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MONOTONIC BEHAVIOUR OF SOME NEW GENERALIZED FUZZY INFORMATION MEASURE & ITS ESSENTIAL PROPERTIES

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ABSTRACT

In literature, a number of measures of fuzzy entropy analogous to the various information measures have been proposed in order to combine the fuzzy set theory and its application to the entropy concept as fuzzy information measurements. In the present communication, we have proposed new generalized measure of fuzzy entropy and discussed their essential and desirable properties.

***Keywords**: Fuzzy information measure, Entropy, Symmetry and Monotonicity.*

I. INTRODUCTION

Information theory is a relatively new branch of Mathematics that was made mathematically rigorous only in the 1940s. Information theory deals with the study of problems concerning any system. This includes information processing, information storage, information retrieval and decision making. , information theory studies all theoretical problems connected with the transmission of information over communication channels. This includes the study of uncertainty (information) measures and various practical and economical methods of coding information for transmission. The first studies in this direction were undertaken by Nyquist [6] in 1924 and 1928 [7] and by Hartley in 1928 [3], who recognized the logarithmic nature of the measure of information.

In 1948, Shannon [14] published a remarkable paper on the properties of information sources and of the communication channels used to transmit the outputs of these sources. Around the same time Wiener [15] also considered the communication situation and came up, independently, with results similar to those of Shannon. In the past fifty years the literature on information theory has grown quite voluminous and apart from communication theory it has found deep applications in many social, physical and biological sciences, for example, economics, statistics, accounting, language, psychology, ecology, pattern recognition, computer sciences, fuzzy sets, etc.

A key feature of Shannon information theory is the term *"information"* that can often be given a mathematical meaning as a numerically measurable quantity, on the basis of a probabilistic model, in such a way that the solutions of many important problems of information storage and the transmission can be formulated in terms of this measure of the amount of information. As pointed out by Renyi [13] in his fundamental paper on generalized information measures, in other sort of problems other quantities may serve just as well, or even better, as measures of information. This should be supported either by their operational significance or by a set of natural postulates characterizing them, or, preferably, by both. Thus the idea of generalized entropies arises in the literature.

It started with Renyi [13] who characterized scalar parametric entropy as entropy of order r, which includes Shannon entropy as a limiting case. Fuzzy set theory has been studied extensively over the past 30 years. Most of the early interest in fuzzy set theory pertained to representing uncertainty in human cognitive processes. Fuzzy set theory is now applied to problems in engineering, business, medical and related health sciences, and the natural sciences. In 1978, Zadeh [17] first created the theory of fuzzy, which is related to fuzzy set theory. His study showed that the importance of the theory of fuzzy is based on the fact that much of the information on which human decisions is possibilistic rather than probabilistic in nature. Fuzzy set theory is being recognized as an important problem modeling and solution technique. The use of fuzzy set theory as a methodology for modeling and analyzing decision systems is of particular interest to researchers. In 1976, Zimmermann [18] first introduced fuzzy set theory into an ordinary linear programming problem with fuzzy objective and constraints. Zadeh [17] introduced the concept of fuzzy sets in which imprecise knowledge can be used to define an event.To explain the concept of fuzzy entropy in general, Kapur [8] considered the following vector:

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$(\mu_A(x_1), \mu_A(x_2), \ldots\ldots\ldots\ldots\mu_A(x_n))$

where $\mu_A(x_i)$ gives the perception of the grade of membership of the i[th] element of set A. Thus if $\mu_A(x_i) = 0$ then the i[th] element certainty does not belong to set A and if $\mu_A(x_i) = 1$, it definitely belongs to set A. If $\mu_A(x_i) = 0.5$, there is maximum certainty whether x_i belongs to set A or not. The above vector in which every element lies between 0 and 1 and has the interpretation given above, is called fuzzy vector and the set A is called a fuzzy set. If every element of the set is 0 or 1, there is no uncertainty about it and the set is said to be a crisp set. Thus there are 2^n crisp sets with n elements and infinity many sets with n elements. If Some elements are 0 or 1 and the others lie between 0 or 1 the set will still said to be a fuzzy set. With the i[th] element, we associate a fuzzy uncertainty f ($\mu_A(x_i)$), where f(x) has following properties:

