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STABILITY ANALYSIS OF TWO COMPETITIVE INTERACTING SPECIES WITH OPTIMAL AND BIONOMIC HARVESTING OF THE FIRST SPECIES

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ABSTRACT

This paper deals with two species competitive model and optimal harvesting of the first species under bionomic conditions. The model is characterized by couple of first order non-linear ordinary differential equations. All the four equilibrium points are identified and their local stability is discussed. Solutions for the linearized perturbed equations are found. The conditions for global stability of the system are derived by using Liapunov function. Biological and Bionomical equilibria of the system are derived. Mathematical formulation of the optimal harvesting policy is given and its solution is derived by using Pontryagin's maximum principle.

1. INTRODUCTION

There is an extensive study on several kinds of prey- predator interactions after it was initiated by Lotka [1] and Volterra[2]. Bionomics of natural resources has played a significant role in all these interactions. There is a strong impact of harvesting on the dynamic evolution of a population. In fishery, forestry, agriculture and wild life management, the exploitation of biological resources and harvesting of population species can be seen. The problems of predator-prey systems in the presence of harvesting were discussed by many authors and attention on economic policies from harvesting have also been analysed. A detailed discussion on the issues and techniques associated with the bionomic exploitation of natural resources was given by Clark [3, 4]. A study on a class of predator-prey models under constant rate of harvesting of both species simultaneously was made by Brauer and Soudack [5, 6]. Multi-species harvesting models are also studied in detail by Chaudhuri [7, 8]. Models on the combined harvesting of a two species prey predator fishery have been discussed by Ragozin and Brown [9], Chaudhuri and Saha Ray [10]. K. Shiva Reddy *et.al* [12] proposed the mathematical model for the three species ecosystem comprising of two predators competing for the prey. They also investigated the stability concepts using various mathematical techniques.

In this connection, a mathematical model based on the system of non-linear equations has been constructed. All the four equilibrium points are identified and their local stability is discussed. The conditions for global stability of the system are derived by using Liapunov function. Biological and Bionomical equilibria of the system are derived.

2. MATHEMATICAL MODEL

The model equations for a two species competitive system are given by the following system of non-linear ordinary differential equations

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^{\,2} - \alpha_{12} N_1 N_2 - q_1 E N_1 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^{\,2} - \alpha_{21} N_1 N_2 \tag{2.2}$$

where N_1 and N_2 are the populations of the first and second species with natural growth rates (bio potentials) a_1 and a_2 respectively, α_{11} is rate of decrease of the first species due to insufficient food, α_{12} is rate of decrease of the first species due to inhibition by the second species, α_{21} is rate of decrease of the second species due to inhibition by the first species, α_{22} is rate of decrease of the second species due to insufficient food other than the first species; q_1 is the catch ability co-efficient of the first species, E is the harvesting effort and $q_1 E N_1$ is the catch-rate functions based on the catch-per-unit-effort hypothesis. Further both the variables N_1 and N_2 are non-negative and the model parameters $a_1, a_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, q_1, E,\ a_1 - q_1 E$ are assumed to be non-negative constants.

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3. EQUILIBRIUM STATES

The system has only four equilibrium states defined by

$$\frac{dN_1}{dt} = 0,\ \frac{dN_2}{dt} = 0 \tag{3.1}$$

E_1: The fully washed out state with the equilibrium point $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3.2)

E_2: The state in which, only the predator survives and the prey is washed out.

