International Journal of Mathematical Archive-4(4), 2013, 179-182
Available online through www.ijma.info ISSN 2229 – 5046

A SUMMATION THEOREM DUE TO RAMANUJAN AND ITS APPLICATION IN FINDING A CURIOUS RESULT FOR $\Gamma\left(\frac{1}{4}\right)$

S. Ismail Azad[1*], Intazar Husain[2] and Mohd. Saeed Akhtar[1]

*[*1]General Studies Department, Yanbu Industrial College, Yanbu, P.O. Box 30436, KSA-51000*
[2]Department of Applied Sciences & Humanities, Delhi Institute of Technology Management and Research, Faridabad, Haryana, India-121004

(Received on: 02-03-13; Revised & Accepted on: 23-03-13)

ABSTRACT

The aim of this paper to provide the simple way for the generalization of a summation due to Ramanujan and then to use them to find a curious result by applying the theory of hypergeometric function. The result is derived with the help of generalized Dixon theorem available in the literature. The result derived in this research note is simple, interesting, easily established and may be potentially useful.

Keywords: *Generalized Hypergeometric Function; Dixon's Summation Theorem; Generalized Dixon's Summation Theorem, Ramanujan's Summations.*

---

1. INTRODUCTION

In the theory of hypergeometric and generalized hypergeometric series, classical summations such as those of Gauss, Gauss second, Kummer and Bailey for the series $_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series $_3F_2$; generalized Dixon's summation theorem for the series $_5F_4$ and others play an important role [6]. Applications of the above mentioned classical summation theorems are now well known. We start with the following interesting summations due to Ramanujan.

$$1+\frac{1}{5}\left(\frac{1}{2}\right)^2+\frac{1}{9}\left(\frac{1\cdot 2}{2\cdot 4}\right)^2+\frac{1}{11}\left(\frac{1\cdot 2\cdot 3}{2\cdot 4\cdot 6}\right)^2+\cdots=\frac{\pi^2}{4\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4} \tag{1.1}$$

As pointed out by Berndt [4] that the above summation due to Ramanujan can be obtained quite simply by employing following classical Dixon theorem [2].

$$_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b, 1+a-c \end{matrix} ;1\right]=\frac{\Gamma\left(1+\frac{a}{2}\right)\cdot\Gamma(1+a-b)\cdot\Gamma(1+a-c)\cdot\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\cdot\Gamma\left(1+\frac{a}{2}-b\right)\cdot\Gamma\left(1+\frac{a}{2}-c\right)\cdot\Gamma(1+a-b-c)} \tag{1.2}$$

Provided, $\mathrm{R}(a-2b-2c)>-2$

By choosing suitable parameters a, b and c.

The aim of this research note is to provide another natural generalization of the Ramanujan's Summation (1.1), and a curious result (2.1) that gives the value for $\Gamma\left(\frac{1}{4}\right)$ which is in compact form. For this, the following generalized Dixon's summation theorem [3] will be required in our present investigations.

Generalized Dixon's Summation Theorem

$$_5F_4\left[\begin{matrix} a, & \frac{a}{2}+1, & b, & c, & d \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d & \end{matrix} ;\ 1\right]=\frac{\Gamma(1+a-b)\cdot\Gamma(1+a-c)\cdot\Gamma(1+a-d)\cdot\Gamma(1+a-b-c-d)}{\Gamma(1+a)\cdot\Gamma(1+a-b-c)\cdot\Gamma(1+a-b-d)\cdot\Gamma(1+a-c-d)} \tag{1.3}$$

Provided, $\mathrm{Re}(a-b-c-d)>-1$.

Corresponding author: *S. Ismail Azad[1*]*
*[*1]General Studies Department, Yanbu Industrial college, Yanbu, P.O. Box 30436, KSA-51000*

Generalized Hypergeometric Summation Theorem

$$ {}_2F_1\ [a,b;c{:}\ 1] = 1 + \frac{ab}{1!\,c}z + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}z^3 \cdots $$

$$ = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{1.4} $$

which converges if c is not a negative integer (1) for all of $|z| < 1$ and (2) on the unit circle $|z| = 1$.

If $(c-a-b) > 0$. Here, $(a)_n$ is the Pochhammer symbol or rising factorial.