I. f(x) = 0 when x = 0 or 1
II. f(x) increases as x goes from 0 to 0.5
III. f(x) decreases as x goes from 0.5 to1.0
IV. f(x)=f(1-x)

If the n elements are independent, the total fuzzy uncertainty is given by

$$H(A) = \sum_{i=1}^{n} f(\mu_A(x_i)$$

The fuzzy uncertainty is called fuzzy entropy.

Klir and Folger [14] stated that the term fuzzy entropy was apparently due to clarity of product terms in the following two expressions:

$$H(A) = \sum_{i=1}^{n} \mu_A(x_i)\ln\mu_A(x_i) - \sum_{i=1}^{n}(1-\mu_A(x_i))\ln(1-\mu_A(x_i))$$

After this development, a large number of measures of fuzzy entropy were discussed, characterized and generalized by various authors.

Bajaj & Hooda [1] introduced two new generalized measures of fuzzy directed divergence with the proof of their validity.

Deshmukh, Khot and Nikhil [2] proposed some new generalized measure of fuzzy entropy based upon real parameters & discussed their properties.

Kumar, Mahajan & R.Kumar [4], [5] introduced some new generalized parametric measures of fuzzy entropy & also they introduced two new generalized parametric measures of fuzzy directed divergence.

Parkash & Gandhi [10],[12] proposed two new generalized fuzzy measures of entropy and also they introduced two new fuzzy measures involving trigonometric functions & simultaneously provided their applications to obtain the basic results already existing in the literature of geometry.

Parkash, Gandhi & Thukral [11] proved that sampling distributions can be used to develop new information measures.

Zarandi [16] introduced the concept of fuzzy information theory using the notion of fuzzy sets.

II. NEW GENERALIZED MEASURE OF FUZZY ENTROPY

In this section, we propose the following generalized parametric measure of fuzzy entropy:

$$H_\alpha^\beta(A) = \frac{1}{(1-\alpha)\beta}\sum_{i=1}^{n}\left(\{\,\mu_A^{\alpha\mu_A(x_i)}(x_i) + (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}\}^\beta - 2^\beta\right)$$

where $\alpha > 0\,, \alpha \neq 1\,, \beta \neq 0.$ (2.1)

Under the assumption, $0^{0.\alpha} = 1$ we study the following properties:

1. $H_\alpha^\beta(A) \geq 0$
2. When $\mu_A(x_i) = 0$, $H_\alpha^\beta(A) = 0$
3. When $\mu_A(x_i) = \frac{1}{2}$,

$$H_\alpha^\beta(A) = \frac{n.2^\beta}{(1-\alpha).\beta}\left(\frac{1-2^{\frac{\alpha\beta}{2}}}{2^{\frac{\alpha\beta}{2}}}\right)$$

Hence, $H_\alpha^\beta(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \le \mu_A(x_i) \le \frac{1}{2}$

4. When $\mu_A(x_i) = 1$, $H_\alpha^\beta(A) = 0$. Hence $H_\alpha^\beta(A)$ is an decreasing function of $\mu_A(x_i)$ for $\frac{1}{2} \le \mu_A(x_i) \le 1$

5. $H_\alpha^\beta(A)$ does not change when $\mu_A(x_i)$ is changed to (1 - $\mu_A(x_i)$)

Under the above conditions, the generalized measure proposed in (2.1) is a valid measure of fuzzy entropy
Next, we have computed different values of $H_\alpha^\beta(A)$ for different values of α and β and presented the generalized measure graphically.

Case I: For $\alpha > 1$, $\beta = 1$, we have compiled the values of $H_\alpha^\beta(A)$ in Table (2.1), (2.2), (2.3) and presented the fuzzy entropy in Fig (2.1), (2.2), (2.3) which clearly shows that the fuzzy entropy is a concave function.

