2.5:** If we put A = B, S = T in the above theorem 2.4, we get the following corollary.

**Corollary 2.6:** Let (X, d) and (Y, e) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$$d(TAx, TAx') \leq c_1.\max\{d(x,x'), d(x,TAx), d(x', TBx'), e(Ax, Ax'), \frac{d(x, TAx') + d(TAx, x')}{2}\}$$

$$e(ATy, ATy') \leq c_2.\max\{e(y, y'), e(y, ATy), e(y',ATy'), d(Ty, Ty'), \frac{e(y, ATy') + e(ATy, y')}{2}\}$$

for all x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings A and T is continuous , then TA has a unique fixed point z in X and AT have a unique fixed point w in Y. Further, Az = w and Tw = z.

**Theorem 2.7:** Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \leq c_1.\max\{d(x, x'), d(x, SAx), d(x', TBx'), e(Ax, Bx'), \frac{d(x, TBx')}{2}, \frac{d(SAx, x')}{2}\} \quad (1)$$

$$e(BSy, ATy') \leq c_2.\max\{e(y, y'), e(y, BSy), e(y', ATy'), d(Sy, Ty'), \frac{e(y, ATy')}{2}, \frac{e(BSy, y')}{2}\} \quad (2)$$

for all x, x′ in X and y,y′ in Y where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

**Proof:** Let $x_0$ be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}$, $Sy_{2n-1} = x_{2n-1}$, $Bx_{2n-1} = y_{2n}$; $Ty_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .

Now we have

$d(x_{2n+1}, x_{2n}) = d(SAx_{2n}, TBx_{2n-1})$

$\leq c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n},SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n},Bx_{2n-1}), \dfrac{d(x_{2n},TBx_{2n-1})}{2}, \dfrac{d(SAx_{2n}, x_{2n-1})}{2}\}$

$= c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \dfrac{d(x_{2n}, x_{2n})}{2}, \dfrac{d(x_{2n+1}, x_{2n-1})}{2}\}$

$\leq c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), 0, \dfrac{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})}{2}\}$

$\leq c_1.\max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\}$

Now

$$e(y_{2n}, y_{2n+1}) = e(BSy_{2n-1}, ATy_{2n})$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n}, ATy_{2n}), d(Sy_{2n-1}, Ty_{2n}), \frac{e(y_{2n-1}, ATy_{2n})}{2}, \frac{e(BSy_{2n-1}, y_{2n})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n+1})}{2}, \frac{e(y_{2n}, y_{2n})}{2}\}$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}, 0\}$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \tag{3}$$

Hence

$$d(x_{2n+1}, x_{2n}) \le c_1c_2.\max\{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n})\} \tag{4}$$

we have

$$d(x_{2n}, x_{2n-1}) = d(x_{2n-1}, x_{2n})$$

$$= d(SAx_{2n-2}, TBx_{2n-1})$$

$$\le c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2},SAx_{2n-2}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n-2}, Bx_{2n-1}), \frac{d(x_{2n-2}, TBx_{2n-1})}{2}, \frac{d(SAx_{2n-2}, x_{2n-1})}{2}\}$$

$$= c_1.\max\{d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n})}{2}, \frac{d(x_{2n-1}, x_{2n-1})}{2}\}$$

$$\le c_1.\max\{d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}, 0\}$$

$$\le c_1.\max\{d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\}$$

Now

$$e(y_{2n},y_{2n-1}) = e(BSy_{2n-1}, ATy_{2n-2})$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n-2}, ATy_{2n-2}), d(Sy_{2n-1}, Ty_{2n-2}), \frac{e(y_{2n-1}, ATy_{2n-2})}{2}, \frac{e(BSy_{2n-1}, y_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n-1}, y_{2n-1})}{2}, \frac{e(y_{2n}, y_{2n-2})}{2}\}$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0, \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}\}$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \tag{5}$$

Hence

$$d(x_{2n}, x_{2n-1}) \le c_1c_2.\max\{ d(x_{2n-1},x_{2n-2}), e(y_{2n-1},y_{2n-2})\} \tag{6}$$

from inequalities (3), (4), (5) and (6), we have

$$d(x_{n+1},x_n) \le c_1(c_2)^n.\max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

$$e(y_{n+1},y_n) \le (c_2)^n.\max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) and (Y, e) respectively. Since (X, d) and (Y, e) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z. .

Suppose SAz $\neq$ z.

We have

$$d(SAz, z) = \lim_{n\to\infty} d(SAz, TBx_{2n-1})$$

$$\leq \lim_{n\to\infty} c_1.\max\left\{ d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, TB\, x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, TBx_{2n-1})}{2}, \frac{d(SAz, x_{2n-1})}{2} \right\}$$

$$\leq \lim_{n\to\infty} c_1.\max\left\{ d(z, x_{2n-1}), d(z,SAz), d(x_{2n-1}, TB\, x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, x_{2n})}{2}, \frac{d(SAz, x_{2n-1})}{2} \right\}$$

$$= c_1.\max\left\{ d(z, z), d(z, SAz), d(z, z), e(w, w), \frac{d(z, z)}{2}, \frac{d(SAz, z)}{2} \right\}$$

$$= c_1.\max\left\{ 0, d(z, SAz), 0, 0, 0, \frac{d(SAz, z)}{2} \right\}$$

$$\leq c_1.\, d(z, SAz)$$

$< d(z, SAz)$ (Since $0 \leq c_1 < 1$)

Which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose BSw $\neq$ w.

