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#### Abstract

A Graph $G=(V, E)$ with $p$ vertices and $q$ edges is called a Harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels from $1,2 \ldots q+1$ in such $a$ way that when each edge $e=u v$ is labeled with $f(e=u v)=\left\lceil\frac{2 f(u) f(v)}{f(u)+f(v)}\right\rceil$ (or) $\left\lfloor\frac{2 f(u) f(v)}{f(u)+f(v)}\right\rceil$, then the edge labels are distinct. In this case $f$ is called Harmonic Mean labeling of $G$.

In this paper we prove the Harmonic mean labeling behaviour for some new graphs.


Key words: Graph, Harmonic mean graph, Path, Cycle, Star graph, Union of graphs, Comb, Crown.

## INTRODUCTION

We begin with simple, finite and undirected graph $G=(V, E)$ with $p$ vertices and $q$ edges. The vertex set is denoted by $\mathrm{V}(\mathrm{G})$ and the edge set is denoted by $\mathrm{E}(\mathrm{G})$. For standard terminology and Notations we follow Harary [1].
S. Somasundaram and S. S. Sandhya introduced Harmonic mean labeling of graphs in [2] and studied their behaviour in [3], [4] and [5]. In this paper, we investigate Harmonic mean labeling for some new graphs.

We will provide brief summary of definitions and other informations which are necessary for the present investigation.
Definition 1.1: A Graph $G$ with $p$ vertices and $q$ edges is called a Harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2 \ldots . . \mathrm{q}+1$ in such a way that when each edge $e=u v$ is labeled with $\mathrm{f}(\mathrm{e}=\mathrm{uv})=\left\lceil\frac{2 f(u) f(v)}{f(u)+f(v)}\right\rceil$ or $\left\lfloor\frac{2 f(u)+f(v)}{f(u)+f(v)}\right\rceil$, then the edge labels are district. In this case f is called Harmonic mean labeling of G.

Definition 1.2: The union of two graphs $G_{1}$ and $G_{2}$ is graph $G_{1} \cup G_{2}$ with $V\left(G_{1} \cup G_{2}\right)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E\left(G_{1} \cup G_{2}\right)=$ E $\left(G_{1}\right) \cup\left(G_{2}\right)$

Definition 1.3: The Corona $G_{1} A G_{2}$ of two graphs $G_{1}$ and $G_{2}$ is defined as the Graph $G$ obtained by taking one copy of $G_{1}$ (which has $p_{1}$ vertices) and $p_{1}$ copies of $G_{2}$ and then joining the $i^{\text {th }}$ vertex of $G_{1}$ to every vertices in the $i^{\text {th }}$ copy of $G_{2}$.

Definition: 1.4: The graph $P_{n} A K_{1, m}$ vertex of $P_{m}$ is obtained by attaching $K_{1, m}$ to each vertex of $P_{n}$.
Definition 1.5: The graph $\mathrm{C}_{\mathrm{n}} A K_{1}$ is called a crown.
Definition 1.6: Comb is a graph obtained by joining a single pendant edge to each vertex of a path.
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We shall make frequent reference to the following results.
Theorem 1.7 [2]: Any path is a Harmonic mean graph.
Theorem 1.8 [2]: Any cycle is a Harmonic mean graph.
Theorem 1.9 [2]: Combs are Harmonic mean graphs.
Theorem 1.10 [3]: Crowns are harmonic mean graphs.

## 2. MAIN RESULTS

Theorem 2.1: $\left.C_{n} A K_{1} \cup P_{n} A K_{1}\right)$ is a Harmonic mean graph.
Proof: Let $G=C_{n} A K_{1} \cup P_{n} A K_{1}$ be the given graph.
Let $\mathrm{u}_{1}, \mathrm{u}_{2}, \ldots . . \mathrm{u}_{\mathrm{m}}, \mathrm{u}_{\mathrm{m}+1}, \ldots . \mathrm{u}_{\mathrm{n}}$ and $\mathrm{v}_{1}, \mathrm{v}_{2} \ldots . \mathrm{v}_{\mathrm{m}}, \mathrm{v}_{\mathrm{m}+1} \ldots \ldots \mathrm{v}_{\mathrm{n}}$ are the vertices of G .
Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots . \mathrm{q}+1\}$
by