For $\alpha = 2$, $\beta = 1$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	0.5419
0.2	0.7749
0.3	0.9075
0.4	0.9778
0.5	1.0
0.6	0.9778
0.7	0.9075
0.8	0.7749
0.9	0.5419
1.0	0.0

Table: 2.1

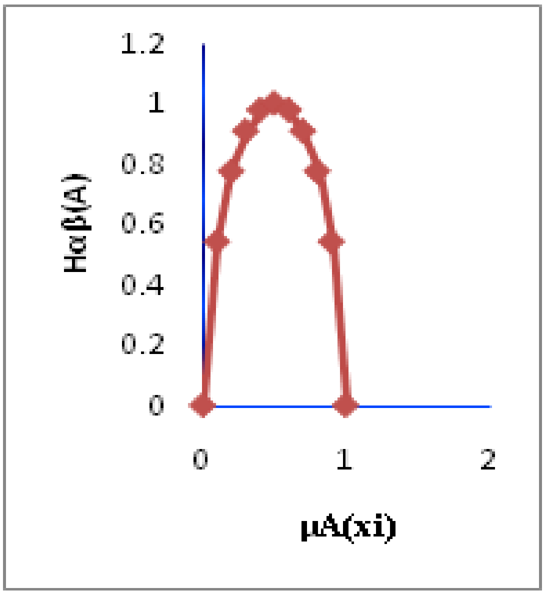

Figure: 2.1

For $\alpha = 4, \beta = 1$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	0.3058
0.2	0.4114
0.3	0.4652
0.4	0.4919
0.5	0.5
0.6	0.4919
0.7	0.4652
0.8	0.4114
0.9	0.3058
1.0	0.0

Table: 2.2

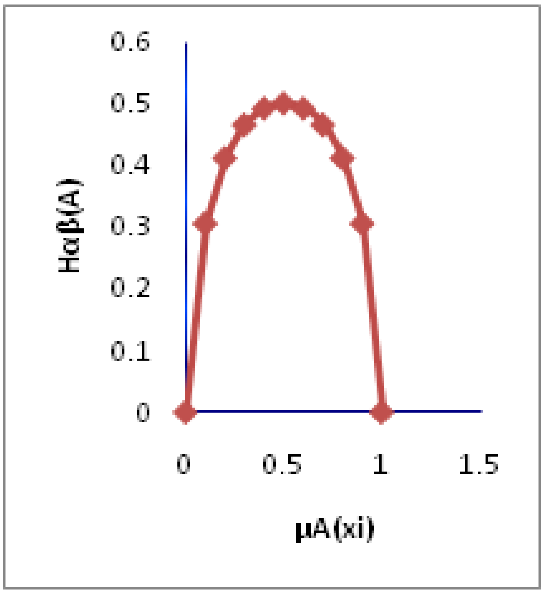

Figure: 2.2

For $\alpha = 6, \beta = 1$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	0.2365
0.2	0.3024
0.3	0.3323
0.4	0.3460
0.5	0.35
0.6	0.3460
0.7	0.3323
0.8	0.3024
0.9	0.2365
1.0	0.0

Table: 2.3

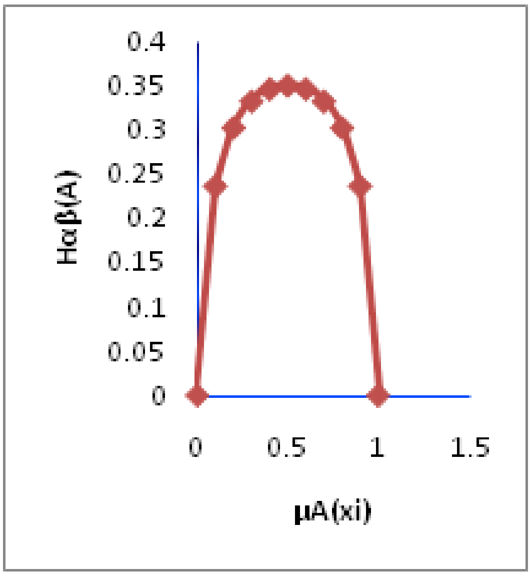

Figure: 2.3

Case II: For $0 < \alpha < 1$ and $0 < \beta < 1$.