The equilibrium point is $\overline{N}_1 = 0\,; \overline{N}_2 = \dfrac{a_2}{\alpha_{22}}$ (3.3)

E_3: The state in which, only the prey survives and the predator is washed out

The equilibrium point is $\overline{N}_1 = \dfrac{(a_1 - q_1E)}{\alpha_{11}}\,; \overline{N}_2 = 0$ (3.4)

E_4: The co-existent state (**normal steady state**). The equilibrium point is

$$\overline{N}_1 = \frac{\alpha_{22}(a_1 - q_1E) - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\,;\ \overline{N}_2 = \frac{a_2\alpha_{11} - \alpha_{21}(a_1 - q_1E)}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \tag{3.5}$$

This state would exit only when $\alpha_{22}\left(a_1 - q_1E\right) > a_2\alpha_{12}$, $a_2\alpha_{11} > \alpha_{21}(a_1 - q_1E)\overline{N}_2$, $\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$

4. STABILITY OF THE EQUILIBRIUM STATES

To investigate the stability of the equilibrium states we consider small perturbations u_1, u_2 in N_1 and N_2 over $\overline{N}_1$ and $\overline{N}_2$ respectively, so that

$$N_1 = \overline{N}_1 + u_1\,;\ N_2 = \overline{N}_2 + u_2 \tag{4.1}$$

By substituting (4.1) in (2.1) & (2.2) and neglecting second and higher powers of the perturbations u_1, u_2 we get the equations of the perturbed state

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} (a_1 - q_1E) - 2\alpha_{11}\overline{N}_1 - \alpha_{12}\overline{N}_2 & -\alpha_{12}\overline{N}_1 \\ -\alpha_{21}\overline{N}_2 & a_2 - \alpha_{21}\overline{N}_1 - 2\alpha_{22}\overline{N}_2 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is

$$det\left[A - \lambda I\right] = 0 \tag{4.4}$$

The characteristic roots of equation (4.4) may be real or complex.

The equilibrium state is **stable** only when the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex.

4.1. Stability of the equilibrium state E_1:

To discuss the stability of equilibrium point $\overline{N}_1 = 0\,; \overline{N}_2 = 0$, we consider slight deviations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{4.5}$$

$$N_2 = \overline{N}_2 + u_2(t). \tag{4.6}$$

Substituting (4.5) and (4.6) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = \left(a_1 - q_1E\right)u_1 - \alpha_{11}u_1^2 - \alpha_{12}u_1u_2 \tag{4.7}$$

$$\frac{du_2}{dt} = a_2u_2 - \alpha_{22}u_2{}^2 - \alpha_{21}u_1u_2 \tag{4.8}$$

On neglecting products and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = (a_1 - q_1E)u_1 \tag{4.9}$$

and

$$\frac{du_2}{dt} = a_2u_2 \tag{4.10}$$

The characteristic equation is

$$[\lambda\text{-}(a_1 - q_1E)][\lambda\text{-}\ a_2] = 0, \tag{4.11}$$

Whose roots $(a_1 - q_1E)$, a_2 are both positive. Hence the equilibrium state is **unstable**.

The solutions for equations (4.9) and (4.10) are

$$u_1 = u_{10}\ e^{(a_1 - q_1E)t} \tag{4.12}$$

$$u_2 = u_{20}\ e^{a_2t} \tag{4.13}$$

where u_{10}, u_{20} are the initial values of u_1 and u_2.

4.2 Stability of the equilibrium state E_2:

Substituting (4.5) and (4.6) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = (a_1 - q_1E)u_1 - \alpha_{11}u_1{}^2 - \alpha_{12}u_1u_2 - \frac{a_2\alpha_{12}}{\alpha_{22}}u_1 \tag{4.14}$$

$$\frac{du_2}{dt} = -a_2u_2 - \alpha_{22}u_2{}^2 - \alpha_{21}u_1u_2 - \frac{a_2\alpha_{21}}{\alpha_{22}}u_1 \tag{4.15}$$

On neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = (a_1 - q_1E)u_1 - \frac{a_2\alpha_{12}}{\alpha_{22}}u_1 \tag{4.16}$$

and

$$\frac{du_2}{dt} = \frac{-a_2\alpha_{21}}{\alpha_{22}}u_1 - a_2u_2 \tag{4.17}$$

The characteristic equation is

$$(\lambda + a_2)\ \left\{\lambda - \left[(a_1 - q_1E) - \frac{a_2\alpha_{12}}{\alpha_{22}}\right]\right\} = 0 \tag{4.18}$$