We have

$$e(BSw, w) = \lim_{n\to\infty} e(BSw, y_{2n+1})$$

$$= \lim_{n\to\infty} e(BSw, ATy_{2n})$$

$$\leq \lim_{n\to\infty} c_2 \max\left\{ e(w, y_{2n}), e(w, BSw), e(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n}), \frac{e(w, ATy_{2n})}{2}, \frac{e(BSw, y_{2n})}{2} \right\}$$

$$= c_2.\max\left\{ e(w, w), e(w, BSw), e(w, w), d(z, z), \frac{e(w,w)}{2}, \frac{e(BSw, w)}{2} \right\}$$

$< e(w, BSw)$ (Since $0 \leq c_2 < 1$)

Which is a contradiction.

Thus BSw = w.

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

Suppose TBz $\neq$ z.

$$d(z, TBz) = \lim_{n\to\infty} d(x_{2n+1}, TBz)$$

$$= \lim_{n\to\infty} d(SAx_{2n}, TBz)$$

 

$\leq \lim_{n\to\infty} c_1.\max\{d(x_{2n},z), d(x_{2n},SAx_{2n}), d(z, TBz), e(Ax_{2n}, Bz), \frac{d(x_{2n},TBz)}{2}, \frac{d(SAx_{2n}, z)}{2}\}$

$= c_1.\max\{d(z, z), d(z, z), d(z, TBz), e(w, Bz), \frac{d(z, TBz)}{2}, \frac{d(z,z)}{2}\}$

$= c_1.\max\{0, 0, d(z, TBz), 0, \frac{d(z, TBz)}{2}, 0\}$

$< d(z, TBz)$ (Since $0 \leq c_1 < 1$)

Which is a contradiction.

Thus $TBz = z$.

Hence $Tw = z$. (Since $Bz = w$)

Now we prove $ATw = w$.

Suppose $ATw \neq w$.

$e(w, ATw) = \lim_{n\to\infty} e(y_{2n}, ATw)$

$= \lim_{n\to\infty} e(BSy_{2n-1}, ATw)$

$\leq \lim_{n\to\infty} c_2.\max\{e(y_{2n-1}, w), e(y_{2n-1}, BSy_{2n-1}), e(w, ATw), d(Sy_{2n-1}, Tw), \frac{e(y_{2n-1}, ATw)}{2}, \frac{e(BSy_{2n-1}, w)}{2}\}$

$= c_2.\max\{e(w, w), e(w, w), e(w, ATw), d(z,z), \frac{e(w,ATw)}{2}, \frac{e(w,w)}{2}\}$

$= c_2.\max\{0, 0, e(w, ATw), 0, \frac{e(w,ATw)}{2}, 0\}$

$< e(w, ATw)$ (Since $0 \leq c_2 < 1$)

Which is a contradiction.

Thus $ATw = w$.

The same results hold if one of the mappings B, S and T is continuous.

**Uniqueness:** Let $z'$ be another common fixed point of SA and TB in X, $w'$ be another common fixed point of BS and AT in Y.

We have $d(z, z') = d(SAz, TBz')$

$\leq c_1.\max\{d(z, z'), d(z, SAz), d(z', TBz'), e(Az, Bz'), \frac{d(z, TBz')}{2}, \frac{d(SAz, z')}{2}\}$

$\leq c_1.\max\{d(z, z'), d(z, z), d(z', z'), e(w, w'), \frac{d(z, z')}{2}, \frac{d(z, z')}{2}\}$

$= c_1.\max\{d(z, z'), d(z, z), d(z', z'), e(w, w'), \frac{d(z, z')}{2}\}$

$= c_1.\max\{d(z, z'), 0, 0, e(w, w')\}$

$\leq e(w, w')$

$e(w, w') = e(BSw, ATw')$

$\leq c_2.\max\{e(w, w'), e(w, BSw), e(w', ATw'), d(Sw, Tw'), \frac{e(w, ATw')}{2}, \frac{e(BSw, w')}{2}\}$

$= c_2.\max\{e(w, w'), e(w, w), e(w', w'), d(z, z'), \frac{e(w, w')}{2}, \frac{e(w, w')}{2}\}$

$= c_2. \max\{e(w, w'), 0, 0, d(z, z'), \frac{e(w, w')}{2}\}$

$< d(z, z')$

Hence $d(z, z') < e(w, w') < d(z, z')$

Which is a contradiction.