We have compiled the values of $H_\alpha^\beta(A)$ in Table (2.4), (2.5) and presented the fuzzy entropy in Fig (2.4), (2.5) which clearly shows that the fuzzy entropy is a concave function.

Let α = 0.2 and $\beta = 0.6$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	-0.0610
0.2	-0.0933
0.3	-0.1135
0.4	-0.1249
0.5	-0.1286
0.6	-0.1249
0.7	-0.1135
0.8	-0.0933
0.9	-0.0610
1.0	0.0

Table: 2.4

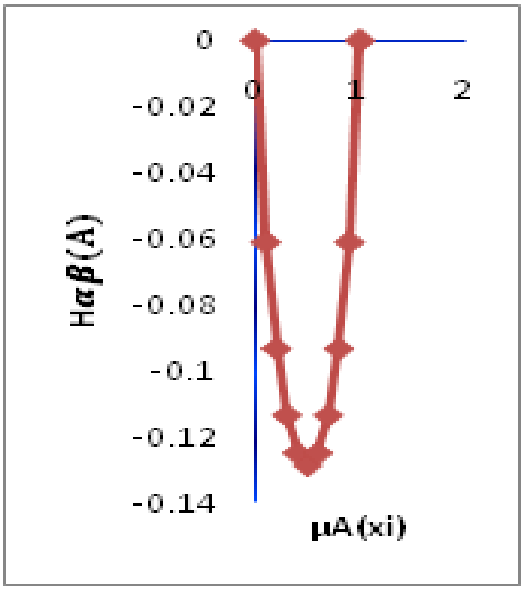

Figure: 2.4

Let α = 0.4 and $\beta = 0.8$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	-0.1829
0.2	-0.2781
0.3	-0.3372
0.4	-0.3701
0.5	-0.3808
0.6	-0.3701
0.7	-0.3372
0.8	-0.2781
0.9	-0.1829
1.0	0.0

Table: 2.5

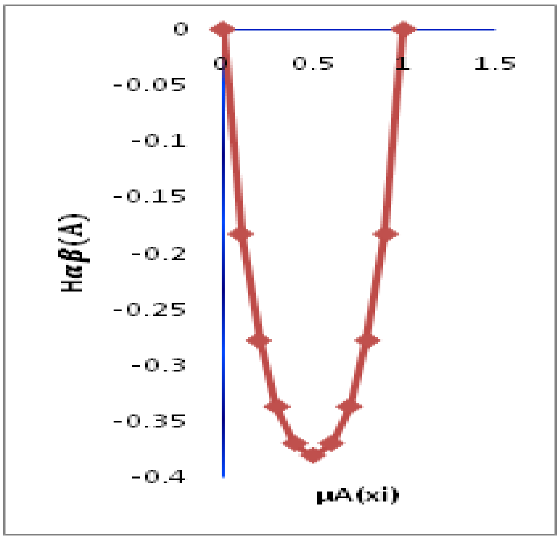

Figure: 2.5

Case III: For $0 < \alpha < 1$ and $\beta > 1$.

We have compiled the values of $H_\alpha^\beta(A)$ in Table (2.6), (2.7) and presented the fuzzy entropy in Fig (2.6), (2.7) which clearly shows that the fuzzy entropy is a concave function.