Case (i): When $(a_1 - q_1E) > \frac{a_2\alpha_{12}}{\alpha_{22}}$, one root of the equation (4.18) is $\lambda_1 = -a_2$ and the other root,

$\lambda_2 = \left[(a_1 - q_1E) - \frac{a_2\alpha_{12}}{\alpha_{22}}\right]$ is positive. Hence the equilibrium state is **unstable**

Case (ii): When $\left(a_1 - q_1E\right) < \frac{a_2\alpha_{12}}{\alpha_{22}}$**,** one root of the equation (4.18) is $\lambda_1 = -a_2$ and the other root,

$\lambda_2 = \left[\left(a_1 - q_1E\right) - \frac{a_2\alpha_{12}}{\alpha_{22}}\right]$ is negative.

As the roots of the equation (4.18) are both negative, the equilibrium state is **stable.**

The solutions for equations (4.16) and (4.17) are

$$u_1 = u_{10}e^{\lambda_2 t}, \tag{4.19}$$

and

$$u_2 = \rho_1 e^{\lambda_2 t} + (u_{20} - \rho_1)\, e^{-a_2 t} \tag{4.20}$$

where $$\rho_1 = \frac{u_{10}a_2\alpha_{21}}{\left(a_1 - q_1E\right)\alpha_{22} + a_2\left[\alpha_{22} - \alpha_{12}\right]} \tag{4.21}$$

Case (iii): When $\left(a_1 - q_1E\right) = \frac{a_2\alpha_{12}}{\alpha_{22}}$, one root of the equation (4.18) is $\lambda_1 = -a_2$ and the other root $\lambda_2 = 0$. Hence the equilibrium state is **"neutrally" stable**.

The solutions for equations (4.16) & (4.17) are

$$u_1 = u_{10} \tag{4.22}$$

and

$$u_2 = \frac{(a_1 - q_1E)\alpha_{21}}{a_2\alpha_{12}}u_{10} + \left[u_{20} - (a_1 - q_1E)\frac{\alpha_{21}}{a_2\alpha_{12}}u_{10}\right]e^{-a_2 t} \tag{4.23}$$

4.3 Stability of the equilibrium state E_3:

Substituting (4.5) and (4.6) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -(a_1 - q_1E)u_1 - \alpha_{11}u_1^2 - \alpha_{12}u_1u_2 - (a_1 - q_1E)\frac{\alpha_{12}u_2}{\alpha_{11}} \tag{4.24}$$

$$\frac{du_2}{dt} = a_2u_2 - \alpha_{22}u_2^2 - \alpha_{21}u_1u_2 - (a_1 - q_1E)\frac{\alpha_{21}u_2}{\alpha_{11}} \tag{4.25}$$

By neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = -(a_1 - q_1E)u_1 - (a_1 - q_1E)\frac{\alpha_{12}u_2}{\alpha_{11}} \tag{4.26}$$

and

$$\frac{du_2}{dt} = \left[a_2 - (a_1 - q_1E)\frac{\alpha_{21}}{\alpha_{11}}\right]u_2 \tag{4.27}$$

and the characteristic equation is

$$\left[\lambda + (a_1 - q_1E)\right]\ \left\{\lambda - \left[a_2 - (a_1 - q_1E)\frac{\alpha_{21}}{\alpha_{11}}\right]\right\} = 0 \tag{4.28}$$

Case (i): When $a_2 > (a_1 - q_1E)\dfrac{\alpha_{21}}{\alpha_{11}}$, one root of the equation (4.28) is $\lambda_1 = -(a_1 - q_1E)$ while the other root $\lambda_2 =$ $\left[a_2 - (a_1 - q_1E)\dfrac{\alpha_{21}}{\alpha_{11}}\right]$ is positive. Hence the equilibrium state is **unstable.**

Case (ii): When $a_2 < (a_1 - q_1E)\dfrac{\alpha_{21}}{\alpha_{11}}$, one root of the equation (4.28) is $\lambda_1 = -(a_1 - q_1E)$ and the other root $\lambda_2 = \left[a_2 - (a_1 - q_1E)\dfrac{\alpha_{21}}{\alpha_{11}}\right]$ is negative.