$$
\begin{aligned}
\mathrm{f}\left(\mathrm{u}_{i}\right) & =2 i, 1 \leq i \leq \mathrm{m} \\
\mathrm{f}\left(\mathrm{u}_{i}\right) & =2 i-1, \mathrm{~m}+1 \leq i \leq \mathrm{n} \\
\mathrm{f}\left(\mathrm{v}_{i}\right) & =2 i-1,1 \leq i \leq \mathrm{m} \\
\mathrm{f}\left(\mathrm{v}_{\mathrm{i}}\right) & =2 i, \mathrm{~m}+1 \leq i \leq \mathrm{n}
\end{aligned}
$$

Edges are labeled with

$$
\begin{aligned}
& \mathrm{f}\left(\mathrm{u}_{1} \mathrm{u}_{2}\right)=2 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=2 i+1,2 \leq i \leq \mathrm{m}-1 . \\
& \mathrm{f}\left(\mathrm{u}_{\mathrm{m}} \mathrm{u}_{1}\right)=4 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=2 i, \mathrm{~m}+1 \leq i \leq \mathrm{n}-1 \\
& \mathrm{f}\left(\mathrm{u}_{\mathrm{i}} \mathrm{v}_{i}\right)=2 i-1,1 \leq i \leq 2 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{i}\right)=2 i, 3 \leq i \leq \mathrm{m} \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{i}\right)=2 i-1, \mathrm{~m}+1 \leq i \leq \mathrm{n}
\end{aligned}
$$

Hence $G=C_{n} A K_{1} \cup P_{n} A K_{1}$ is a harmonic mean graph.
Example 2.2: A Harmonic mean labeling $C_{8} A K_{1} \cup P_{5} A K_{1}$ is given below.


Figure: 1

## Next we have

Theorem 2.3: A graph obtained by attaching $\mathrm{K}_{1,2}$ at each pendant vertex of a comb is a Harmonic mean graph.
Proof: Let $G_{1}$ be a comb and $G$ be the graph obtained by attaching $K_{1,2}$ at each pendant vertex of $G_{1}$
Let its vertices be $\mathrm{u}_{i}, \mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i} 1}, \mathrm{v}_{\mathrm{i} 2}, 1 \leq i \leq \mathrm{n}$
Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots . . \mathrm{q}+1\}$
by

$$
\begin{aligned}
& \mathrm{f}\left(\mathrm{u}_{\mathrm{i}}\right)=4 i-2,1 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{1}\right)=3 \\
& \mathrm{f}\left(\mathrm{v}_{i}\right)=4 i-3,2 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{11}\right)=1 \\
& \mathrm{f}\left(\mathrm{v}_{11}\right)=4 i-1,2 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{\mathrm{i} 2}\right)=4 \mathrm{i}, 1 \leq i \leq \mathrm{n}
\end{aligned}
$$

Edges are labeled with

$$
\begin{aligned}
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=4 i, 1 \leq i \leq \mathrm{n}-1 \\
& \mathrm{f}\left(\mathrm{u}_{1} \mathrm{v}_{1}\right)=2 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{i}\right)=4 i-3,2 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{1} \mathrm{v}_{11}\right)=1 \\
& \mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{i 1}\right)=4 i-2,2 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{\mathrm{i}}\right)=4 i-1,1 \leq i \leq \mathrm{n}
\end{aligned}
$$

Hence f provide a Harmonic mean labeling for G.
Example 2.4: A Harmonic mean labeling for $G$ is given below


Figure: 2
The same argument gives the following.
Theorem 2.5: A graph obtained by attaching $\mathrm{K}_{1,2}$ at each pendant vertex of the crown is a Harmonic men graphs.
Proof: Let $G_{1}$ be the Crown and $G$ be the graph obtained by attaching $K_{1,2}$ at each pendant vertex of the crown. Let it vertices be $\mathrm{u}_{i}, \mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i} 1}, \mathrm{v}_{\mathrm{i} 2}, 1 \leq i \leq \mathrm{n}$

Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots . \mathrm{q}+1\}$
by $\quad f\left(u_{1}\right)=2$
$f\left(\mathrm{u}_{\mathrm{i}}\right)=4 i-2,2 \leq i \leq n$
$f\left(v_{1}\right)=4$
$f\left(v_{i}\right)=4 i-2,2 \leq i \leq n$
$f\left(v_{11}\right)=1$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i} 1}\right)=4 i, 2 \leq i \leq \mathrm{n}$
$f\left(v_{i 2}\right)=4 i+1,1 \leq i \leq n$
Edges are labeled with

$$
\begin{aligned}
& \mathrm{f}\left(\mathrm{u}_{1} \mathrm{u}_{2}\right)=3 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=4 i+1,2 \leq i \leq \mathrm{n}-1 \\
& \mathrm{f}\left(\mathrm{u}_{\mathrm{n}} \mathrm{u}_{1}\right)=4 \\
& \mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{i}\right)=4 i-2,1 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{1} \mathrm{v}_{11}\right)=1 \\
& \mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{i 1}\right)=4 i-1,2 \leq i \leq \mathrm{n} \\
& \mathrm{f}\left(\mathrm{v}_{1} \mathrm{v}_{12}\right)=5 \\
& \mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{\mathrm{i} 2}\right)=4 i, 2 \leq i \leq \mathrm{n}
\end{aligned}
$$

Example 2.6: The following figure shows the Harmonic mean labeling behaviour for G.


Figure: 3
Now we prove the following
Theorem 2.7: A Graph obtained by attaching a triangle at each pendant vertex of a comb is a Harmonic mean graph.
Proof: Let $\mathrm{G}^{1}$ be a comb and $G$ be a graph obtained by attaching a triangle at each pendant vertex of a comb. Let its vertices be $\mathrm{u}_{i}, \mathrm{v}_{\mathrm{i}}, \mathrm{u}_{i 1}, \mathrm{u}_{i 2}, 1 \leq i \leq \mathrm{n}$.

Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots . . \mathrm{q}+1\}$
by $\mathrm{f}\left(\mathrm{u}_{\mathrm{i}}\right)=5 i-2,1 \leq i \leq \mathrm{n}$
$f\left(v_{i}\right)=5 i-1,1 \leq i \leq n$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i} 1}\right)=5 i-4,1 \leq i \leq \mathrm{n}$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i} 2}\right)=5 i-3,1 \leq i \leq \mathrm{n}$
Edges are labeled with
$\mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=5 i, 1 \leq i \leq \mathrm{n}-1$
$\mathrm{f}\left(\mathrm{u}_{\mathrm{i}} \mathrm{v}_{\mathrm{i}}\right)=5 i-1,1 \leq i \leq \mathrm{n}$
$\mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{i 1}\right)=5 i-3,1 \leq i \leq \mathrm{n}$
$\mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{\mathrm{i} 2}\right)=5 i-2,1 \leq i \leq \mathrm{n}$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i} 1} \mathrm{v}_{\mathrm{i} 2}\right)=5 i-4,1 \leq i \leq \mathrm{n}$
Hence f provide a Harmonic mean labeling for G.
Example 2.8: Harmonic mean labeling pattem of G is shown below.


Figure: 4
Consequently we have the following
Theorem 2.9: A graph obtained by attaching a triangle at each pendant vertex of a crown is Harmonic mean graph.
Proof: Let $G_{1}$ be a crown and $G$ be a graph obtained by attaching a triangle at each pendant vertex of a crown.
Let the vertices of the graph be $\mathrm{u}_{i}, \mathrm{v}_{i} \mathrm{v}_{\mathrm{i} 1}, \mathrm{v}_{\mathrm{i} 2}, 1 \leq i \leq \mathrm{n}$
Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots \ldots \ldots . \mathrm{q}+1\}$
by $f\left(u_{1}\right)=2$
$\mathrm{f}\left(\mathrm{u}_{i}\right)=4 i-1,2 \leq i \leq \mathrm{n}$
$f\left(v_{1}\right)=4$
$f\left(v_{i}\right)=4 i-2,2 \leq i \leq n$
$f\left(v_{11}\right)=1$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i} 1}\right)=4 i, 2 \leq i \leq \mathrm{n}$
$f\left(v_{i 2}\right)=4 i+1,1 \leq i \leq n$
Edges are labeled with
$f\left(\mathrm{u}_{1} \mathrm{u}_{2}\right)=3$
$\mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=4 i+1,2 \leq i \leq \mathrm{n}-1$,
$f\left(u_{n} u_{1}\right)=6$
$f\left(u_{i} v_{i}\right)=4 i-2,1 \leq i \leq n$
$f\left(v_{1} v_{11}\right)=1$
$\mathrm{f}\left(\mathrm{v}_{i} \mathrm{v}_{\mathrm{i} 1}\right)=5 i-22 \leq i \leq \mathrm{n}$.
$f\left(v_{1} v_{12}\right)=3$
$f\left(v_{i} v_{i 2}\right)=5 i-1,2 \leq i \leq n$
$f\left(\mathrm{v}_{11} \mathrm{v}_{12}\right)=2$
$\mathrm{f}\left(\mathrm{v}_{\mathrm{i1}} \mathrm{v}_{\mathrm{i} 2}\right)=5 i-3,2 \leq i \leq n$
Thus $f$ provide a Harmonic mean labeling for G.
Example 2.10: The Harmonic mean labeling pattern of $G$ is shown in the following figure