Thus $z = z'$.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

**Remark 2.8:** If we put A = B, S = T in the above theorem 2.7, we get the following corollary.

**Corollary 2.9:** Let (X, d) and (Y, e) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$$d(TAx, TAx') \le c_1. \max\{d(x, x'), d(x, TAx), d(x', TAx'), e(Ax, Ax'), \frac{d(x, TAx')}{2}, \frac{d(TAx, x')}{2}\}$$

$$e(ATy, ATy') \le c_2. \max\{e(y, y'), e(y, ATy), e(y', ATy'), d(Ty, Ty'), \frac{e(y, ATy')}{2}, \frac{e(ATy, y')}{2}\}$$

for all x, x′ in X and y, y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT have a unique fixed point w in Y. Further, Az = w and Tw = z.

**Theorem 2.10:** Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \le c_1. \max\{d(x, x'), d(x, SAx), d(x', TBx'), e(Ax, Bx'), \frac{d(x, TBx')}{2}, \frac{d(SAx, x')}{2}, \frac{d(x, TBx') + d(SAx, x')}{2}\} \quad (1)$$

$$e(BSy, ATy') \le c_2. \max\{e(y, y'), e(y, BSy), e(y', ATy'), d(Sy, Ty'), \frac{e(y, ATy')}{2}, \frac{e(BSy, y')}{2}, \frac{e(y, ATy') + e(BSy, y')}{2}\} \quad (2)$$

for all x, x′ in X and y,y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

**Proof:** Let $x_0$ be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}, Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}; Ty_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .

Now we have

$d(x_{2n+1}, x_{2n}) = d(SAx_{2n}, TBx_{2n-1})$

$$\le c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n},SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n},Bx_{2n-1}), \frac{d(x_{2n}, TBx_{2n-1})}{2}, \frac{d(SAx_{2n}, x_{2n-1})}{2}, \frac{d(x_{2n}, TBx_{2n-1}) + d(SAx_{2n}, x_{2n-1})}{2}\}$$

$$= c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \frac{d(x_{2n}, x_{2n})}{2}, \frac{d(x_{2n+1}, x_{2n-1})}{2}, \frac{d(x_{2n}, x_{2n}) + d(x_{2n+1}, x_{2n-1})}{2}\}$$

$$\leq c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), 0, \frac{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})}{2}, \frac{d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})}{2}\}$$

$$\leq c_1. \max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\}$$

Now

$$e(y_{2n}, y_{2n+1}) = e(BSy_{2n-1}, ATy_{2n})$$

$$\leq c_2. \max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n}, ATy_{2n}), d(Sy_{2n-1}, Ty_{2n}). \frac{e(y_{2n-1}, ATy_{2n})}{2}, \frac{e(BSy_{2n-1}, y_{2n})}{2}, \frac{e(y_{2n-1}, ATy_{2n}) + e(BSy_{2n-1}, y_{2n})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n+1})}{2}, \frac{e(y_{2n}, y_{2n})}{2}, \frac{e(y_{2n-1}, y_{2n+1}) + e(y_{2n}, y_{2n})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}, 0, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}\}$$

$$\leq c_2. \max\{e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \tag{3}$$

Hence

$$d(x_{2n+1}, x_{2n}) \leq c_1c_2. \max\{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n})\} \tag{4}$$

We have

$$d(x_{2n}, x_{2n-1}) = d(x_{2n-1}, x_{2n})$$

$$= d(SAx_{2n-2}, TBx_{2n-1})$$

$$\leq c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2},SAx_{2n-2}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n-2}, Bx_{2n-1}), \frac{d(x_{2n-2}, TBx_{2n-1})}{2}, \frac{d(SAx_{2n-2}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, TBx_{2n-1}) + d(SAx_{2n-2}, x_{2n-1})}{2}\}$$

$$= c_1. \max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n})}{2}, \frac{d(x_{2n-1}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, x_{2n}) + d(x_{2n-1}, x_{2n-1})}{2}\}$$

$$= c_1. \max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}, 0, \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}\}$$

$$\leq c_1. \max\{ d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\}$$

Now

$$e(y_{2n},y_{2n-1}) = e(BSy_{2n-1}, ATy_{2n-2})$$

$$\leq c_2.\max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n-2}, ATy_{2n-2}), d(Sy_{2n-1}, Ty_{2n-2}), \frac{e(y_{2n-1}, ATy_{2n-2})}{2}, \frac{e(BSy_{2n-1}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, ATy_{2n-2}) + e(BSy_{2n-1}, y_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}),\ e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n-1}, y_{2n-1})}{2}, \frac{e(y_{2n}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, y_{2n-1}) + e(y_{2n}, y_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}),\ e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0, \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}, \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}\}$$

$$\leq c_2.\max\{e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \tag{5}$$

Hence

$$d(x_{2n}, x_{2n-1}) \leq c_1c_2.\max\{ d(x_{2n-1},x_{2n-2}), e(y_{2n-1},y_{2n-2})\} \tag{6}$$

From inequalities (3), (4), (5) and (6), we have

$$d(x_{n+1},x_n) \leq c_1(c_2)^n.\max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

$$e(y_{n+1},y_n) \leq (c_2)^n.\max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$and $\{y_n\}$are Cauchy sequences in (X, d) and (Y, e) respectively. Since (X, d) and (Y, e) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z.