As the roots of the equation (4.28) are both negative, the equilibrium state is **stable.**

The solutions for equations (4.26) & (4.27) are

$$u_1 = \rho_2 u_{20} e^{dt} + (u_{10} + \rho_2)\, e^{-(a_1 - q_1E)t} \tag{4.29}$$

$$u_2 = u_{20} e^{dt} \tag{4.30}$$

where

$$d = \left[a_2 + (a_1 - q_1E)\frac{\alpha_{21}}{\alpha_{11}}\right] \quad \text{and} \quad \rho_2 = \frac{(a_1 - q_1E)\alpha_{12}}{a_2\alpha_{11} + (a_1 - q_1E)\left[\alpha_{11} + \alpha_{21}\right]} \tag{4.31}$$

4.4 Stability of the equilibrium state E_4:

Substituting (4.5) and (4.6) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -\alpha_{11}u_1^{\,2} - \alpha_{12}u_1u_2 - \alpha_{11}\overline{N}_1u_1 - \alpha_{12}\overline{N}_1u_2 \tag{4.32}$$

$$\frac{du_2}{dt} = -\alpha_{22}u_2^{\,2} + \alpha_{21}u_1u_2 - \alpha_{22}\overline{N}_2u_2 - \alpha_{21}\overline{N}_2u_1 \tag{4.33}$$

By neglecting products, and higher powers of u_1 and u_2, we get

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1u_1 - \alpha_{12}\overline{N}_1u_2 \tag{4.34}$$

$$\frac{du_2}{dt} = -\alpha_{21}\overline{N}_2u_1 - \alpha_{22}\overline{N}_2u_2 \tag{4.35}$$

The characteristic equation is

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + [\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}]\,\overline{N}_1\overline{N}_2 = 0 \tag{4.36}$$

Since the sum of the roots of (4.36) is negative and the product of the roots is positive, the roots of which can be noted to be negative .Hence the co-existent equilibrium state is **stable.**

The solutions for equations (4.34) & (4.35) are

$$u_1 = \left[\frac{u_{10}(\lambda_1 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}(\lambda_2 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) - u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) - u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

where λ_1, λ_2 are the roots of the equation (4.36)

5. GLOBAL STABILITY

Theorem: The Equilibrium point $(\overline{N}_1, \overline{N}_2)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov function

$$V(N_1,N_2) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left[\frac{N_1}{\overline{N}_1}\right] + l\left\{N_2 - \overline{N}_2 - \overline{N}_2 \ln\left[\frac{N_2}{\overline{N}_2}\right]\right\} \tag{5.1}$$

where 'l' is positive constant

Differentiating V w.r.to 't' we get

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + l\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\frac{dN_2}{dt} \tag{5.2}$$

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\left\{N_1\left[(a_1 - q_1E) - \alpha_{11}N_1 - \alpha_{12}N_2\right]\right\} + l\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\left\{N_2\left[a_2 - \alpha_{22}N_2 - \alpha_{21}N_1\right]\right\}$$