Figure: 5
Next we have
Definition 2.11: Let $v$ be a vertex of a graph $G$. Then the duplication of $v$ is a graph $G(v)$ obtained from $G$ by adding a new vertex $\mathrm{v}^{1}$ with $\mathrm{N}\left(\mathrm{v}^{1}\right)=\mathrm{N}(\mathrm{v})$

Definition 2.12: Let $\mathrm{e}=\mathrm{uv}$ be an edge of G . Then the duplication of an edge $\mathrm{e}=\mathrm{uv}$ is a graph $\mathrm{G}(u v)$ obtained by adding a new edge u'v' s.t
$N\left(u^{\prime}\right)=N(u) \cup\left\{v^{\prime}\right\}-\{v\}$ and
$\mathrm{N}\left(\mathrm{v}^{\prime}\right)=\mathrm{N}(\mathrm{v}) \cup\left\{\mathrm{u}^{\prime}\right\}-\{\mathrm{u}\}$
Now we prove the following
Theorem 2.13: The graph obtained by duplicating each edge in a cycle $C_{n}$ is a Harmonic mean graph.
Proof: Let $G$ be a graph obtained by duplicating each edge in a cycle $C_{n}$.
Let $\mathrm{u}_{1}, \mathrm{u}_{2}$. $\qquad$ . $\mathrm{u}_{\mathrm{n}} \mathrm{u}_{1}$ be the given cycle.

Let $\mathrm{e}_{1}, \mathrm{e}_{2} \ldots \ldots .$. . $\mathrm{e}_{\mathrm{n}}$ are the edges obtained by duplication.

Define a function $\mathrm{f}: \mathrm{V}(\mathrm{G}) \rightarrow\{1,2 \ldots \ldots \mathrm{q}+1\}$
by $\mathrm{f}\left(\mathrm{u}_{i}\right)=3 i, 1 \leq i \leq \mathrm{n}$
$\mathrm{f}\left(\mathrm{v}_{1}\right)=1$
$f\left(v_{i}\right)=3 i-1,2 \leq i \leq n$
Edges are labeled with $\mathrm{f}\left(\mathrm{u}_{1} \mathrm{u}_{2}\right)=4$
$\mathrm{f}\left(\mathrm{u}_{i} \mathrm{u}_{i+1}\right)=3 i+2,2 \leq i \leq \mathrm{n}-1$
$\mathrm{f}\left(\mathrm{u}_{\mathrm{n}} \mathrm{u}_{1}\right)=6$
$\mathrm{f}\left(\mathrm{u}_{\mathrm{i}} \mathrm{v}_{\mathrm{i}}\right)=3 i-1,1 \leq i \leq 2$
$\mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{\mathrm{i}}\right)=3 i, 3 \leq i \leq n$
$f\left(u_{1} v_{2}\right)=3$
$\mathrm{f}\left(\mathrm{u}_{i} \mathrm{v}_{i+1}\right)=3 i+1,2 \leq i \leq \mathrm{n}-1$
$\mathrm{f}\left(\mathrm{u}_{\mathrm{n}} \mathrm{v}_{1}\right)=1$
Hence G is a Harmonic mean graph.
Example 2.14: The Harmonic mean labeling pattern of $G$ is given below


Figure: 6

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