Suppose SAz $\neq$ z.

We have

$$d(SAz, z) = \lim_{n\to\infty} d(SAz, TBx_{2n-1})$$

$$\leq \lim_{n\to\infty} c_1.\max\{d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1},TB\,x_{2n-1}), e(Az, Bx_{2n-1}), \frac{d(z, TBx_{2n-1})}{2}, \frac{d(SAz, x_{2n-1})}{2}, \frac{d(z, TBx_{2n-1}) + d(SAz, x_{2n-1})}{2}\}$$

$$= \lim_{n\to\infty} c_1.\max\{d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, x_{2n}), e(Az, y_{2n}), \frac{d(z, x_{2n})}{2}, \frac{d(SAz, x_{2n-1})}{2}, \frac{d(z, x_{2n}) + d(SAz, x_{2n-1})}{2}\}$$

$$= c_1.\max\{ d(z, z), d(z, SAz), d(z, z), e(w, w), \frac{d(z, z)}{2}, \frac{d(SAz, z)}{2}, \frac{d(SAz, z)}{2}\}$$

$$= c_1.\max\{ 0, d(z, SAz), 0, 0, 0, \frac{d(SAz, z)}{2}\}$$

$$\leq c_1.d(z, SAz)$$

$$< d(z, SAz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose BSw ≠ w.

We have

$$e(BSw, w) = \lim_{n\to\infty} e(BSw , y_{2n+1})$$
$$= \lim_{n\to\infty} e(BSw, ATy_{2n})$$
$$\le \lim_{n\to\infty} c_2 \max\{e(w, y_{2n}), e(w, BSw), e(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n}), \frac{e(w, ATy_{2n})}{2}, \frac{e(BSw, y_{2n})}{2}, \frac{e(w, ATy_{2n}) + e(BSy, y_{2n})}{2}\}$$
$$= c_2.\max \{e(w, w), e(w, BSw), e(w, w), d(z, z), \frac{e(w,w)}{2}, \frac{e(BSw,w)}{2}, \frac{e(BSw,w)}{2}\}$$
$$< e(w,BSw) \quad (\text{Since } 0 \le c_2 < 1)$$

Which is a contradiction.

Thus BSw = w.

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

Suppose TBz ≠ z.

$$d(z,TBz) = \lim_{n\to\infty} d(x_{2n+1}, TBz)$$
$$= \lim_{n\to\infty} d(SAx_{2n}, TBz)$$
$$\le \lim_{n\to\infty} c_1. \max\{ d(x_{2n} ,z), d(x_{2n},SAx_{2n}), d(z, TBz), e(Ax_{2n}, Bz), \frac{d(x_{2n}, TBz)}{2}, \frac{d(SAx_{2n}, z)}{2}, \frac{d(x_{2n}, TBz) + d(SAx_{2n}, z)}{2}\}$$
$$= c_1. \max\{d(z, z), d(z, z), d(z, TBz), e(w, w), \frac{d(z, TBz)}{2}, \frac{d(z,z)}{2}, \frac{d(z, TBz) + d(z,z)}{2}\}$$
$$= c_1. \max\{0, 0, d(z, TBz), 0, \frac{d(z, TBz)}{2}, 0, \frac{d(z, TBz)}{2}\}$$
$$< d(z, TBz) \quad (\text{Since } 0 \le c_1 < 1)$$

Which is a contradiction.

Thus TBz = z.

Hence Tw = z. (Since Bz = w)

Now we prove ATw = w.

Suppose ATw ≠ w.

$$e(w, ATw) = \lim_{n\to\infty} e(y_{2n}, ATw)$$
$$= \lim_{n\to\infty} e(BSy_{2n-1}, ATw)$$
$$\le \lim_{n\to\infty} c_2.\max\{ e(y_{2n-1}, w), e(y_{2n-1}, BSy_{2n-1}), e(w, ATw), d(Sy_{2n-1}, Tw), \frac{e(y_{2n-1}, ATw)}{2}, \frac{e(BSy_{2n-1}, w)}{2}, \frac{e(y_{2n-1}, ATw) + e(BSy_{2n-1}, w)}{2}\}$$

$$= c_2.\max\{ e(w, w), e(w, w), e(w, ATw), d(z,z), \frac{e(w,ATw)}{2}, \frac{e(w,w)}{2}, \frac{e(w,ATw)+e(w,w)}{2}\}$$

$$= c_2.\max\{ 0, 0, e(w, ATw), 0, \frac{e(w,ATw)}{2}, 0, \frac{e(w,ATw)}{2}\}$$

$$< e(w, ATw) \quad (\text{Since } 0 \le c_2 < 1)$$

Which is a contradiction.