$$\begin{aligned}
\frac{dV}{dt} &= \left(N_1 - \overline{N}_1\right)\left\{(a_1 - q_1E) - \alpha_{11}N_1 - \alpha_{12}N_2\right\} + l\left(N_2 - \overline{N}_2\right)\left\{a_2 - \alpha_{22}N_2 - \alpha_{21}N_1\right\} \\
&= \left(N_1 - \overline{N}_1\right)\left\{\alpha_{11}\overline{N}_1 + \alpha_{12}\overline{N}_2 - \alpha_{11}N_1 - \alpha_{12}N_2\right\} + l\left(N_2 - \overline{N}_2\right)\left\{\alpha_{22}\overline{N}_2 - \alpha_{21}\overline{N}_1 - \alpha_{22}N_2 - \alpha_{21}N_1\right\} \\
&= \left(N_1 - \overline{N}_1\right)\left\{-\alpha_{11}\left(N_1 - \overline{N}_1\right) - \alpha_{12}\left(N_2 - \overline{N}_2\right)\right\} + l\left(N_2 - \overline{N}_2\right)\left\{-\alpha_{21}\left(N_1 - \overline{N}_1\right) - \alpha_{22}\left(N_2 - \overline{N}_2\right)\right\} \\
&= \left\{-\alpha_{11}\left(N_1 - \overline{N}_1\right)^2 - \alpha_{12}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right)\right\} + l\left\{-\alpha_{21}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) - \alpha_{22}\left(N_2 - \overline{N}_2\right)^2\right\} \\
&= -\alpha_{11}\left(N_1 - \overline{N}_1\right)^2 - \alpha_{12}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) - l\alpha_{21}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) - l\alpha_{22}\left(N_2 - \overline{N}_2\right)^2 \\
&= -\alpha_{11}\left(N_1 - \overline{N}_1\right)^2 - \frac{\alpha_{12}}{2}\left[\left(N_1 - \overline{N}_1\right)^2 + \left(N_2 - \overline{N}_2\right)^2\right] - \frac{l\alpha_{21}}{2}\left[\left(N_1 - \overline{N}_1\right)^2 + \left(N_2 - \overline{N}_2\right)^2\right] - l\alpha_{22}\left(N_2 - \overline{N}_2\right)^2
\end{aligned}$$

$$\frac{dV}{dt} = -\left(\frac{\alpha_{12} + l\alpha_{21}}{2} + \alpha_{11}\right)\left(N_1 - \overline{N}_1\right)^2 - \left(\frac{\alpha_{12} + l\alpha_{21}}{2} + l\alpha_{22}\right)\left(N_2 - \overline{N}_2\right)^2 \tag{5.3}$$

$$< 0$$

Therefore, the equilibrium point $(\overline{N}_1, \overline{N}_2)$ is globally **asymptotically stable**.

6. BIONOMIC EQUILIBRIUM

The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium as well as economic equilibrium. The economic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost for the effort devoted to harvesting.

Let $c_1 =$ fishing cost per unit effort of the prey, $p_1 =$ price per unit biomass of the prey. The net economic revenue for the prey at any time t is given by

$$R_1 = \left(p_1q_1N_1 - c_1\right)E \tag{6.1}$$

The biological equilibrium is $\left((N_1)_\infty, (N_2)_\infty, (E)_\infty\right)$, where $(N_1)_\infty, (N_2)_\infty, (E)_\infty$ are the positive solutions of

$$\left(a_1 - q_1E\right)N_1 - \alpha_{11}N_1^{\,2} - \alpha_{12}N_1N_2 = 0 \tag{6.2}$$

$$a_2N_2 - \alpha_{22}N_2^{\,2} - \alpha_{21}N_1N_2 = 0 \tag{6.3}$$

and $\left(p_2q_2N_2 - c_2\right)E = 0$ (6.4)

From (6.4), we have

$$\left\{ p_1 q_1 (N_1)_\infty - c_1 \right\} (E)_\infty = 0$$

$$\Rightarrow \left\{ p_1 q_1 (N_1)_\infty - c_1 \right\} = 0$$

$$\Rightarrow \left(N_1 \right)_\infty = \frac{c_1}{p_1 q_1} \tag{6.5}$$

From (6.3), we have

$$\left(N_2 \right)_\infty \left\{ a_2 - \alpha_{22} \left(N_2 \right)_\infty - \alpha_{21} \left(N_1 \right)_\infty \right\} = 0$$

$$\Rightarrow \left\{ a_2 - \alpha_{22} \left(N_2 \right)_\infty - \alpha_{21} \left(N_1 \right)_\infty \right\} = 0$$