Let α = 0.2 and $\beta = 1.5$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	-0.1121
0.2	-0.1702
0.3	-0.2061
0.4	-0.2262
0.5	-0.2327
0.6	-0.2262
0.7	-0.2061
0.8	-0.1702
0.9	-0.1121
1.0	0.0

Table: 2.6

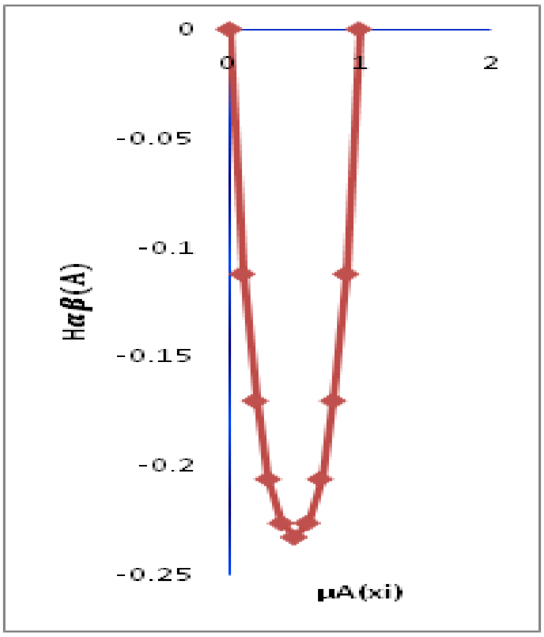

Figure: 2.6

Let α = 0.4 and β = 2.0

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	-0.4043
0.2	-0.6026
0.3	-0.7215
0.4	-0.7864
0.5	-0.8072
0.6	-0.7864
0.7	-0.7215
0.8	-0.6026
0.9	-0.4043
1.0	0.0

Table: 2.7

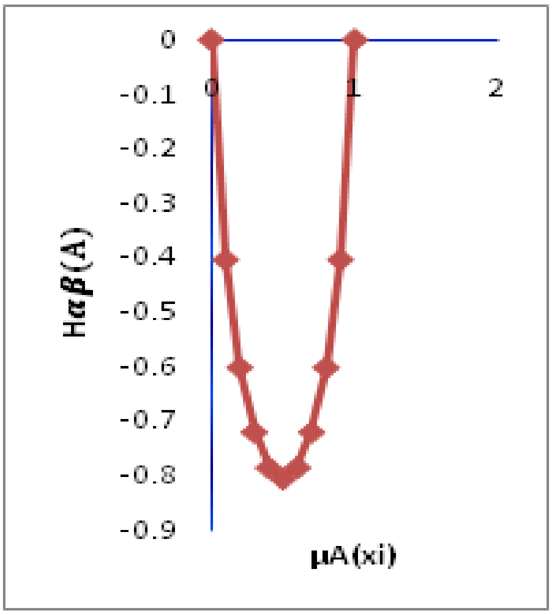

Figure :2.7

Case 4: For $\alpha > 1$ and $0 < \beta < 1$.

We have compiled the values of $H_\alpha^\beta(A)$ in Table (2.8), (2.9) and presented the fuzzy entropy in Fig (2.8), (2.9) which clearly shows that the fuzzy entropy is a concave function.

Let α = 1.5 and $\beta = 0.1$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	0.5074
0.2	0.7763
0.3	0.9534
0.4	1.0517
0.5	1.0844
0.6	1.0517
0.7	0.9534
0.8	0.7763
0.9	0.5074
1.0	0.0