Thus ATw = w.

The same results hold if one of the mappings B, S and T is continuous.

**Uniqueness:** Let z′ be another common fixed point of SA and TB in X, w′ be another common fixed point of BS and AT in Y.

We have d(z, z′) = d(SAz, TBz′)

$$\le c_1.\max\{ d(z, z'), d(z,SAz), d(z',TBz'), e(Az,Bz'), \frac{d(z,TBz')}{2}, \frac{d(SAz,z')}{2}, \frac{d(z,TBz')+d(SAz,z')}{2}\}$$

$$= c_1.\max\{ d(z, z'), d(z, z), d(z',z'), e(w,w'), \frac{d(z,z')}{2}, \frac{d(z,z')}{2}, \frac{d(z,z')+d(z,z')}{2}\}$$

$$= c_1.\max\{d(z, z'), d(z, z), d(z', z'), e(w,w'), \frac{d(z,z')}{2}, d(z, z')\}$$

$$= c_1.\max\{d(z, z'), 0, 0, e(w,w'), \frac{d(z,z')}{2}, d(z, z')\}$$

$$\le e(w, w')$$

$$e(w, w') = e(BSw, ATw')$$

$$\le c_2.\max\{ e(w, w'), e(w, BSw), e(w', ATw'), d(Sw, Tw'), \frac{e(w,ATw')}{2}, \frac{e(BSw,w')}{2}, \frac{e(w,ATw')+e(BSw,w')}{2}\}$$

$$= c_2.\max\{e(w, w'), e(w, w), e(w', w'), d(z, z'), \frac{e(w,w')}{2}, \frac{e(w,w')}{2}, \frac{e(w,w')+e(w,w')}{2}\}$$

$$= c_2.\max\{e(w, w'), 0, 0, d(z, z'), \frac{e(w,w')}{2}, e(w, w')\}$$

$$< d(z, z')$$

Hence d(z, z′) < e(w, w′) < d(z, z′)

Which is a contradiction.

Thus z = z′.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

**Remark 2.11:** If we put A = B, S = T in the above theorem 2.10, we get the following corollary.

**Corollary 2.12:** Let (X,d) and (Y,e) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

 

$$d(TAx, TAx') \le c_1. \max\{ d(x, x'), d(x, TAx), d(x', TAx'), e(Ax, Ax'), \frac{d(x,TAx')}{2}, \frac{d(TAx,x')}{2}, \frac{d(x,TAx')+d(TAx,x')}{2}\}$$

$$e(ATy, ATy') \le c_2. \max\{ e(y, y'), e(y, ATy), e(y', ATy'), d(Ty, Ty'), \frac{e(y,ATy')}{2}, \frac{e(ATy,y')}{2}, \frac{e(y,ATy')+e(ATy,y')}{2}\}$$

for all x, x′ in X and y,y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A and T is continuous , then TA has a unique fixed point z in X and AT have a unique fixed point w in Y. Further, Az = w and Tw = z.

**Theorem 2.13:** Let (X, d) and (Y, e) be complete metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$d(SAx, TBx') \le c_1. \max\{ d(x,x'), d(x, SAx), d(x', TBx'), e(Ax, Bx'), \frac{d(x,TBx')}{2}, \frac{d(SAx,x')}{2}, \frac{d(x,TBx')+d(SAx,x')}{2}, \frac{d(x,SAx)+d(x',TBx')}{2}\} \quad (1)$$

$$e(BSy, ATy') \le c_2. \max\{ e(y, y'), e(y, BSy), e(y', ATy'), d(Sy, Ty'), \frac{e(y,ATy')}{2}, \frac{e(BSy,y')}{2}, \frac{e(y,ATy')+e(BSy,y')}{2}, \frac{e(y,BSy)+e(y',ATy')}{2}\} \quad (2)$$

for all x, x′ in X and y,y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

**Proof:** Let $x_0$ be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$Ax_{2n-2} = y_{2n-1}, Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}; Ty_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .

Now we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(SAx_{2n}, TBx_{2n-1}) \\ &\le c_1.\max\{d(x_{2n}, x_{2n-1}), d(x_{2n},SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n},Bx_{2n-1}), \frac{d(x_{2n},TBx_{2n-1})}{2}, \\ &\qquad \frac{d(SAx_{2n},x_{2n-1})}{2}, \frac{d(x_{2n},TBx_{2n-1})+d(SAx_{2n},x_{2n-1})}{2}, \\ &\qquad \frac{d(x_{2n},SAx_{2n})+d(x_{2n-1},TBx_{2n-1})}{2}\} \\ &= c_1. \max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), \frac{d(x_{2n},x_{2n})}{2}, \frac{d(x_{2n+1},x_{2n-1})}{2}, \\ &\qquad \frac{d(x_{2n},x_{2n})+d(x_{2n+1},x_{2n-1})}{2}, \frac{d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n})}{2}\} \\ &\le c_1.\max\{ d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n}), 0, \frac{d(x_{2n+1},x_{2n})+d(x_{2n},x_{2n-1})}{2}, \\ &\qquad \frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2}, \frac{d(x_{2n},x_{2n+1})+d(x_{2n-1},x_{2n})}{2}\} \\ &\le c_1. \max\{d(x_{2n-1}, x_{2n}), e(y_{2n+1},y_{2n})\} \end{aligned}$$