$$\Rightarrow \left\{ a_2 - \alpha_{22} \left(N_2 \right)_\infty - \alpha_{21} \frac{c_1}{p_1 q_1} \right\} = 0$$

$$\Rightarrow \left\{ \left(a_2 - \alpha_{21} \frac{c_1}{p_1 q_1} \right) - \alpha_{22} \left(N_2 \right)_\infty \right\} = 0$$

$$\Rightarrow \left(N_2 \right)_\infty = \frac{1}{\alpha_{22}} \left(a_2 - \alpha_{21} \frac{c_1}{p_1 q_1} \right) \tag{6.6}$$

From (6.2), (6.5) & (6.6), we get

$$\left(a_1 \frac{c_1}{p_1 q_1} - \alpha_{11} \frac{c_1^{\ 2}}{p_1^{\ 2} q_1^{\ 2}} - \alpha_{12} \frac{c_1}{p_1 q_1} \left(N_1 \right)_\infty - q_1 \left(E \right)_\infty \frac{c_1}{p_1 q_1} \right) = 0$$

$$\Rightarrow \left(E \right)_\infty = \frac{1}{q_1} \left[\left(a_1 - \alpha_{11} \frac{c_1}{p_1 q_1} \right) - \alpha_{12} \left(N_1 \right)_\infty \right] \tag{6.7}$$

It is clear that $\left(E \right)_\infty > 0$ if $\left(a_1 - \alpha_{11} \frac{c_1}{p_1 q_1} \right) > \alpha_{12} \left(N_2 \right)_\infty$ (6.8)

Thus the bionomic equilibrium $\left(\left(N_1 \right)_\infty, \left(N_2 \right)_\infty, \left(E \right)_\infty \right)$ exists, if inequality (6.8) holds.

7. OPTIMAL HARVESTING POLICY

The present value J of a continuous time-stream of revenues is given by

$$J = \int_0^\infty e^{-\delta t} \left(p_1 q_1 N_1 - c_1 \right) E dt \tag{7.1}$$

Where δ denotes the instantaneous annual rate of discount. Our problem is to maximize J subject to the state equations (2.1) & (2.2) and control constraints $0 \le E \le \left(E \right)_{\max}$ by invoking Pontryagin's maximum principle [11].

The Hamiltonian for the problem is given by

$$H = e^{-\delta t} \left(p_1 q_1 N_1 - c_1 \right) E + \lambda_1 \left(a_1 N_1 - \alpha_{11} N_1^{\ 2} - \alpha_{12} N_1 N_2 - q_1 E N_1 \right) + \lambda_2 \left(a_2 N_2 - \alpha_{22} N_2^{\ 2} - \alpha_{21} N_1 N_2 \right) \tag{7.2}$$

where λ_1, λ_2 are the adjoint variables.

Let us assume that the control constraints are not binding i.e. the optimal solution does not occur at $(E)_{\max}$. At $(E)_{\max}$ we have a singular control.

By Pontryagin's maximum principle,

$$\frac{\partial H}{\partial E}=0 \;;\quad \frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial N_1} \;;\; \frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial N_2}$$

$$\frac{\partial H}{\partial E}=0 \Rightarrow e^{-\delta t}\left(p_1q_1N_1-c_1\right)-\lambda_1q_1N_1=0$$

$$\Rightarrow \lambda_1=e^{-\delta t}\left(p_1-\frac{c_1}{q_1N_1}\right) \tag{7.3}$$

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial N_1}=-\left\{e^{-\delta t}p_1q_1E+\lambda_1\left[(a_1-q_1E)-2\alpha_{11}N_1-\alpha_{12}N_2\right]+\lambda_2\left(-\alpha_{21}N_2\right)\right\}$$

$$\Rightarrow \frac{d\lambda_1}{dt}=\left(\lambda_1\alpha_{11}N_1+\lambda_2\alpha_{21}N_2-e^{-\delta t}p_1q_1E\right) \tag{7.4}$$