Table: 2.8

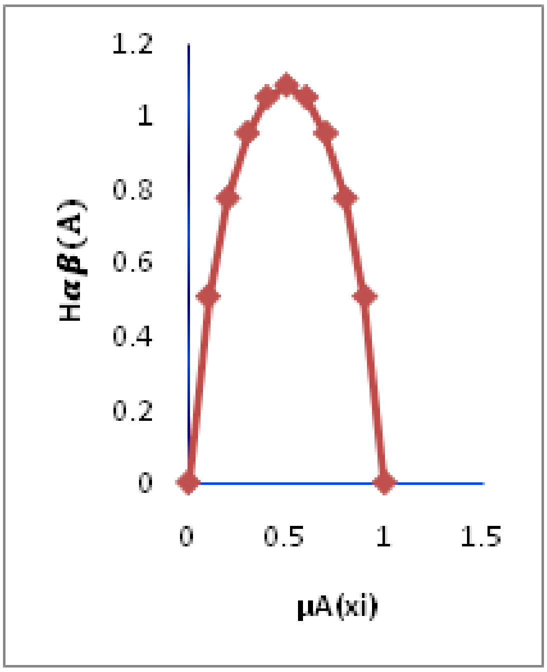

Figure: 2.8

Let α = 2.0 and $\beta = 0.2$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	0.3519
0.2	0.5358
0.3	0.6538
0.4	0.7210
0.5	0.7430
0.6	0.7210
0.7	0.6538
0.8	0.5358
0.9	0.3510
1.0	0.0

Table: 2.9

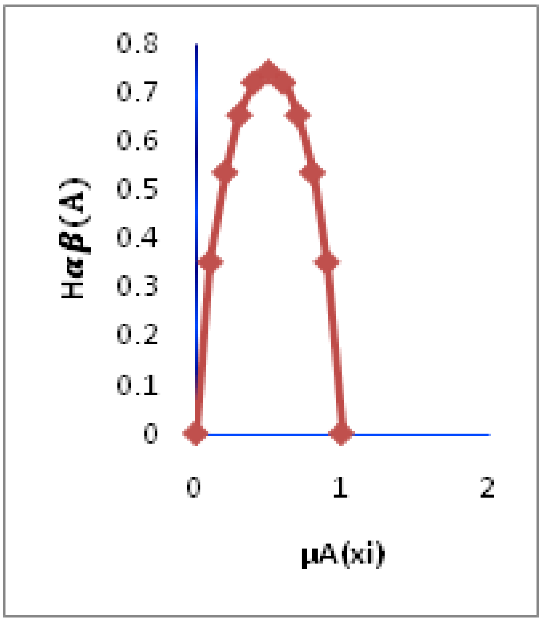

Figure: 2.9

Case 5: For $\alpha > 1$ and $\beta > 1$ we have compiled the values of $H_\alpha^\beta(A)$ in Table (2.10), (2.11) and presented the fuzzy entropy in Fig (2.10), (2.11) which clearly shows that the fuzzy entropy is a concave function.

Let α = 1.5 and β = 2.5

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	2.0337
0.2	2.7289
0.3	3.0732
0.4	3.2411
0.5	3.2916
0.6	3.2411
0.7	3.0732
0.8	2.7289
0.9	2.0337
1.0	0.0

Table: 2.10

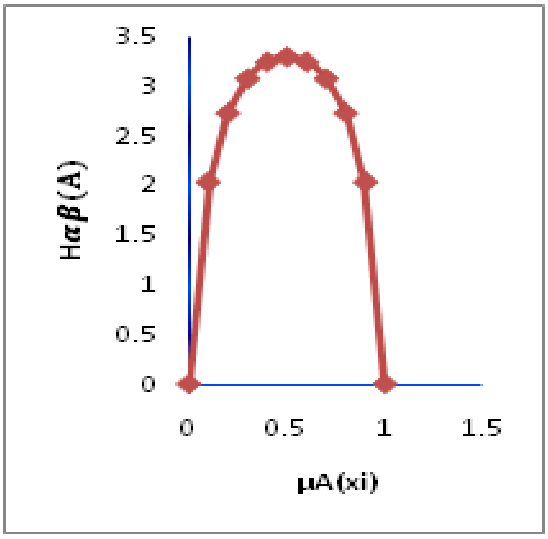

Figure: 2.10

Let α = 2.0 and $\beta = 3.5$

$\mu_A(x_i)$	$H_\alpha^\beta(A)$
0.0	0.0
0.1	2.1628
0.2	2.6510
0.3	2.8431
0.4	2.9240
0.5	2.9467
0.6	2.9240
0.7	2.8431
0.8	2.6510
0.9	2.1628
1.0	0.0