Pochhammer Symbol
The Pochhammer symbol or shifted factorial is defined by

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & ; \quad n = 0 \\ a(a+1)(a+2)\ldots\ldots(a+n-1)\,; & \quad n = 1,2,\ldots\ldots \end{cases} $$

$a \neq 0,-1,-2,\ldots.$ and the notation Γ stands for Gamma function.

Note that $(0)_0 = 1$.

Double Factorial
The double factorial of a positive integer n is a generalization of the usual factorial $n!$ defined by

$$ n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1 & n > 0 \quad \text{odd} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2 & n > 0 \quad \text{even} \\ 1 & n = -1, 0 \end{cases} $$

Note that $(-1)!! = 0!! = 1$, by definition [1], Arfken 1985, p. 547).

Recurrence Relations and Identities

$$ \Gamma(x)\cdot\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)} $$
$$ \Gamma(x)\cdot\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} $$
$$ \Gamma(1+z) = z\cdot\Gamma(z) $$
$$ \Gamma(1-z) = -z\cdot\Gamma(-z) $$
$$ \Gamma(n) = (n-1)! $$
$$ a + pn = \frac{a\left(\frac{a+p}{p}\right)_n}{\left(\frac{a}{p}\right)_n} $$

2. MAIN RESULTS
The following curious summation will be established in this research note.

$$ \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left|\frac{(2n-1)!!}{(4n+1)}\right|^2 = \sqrt{\frac{\pi}{32}}\cdot\Gamma\left(\frac{1}{4}\right) $$

$$ \Gamma\left(\frac{1}{4}\right) = \left(\sqrt{32/\pi}\right)\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left|\frac{(2n-1)!!}{(4n+1)}\right|^2 \tag{2.1} $$

Here two aspects can be illustrated. One is the sum of a curious infinite series in terms of Gamma function and the other is the value of $\Gamma\left(\frac{1}{4}\right)$ in compact form which may be a new entry to the literature of computation of Gamma function.

It has long been known [8], $\pi^{0.25}\Gamma\left(\frac{1}{4}\right)$ that is transcendental (Davis 1959), as is $\Gamma\left(\frac{1}{3}\right)$ (Le Lionnais 1983; [5] Borwein and Bailey 2003, p. 138), and Chudnovsky has apparently recently proved that is $\Gamma\left(\frac{1}{4}\right)$ itself transcendental (Borwein and Bailey 2003, p. 138).

There exist efficient iterative algorithms for $\Gamma\left(\frac{k}{4}\right)$ for all integers k (Borwein and Bailey 2003, p. 137). For example, a quadratically converging iteration for $\Gamma\left(\frac{1}{4}\right) = 3.6256099\cdots$ (Sloane's A068466) is given by defining

$$x_n = \frac{1}{2}\left(x_{n-1}^{1/2} + x_{n-1}^{-1/2}\right)$$

$$y_n = \left(\frac{y_{n-1}x_{n-1}^{1/2} + x_{n-1}^{-1/2}}{y_{n-1}+1}\right)$$

setting $x_0 = \sqrt{2}$ and $y_1 = 2^{0.25}$, and then

$$\Gamma\left(\frac{1}{4}\right) = 2(1+\sqrt{2})^{3/4}\left[\prod_{n=1}^{\infty} x_n^{-1}\left(\frac{1+x_n}{1+y_n}\right)^3\right]^{1/4}$$

(Borwein and Bailey 2003, pp. 137-138).

3. DERIVATION ANALYSIS AND SPECIAL CASE

Without the loss of generality of the generalized Dixon theorem (1.3), one can choose suitable numerator and denominator parameters in various ways to apply in (1.3) to get many [7] convergent infinite series and after a little algebraic manipulation one can deduce many curious results. For example if we put $a = \frac{1}{2}, b = c = \frac{1}{4}$ and $d = x$ in the left hand side as well as right hand side of (1.3) and applying the Pochhammer symbol or shifted factorials and the Gamma function expansions and the recurrence formulas listed in the introduction part are enough to get (3.1).