Now

$$\begin{aligned} e(y_{2n}, y_{2n+1}) &= e(BSy_{2n-1}, ATy_{2n}) \\ &\le c_2. \max\{ e(y_{2n-1}, y_{2n}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n}, ATy_{2n}), d(Sy_{2n-1}, Ty_{2n}), \end{aligned}$$

$$\frac{e(y_{2n-1}, ATy_{2n})}{2}, \frac{e(BSy_{2n-1}, y_{2n})}{2}, \frac{e(y_{2n-1}, ATy_{2n}) + e(BSy_{2n-1}, y_{2n})}{2}, \frac{e(y_{2n-1}, BSy_{2n-1}) + e(y_{2n}, ATy_{2n})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n+1})}{2}, \frac{e(y_{2n}, y_{2n})}{2}, \frac{e(y_{2n-1}, y_{2n+1}) + e(y_{2n}, y_{2n})}{2}, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}\}$$

$$\le c_2.\max\{e(y_{2n-1}, y_{2n}), e(y_{2n-1}, y_{2n}), e(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n}), \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}, 0, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n}, y_{2n+1})}{2}\}$$

$$\le c_2. \max\{e(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\} \tag{3}$$

Hence

$$d(x_{2n+1}, x_{2n}) \le c_1c_2. \max\{ d(x_{2n-1}, x_{2n}), e(y_{2n-1}, y_{2n})\} \tag{4}$$

We have

$$d(x_{2n}, x_{2n-1}) = d(x_{2n-1}, x_{2n})$$

$$= d(SAx_{2n-2}, TBx_{2n-1})$$

$$\le c_1.\max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2},SAx_{2n-2}), d(x_{2n-1}, TBx_{2n-1}), e(Ax_{2n-2}, Bx_{2n-1}), \frac{d(x_{2n-2}, TBx_{2n-1})}{2}, \frac{d(SAx_{2n-2}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, TBx_{2n-1}) + d(SAx_{2n-2}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, SAx_{2n-2}) + d(x_{2n-1}, TBx_{2n-1})}{2}\}$$

$$= c_1. \max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n})}{2}, \frac{d(x_{2n-1}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, x_{2n}) + d(SAx_{2n-1}, x_{2n-1})}{2}, \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}\}$$

$$= c_1. \max\{ d(x_{2n-2},x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n}), e(y_{2n-1},y_{2n}), \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}, 0, \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{2}\}$$

$$\le c_1. \max\{ d(x_{2n-2},x_{2n-1}), e(y_{2n-1},y_{2n})\}$$

Now

$$e(y_{2n},y_{2n-1}) = e(BSy_{2n-1}, ATy_{2n-2})$$

$$\le c_2. \max\{ e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, BSy_{2n-1}), e(y_{2n-2}, ATy_{2n-2}), d(Sy_{2n-1}, Ty_{2n-2}), \frac{e(y_{2n-1}, ATy_{2n-2})}{2}, \frac{e(BSy_{2n-1}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, ATy_{2n-2}) + e(BSy_{2n-1}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, BSy_{2n-1}) + e(y_{2n-2}, ATy_{2n-2})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), \frac{e(y_{2n-1}, y_{2n-1})}{2}, \frac{e(y_{2n}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, y_{2n-1}) + e(y_{2n}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n-1}, y_{2n-1})}{2}\}$$

$$= c_2.\max\{e(y_{2n-1}, y_{2n-2}), e(y_{2n-1}, y_{2n}), e(y_{2n-2}, y_{2n-1}), d(x_{2n-1}, x_{2n-2}), 0, \frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2},$$

$$\frac{e(y_{2n}, y_{2n-1}) + e(y_{2n-1}, y_{2n-2})}{2}, \frac{e(y_{2n-1}, y_{2n}) + e(y_{2n-1}, y_{2n-1})}{2}\}$$

$$\leq c_2. \max\{e(y_{2n-1}, y_{2n-2}), d(x_{2n-1}, x_{2n-2})\} \tag{5}$$

Hence

$$d(x_{2n}, x_{2n-1}) \leq c_1c_2. \max\{ d(x_{2n-1},x_{2n-2}), e(y_{2n-1},y_{2n-2})\} \tag{6}$$

from inequalities (3), (4), (5) and (6), we have

$$d(x_{n+1},x_n) \leq c_1(c_2)^n \max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

$$e(y_{n+1},y_n) \leq (c_2)^n. \max\{ d(x_1,x_0), e(y_1, y_0)\} \to 0 \text{ as } n\to\infty$$

Thus $\{x_n\}$and $\{y_n\}$are Cauchy sequences in (X,d) and (Y,e) respectively. Since (X,d) and (Y,e) are complete, $\{x_n\}$ converges to a point z in X and $\{y_n\}$ converges to a point w in Y.