Table: 2.11

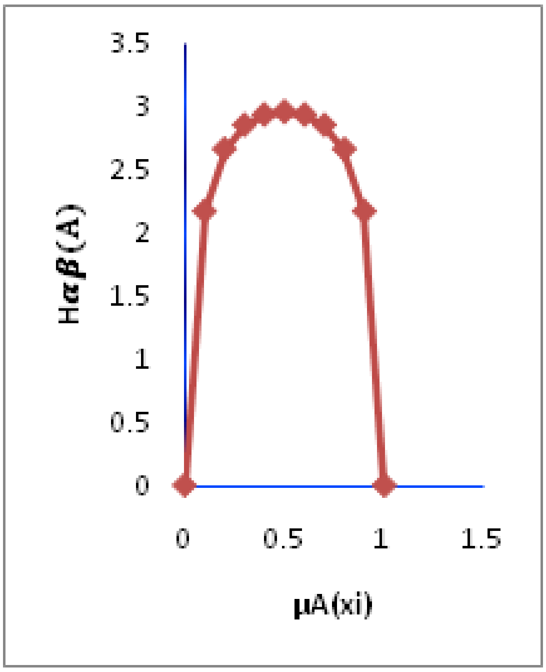

Figure:2.11

In addition to the above essential properties, the measure (2.1) satisfies the following property:

III. MAXIMALITY

We have,

$$\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} = \frac{\alpha}{1-\alpha}[\{\mu_A(x_i)^{\alpha\mu_A(x_i)} + (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}\}]^{\beta-1}$$
$$.[\mu_A(x_i)^{\alpha\mu_A(x_i)}(1+log\mu_A(x_i)) - (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}(1+\log(1-\mu_A(x_i)))]$$

Taking, $\frac{\partial H_\alpha^\beta(A)}{\partial \mu_A(x_i)} = 0$

which is possible iff $\mu_A(x_i) = 1 - \mu_A(x_i)$, that is iff $\mu_A(x_i) = \frac{1}{2}$.

Consider,

$$\frac{\partial^2 H_\alpha^\beta(A)}{\partial^2 \mu_A(x_i)} = \frac{\alpha^2(\beta-1)}{1-\alpha}[\mu_A(x_i)^{\alpha\mu_A(x_i)} + (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}]^{\beta-2}[\mu_A(x_i)^{\alpha\mu_A(x_i)}(1+log\mu_A(x_i))$$
$$-(1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}(1+\log(1-\mu_A(x_i)))]^2$$
$$+\frac{\alpha}{1-\alpha}[\mu_A(x_i)^{\alpha\mu_A(x_i)} + (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}]^{\beta-1}[\mu_A(x_i)^{\alpha\mu_A(x_i)-1}$$
$$+\alpha\mu_A(x_i)^{\alpha\mu_A(x_i)}\{1+log\mu_A(x_i)\}^2 + (1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))-1}$$
$$+\alpha(1-\mu_A(x_i))^{\alpha(1-\mu_A(x_i))}\{1+\log(1-\mu_A(x_i))\}^2]$$

Thus, at $\mu_A(x_i) = \frac{1}{2}$.

$$\frac{\partial^2 H_\alpha^\beta(A)}{\partial^2 \mu_A(x_i)} = \frac{-1}{1-\frac{1}{\alpha}}(2^{1-\frac{\alpha}{2}})^{\beta-1}[2^{2-\frac{\alpha}{2}} + \alpha.2^{1-\frac{\alpha}{2}}(1-log2)^2] < 0$$

Hence, the maximum value exists at $\mu_A(x_i) = \frac{1}{2}$.

IV. CONCLUSION

We have introduced new generalized measure of fuzzy entropy and proved its validity. We have also discussed the particular cases of α and β and presented the fuzzy entropy which clearly shows that fuzzy entropy is a concave function. Further, important property maximality has also been introduced.

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