For $x < 1$:

$$1 + \frac{1}{5}\cdot\frac{x}{(3-2x)} + \frac{1}{6}\cdot\frac{x(x+1)}{(3-2x)(5-2x)} + \frac{5}{26}\cdot\frac{x(x+1)(x+2)}{(3-2x)(5-2x)(7-2x)} + \cdots = \frac{\pi^{1.5}\Gamma(1-x)\Gamma(1.5-x)}{4\{\Gamma(0.75)\}^2\{\Gamma(1.25-x)\}^2} \tag{3.1}$$

Let us discuss some of the special cases of each of the equation listed in section 2.

Using $x = 0.5$ in (3.1) we get (1.1)

Using $x = 0.25$ in (3.1) we get

$$1 + \frac{1}{50} + \frac{1}{216} + \frac{5}{2704} + \frac{35}{36992} + \cdots = \frac{\pi^{1.5}}{4\Gamma(0.75)} \tag{3.2}$$

Which gives a interesting and curious results. We will discuss it as special case as follows.

Special Case:

The most interesting situation is the (3.2)

$$1 + \frac{1}{50} + \frac{1}{216} + \frac{5}{2704} + \frac{35}{36992} + \cdots = \frac{\pi^{1.5}}{4\Gamma(0.75)}$$

$$1 + \frac{1}{50} + \frac{1}{216} + \frac{5}{2704} + \frac{35}{36992} + \cdots = \frac{\pi^{0.5}\Gamma(0.25)}{4\sqrt{2}}$$

$$1 + \frac{3^1}{3!\,5^2} + \frac{3^2 5^1}{5!\,9^2} + \frac{3^2 5^2 7^1}{7!\,13^2} + \frac{3^2 5^2 7^2 9^1}{9!\,17^2} + \cdots = \sqrt{\frac{\pi}{32}}\cdot\Gamma(0.25)$$

$$1 + \frac{3^1}{3!\,5^2} + \frac{3^2 5^1}{5!\,9^2} + \frac{3^2 5^2 7^1}{7!\,13^2} + \frac{3^2 5^2 7^2 9^1}{9!\,17^2} + \cdots = \sqrt{\frac{\pi}{32}}\cdot\Gamma(0.25)$$

$$1 + \frac{1}{2!\,5^2} + \frac{3^2}{4!\,9^2} + \frac{3^2 5^2}{6!\,13^2} + \frac{3^2 5^2 7^2}{8!\,17^2} + \cdots = \sqrt{\frac{\pi}{32}}\cdot\Gamma(0.25)$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} \left| \frac{(2n-1)!!}{(4n+1)} \right|^2 = \sqrt{\frac{\pi}{32}} \cdot \Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right) = \left(\sqrt{\frac{32}{\pi}}\right) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left| \frac{(2n-1)!!}{(4n+1)} \right|^2$$

5. SCOPE

Being open for further work by choosing different possible combinations we can generate many more important and curious relations that will add to the existing literature of special functions and its applications.

ACKNOWLEDGEMENT

The authors are grateful for the useful comments of the anonymous referee. The authors are also thankful to Prof. M. I. Qureshi for his valuable suggestions in the preparation of this paper.

REFERENCES

[1] Arfken, G. Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 544-545 and 547-548, 1985.

[2] Abramowitz, M. and Stegun, I.A.; Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. National Bureau of Standards, Washing- ton, D.C., 1964, Reprinted by Dover Publication, New York, 1972.

[3] Bailey, W. N. Generalized Hypergeometric Series, Cambridge University Press, Cambridge (1935).

[4] Berndt, B. C. Ramanujan's notebooks, Part-II, Springer – Verlag, New York, 1987.

[5] Borwein, J. and Bailey, D. Mathematics by Experiment: Plausible Reasoning in the 21st Century. Wellesley, MA: A K Peters, 2003.

[6] Rainville, E.D. Special Functions, Macmillan, New York, 1960.

[7] Shekhawat, Nidhi, On two summations due to Ramanujan and their generalizations, *Advances in computational mathematics and its applications*, 2(1) pp. 237-238 (2012).

[8] Weisstein, Eric W. "Gamma Function." From Math World-A Wolfram Web Resource. http://mathworld.wolfram.com/GammaFunction.html

Source of support: Nil, Conflict of interest: None Declared