Suppose A is continuous, then

$$\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w.$$

Now we prove SAz = z.

Suppose SAz ≠ z.

We have

$$d(SAz, z) = \lim_{n\to\infty} d(SAz, TBx_{2n-1})$$

$$\leq \lim_{n\to\infty} c_1. \max\{ d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1},TB\, x_{2n-1}), e(Az, Bx_{2n-1}),$$

$$\frac{d(z, TBx_{2n-1})}{2}, \frac{d(SAz, x_{2n-1})}{2}, \frac{d(z, TBx_{2n-1}) + d(SAz, x_{2n-1})}{2},$$

$$\frac{d(z, SAz) + d(x_{2n-1}, TBx_{2n-1})}{2}\}$$

$$= \lim_{n\to\infty} c_1. \max\{d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, x_{2n}), e(Az, y_{2n}), \frac{d(z, x_{2n})}{2}, \frac{d(SAz, x_{2n-1})}{2},$$

$$\frac{d(z, x_{2n}) + d(SAz, x_{2n-1})}{2}, \frac{d(z, SAz) + d(x_{2n-1}, x_{2n})}{2}\}$$

$$= c_1. \max\{ d(z, z), d(z, SAz), d(z, z), e(w, w), \frac{d(z, z)}{2}, \frac{d(SAz, z)}{2}, \frac{d(SAz, z)}{2}, \frac{d(z, SAz)}{2}\}$$

$$= c_1. \max\{ 0, d(z, SAz), 0, 0, 0, \frac{d(SAz, z)}{2}, \frac{d(SAz, z)}{2}, \frac{d(z, SAz)}{2}\}$$

$$\leq c_1. d(z, SAz)$$

$$< d(z, SAz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus SAz = z.

Hence Sw = z. (Since Az = w)

Now we prove BSw = w.

Suppose BSw ≠ w.

We have

$$e(BSw, w) = \lim_{n\to\infty} e(BSw, y_{2n+1})$$

$$= \lim_{n\to\infty} e(BSw, ATy_{2n})$$

$$\leq \lim_{n\to\infty} c_2 \max\{ e(w, y_{2n}), e(w, BSw), e(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n}), \frac{e(w, ATy_{2n})}{2}, \frac{e(BSw, y_{2n})}{2}, \frac{e(w, ATy_{2n}) + e(BSy, y_{2n})}{2}, \frac{e(w, BSw) + e(y_{2n}, ATy_{2n})}{2}\}$$

$$= c_2.\max\{e(w, w), e(w, BSw), e(w, w), d(z, z), \frac{e(w, w)}{2}, \frac{e(BSw, w)}{2}, \frac{e(BSw, w)}{2}, \frac{e(BSw, w)}{2}\}$$

$$< e(w, BSw) \quad (\text{Since } 0 \leq c_2 < 1)$$

Which is a contradiction.

Thus BSw = w.

Hence Bz = w. (Since Sw = z)

Now we prove TBz = z.

Suppose TBz $\neq$ z.

$$d(z,TBz) = \lim_{n\to\infty} d(x_{2n+1}, TBz)$$

$$= \lim_{n\to\infty} d(SAx_{2n}, TBz)$$

$$\leq \lim_{n\to\infty} c_1. \max\{ d(x_{2n} ,z), d(x_{2n},SAx_{2n}), d(z, TBz), e(Ax_{2n}, Bz), \frac{d(x_{2n}, TBz)}{2}, \frac{d(SAx_{2n}, z)}{2}, \frac{d(x_{2n}, TBz) + d(SAx_{2n}, z)}{2}, \frac{d(x_{2n}, SAx_{2n}) + d(z, TBz)}{2}\}$$

$$= c_1.\max\{d(z, z), d(z, z), d(z, TBz), e(w, Bz), \frac{d(z, TBz)}{2}, \frac{d(z, z)}{2}, \frac{d(z, TBz) + d(z, z)}{2}, \frac{d(z, z) + d(z, TBz)}{2}\}$$

$$= c_1.\max\{0, 0, d(z, TBz), 0, \frac{d(z, TBz)}{2}, 0, \frac{d(z, TBz)}{2}, \frac{d(z, TBz)}{2}\}$$

$$< d(z, TBz) \quad (\text{Since } 0 \leq c_1 < 1)$$

Which is a contradiction.

Thus TBz = z.

Hence Tw = z. (Since Bz = w)

Now we prove ATw = w.

Suppose ATw $\neq$ w.

$$e(w, ATw) = \lim_{n\to\infty} e(y_{2n}, ATw)$$

$$= \lim_{n\to\infty} e(BSy_{2n-1,} ATw)$$

$$\leq \lim_{n\to\infty} c_2.\max\{ e(y_{2n-1}, w), e(y_{2n-1}, BSy_{2n-1}), e(w, ATw), d(Sy_{2n-1}, Tw), \frac{e(y_{2n-1}, ATw)}{2}, \frac{e(BSy_{2n-1}, w)}{2}, \frac{e(y_{2n-1}, ATw) + e(BSy_{2n-1}, w)}{2}, \frac{e(y_{2n-1}, BSy_{2n-1}) + e(w, ATw)}{2}\}$$

$$= c_2. \max\{e(w, w), e(w, w), e(w, ATw), d(z,z), \frac{e(w, ATw)}{2}, \frac{e(w, w)}{2},$$

$$\frac{e(w, ATw) + e(w, w)}{2}, \frac{e(w, w) + e(w, ATw)}{2}\}$$

$$= c_2.\ \max\{\ 0\ ,\ 0,\ e(w,\ ATw),\ 0\ ,\ \frac{e(w, ATw)}{2},\ 0,\ \frac{e(w, ATw)}{2}, \frac{e(w, ATw)}{2}\}$$

$$< e\ (w,\ ATw) \quad (\text{Since } 0 \leq c_2 < 1)$$

Which is a contradiction.

Thus ATw = w.

The same results hold if one of the mappings B, S and T is continuous.

**Uniqueness:** Let z′ be another common fixed point of SA and TB in X, w′ be another common fixed point of BS and AT in Y.

We have d(z, z′) = d(SAz, TBz′)

$$\leq c_1.\max\{\ d(z,\ z'),\ d(z,\ SAz),\ \ d(z',\ TBz'),\ e(Az,\ Bz'), \frac{d(z, TBz')}{2}, \frac{d(SAz, z')}{2},$$

$$\frac{d(z, TBz') + d(SAz, z')}{2}, \frac{d(z, SAz) + d(z', TBz')}{2}\}$$

$$\leq c_1.\max\{\ d(z,\ z'),\ d(z,\ z),\ \ d(z',\ z'),\ e(w,\ w'), \frac{d(z, z')}{2}, \frac{d(z, z')}{2},$$

$$\frac{d(z, z') + d(z, z')}{2}, \frac{d(z, z) + d(z', z')}{2}\ \}$$

$$= c_1.\max\{d(z,\ z'),\ 0\ ,\ \ 0\ ,\ e(w,\ w'), \frac{d(z, z')}{2}, \frac{d(z, z')}{2},\ d(z,\ z'),\ 0\ \}$$

$$\leq e(w,\ w')$$

$$e(w,\ w') = e(BSw,\ ATw')$$

$$\leq c_2.\ \max\{\ e(w,w'),\ e(w,\ BSw),\ e(w',\ ATw'),\ d(Sw,\ Tw'),$$

$$\frac{e(w, ATw')}{2}, \frac{e(BSw, w')}{2}, \frac{e(w, ATw') + e(BSw\ , w')}{2},$$

$$\frac{e(w, BSw) + e(w', ATw')}{2}\ \}$$

$$= c_2.\ \max\{e(w,\ w'),\ e(w,\ w),\ e(w',\ w'),\ d(z,\ z'),\ \frac{e(w, w')}{2}, \frac{e(w, w')}{2},$$

$$\frac{e(w, w) + e(w', w')}{2}\ ,\ \frac{e(w, w') + e(w\ , w')}{2}\}$$

$$= c_2.\ \max\{e(w,\ w'),\ 0,\ 0,\ d(z,\ z'), \frac{e(w, w')}{2},\ \frac{e(w, w')}{2},\ 0,\ e(w,\ w')\}$$

$$< d(z,\ z')$$

Hence d(z, z′) < e(w, w′) < d(z, z′)

Which is a contradiction.

Thus z = z′.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

**Remark 2.14:** If we put A = B, S = T in the above theorem 2.13, we get the following corollary.

**Corollary 2.15:** Let (X, d) and (Y, e) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$$d(TAx, TAx') \le c_1.\max\{ d(x, x'), d(x, TAx), d(x', TAx'), e(Ax, Ax'), \frac{d(x,TAx')}{2}, \frac{d(TAx,x')}{2}, \frac{d(x,TAx')+d(TAx,x')}{2}, \frac{d(x,TAx)+d(x',TAx')}{2}\}$$

$$e(ATy, ATy') \le c_2.\max\{ e(y, y'), e(y, ATy), e(y', ATy'), d(Ty, Ty'), \frac{e(y,ATy')}{2}, \frac{e(ATy,y')}{2}, \frac{e(y,ATy')+e(ATy,y')}{2}, \frac{e(y,ATy)+e(y',ATy')}{2}\}$$

for all x, x′ in X and y, y′ in Y where $0 \le c_1 < 1$ and $0 \le c_2 < 1$. If one of the mappings A and T is continuous , then TA has a unique fixed point z in X and AT have a unique fixed point w in Y. Further, Az = w and Tw = z.

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