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial N_2}=-\left\{\lambda_1\left(-\alpha_{12}N_1\right)+\lambda_2\left[a_2-\alpha_{21}N_1-2\alpha_{22}N_2\right]\right\}$$

$$\Rightarrow \frac{d\lambda_2}{dt}=\left(\lambda_1\alpha_{12}N_1+\lambda_2\alpha_{22}N_2\right) \tag{7.5}$$

From (7.3) & (7.5), we get $\dfrac{d\lambda_2}{dt}-\lambda_2\alpha_{22}N_2=A_1e^{-\delta t}$

where

$$A_1=\alpha_{12}\overline{N_1}\left(p_1-\frac{c_1}{q_1\overline{N_1}}\right)$$

Whose solution is given by $\lambda_2=\dfrac{A_1}{\left(\alpha_{22}\overline{N_2}+\delta\right)}e^{-\delta t}$ (7.6)

From (7.4) & (7.6), we get

$$\frac{d\lambda_1}{dt}-\lambda_1\alpha_{11}N_1=-A_2e^{-\delta t}$$

where

$$A_2=\left[p_1q_1E+\frac{A_1\alpha_{21}\overline{N_2}}{\left(\alpha_{22}\overline{N_2}+\delta\right)}\right],$$

whose solution is given by $\lambda_1=\dfrac{A_2}{\left(\alpha_{11}\overline{N_1}+\delta\right)}e^{-\delta t}$ (7.7)

From (7.3) & (7.7), we get a singular path

$$\left(p_1-\frac{c_1}{q_1\overline{N_1}}\right)=\frac{A_2}{\left(\alpha_{11}\overline{N_1}+\delta\right)} \tag{7.8}$$

Thus (7.8) can be written as

$$F(\overline{N_1}) = \left(p_1 - \frac{c_1}{q_1\overline{N_1}}\right) - \frac{A_2}{\left(\alpha_{11}\overline{N_1} + \delta\right)}$$

There exist a unique positive root $\overline{N_1} = \left(N_1\right)_\delta$ of $F(\overline{N_1}) = 0$ in the interval $0 < \overline{N_1} < k_1$, if the following hold $F(0) < 0$, $F(k_1) > 0$, $F'(\overline{N_1}) > 0$ for $\overline{N_1} > 0$.

For $\overline{N_1} = \left(N_1\right)_\delta$, we get

$$\left(N_2\right)_\delta = \frac{1}{\alpha_{22}}\left(a_2 - \alpha_{21}\frac{c_1}{p_1q_1}\right) \tag{7.9}$$

and

$$\left(E\right)_\delta = \frac{1}{q_1}\left[\left(a_1 - \alpha_{11}\frac{c_1}{p_1q_1}\right) - \alpha_{12}\left(N_2\right)_\infty\right] \tag{7.10}$$

Hence once the optimal equilibrium $\left(\left(N_1\right)_\delta, \left(N_2\right)_\delta\right)$ is determined, the optimal harvesting effort $\left(E\right)_\delta$ can be determined.

From (7.3), (7.6) and (7.7), we found that λ_1, λ_2 do not vary with time in optimal equilibrium.

Hence they remain bounded as $t \to \infty$.

From (7.8), we also note that

$$\left(p_1 - \frac{c_1}{q_1\overline{N_1}}\right) = \frac{A_2}{\left(\alpha_{11}\overline{N_1} + \delta\right)} \to 0 \quad \text{as } \delta \to \infty$$

Thus, the net economic revenue of the prey $R_1 = 0$.

This implies that if the discount rate increases, then the net economic revenue decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate.

8. CONCLUSIONS

In this paper, the consequences of two species competitive model with optimal harvesting of the first species under bionomic conditions have been studied. The existence of the possible steady states along with their local stability is discussed and also conditions for global stability of the system are derived by using Liapunov function. The conditions for the existence of Biological and Bionomical equilibria of the system are derived. Further, the optimal harvesting policy has been discussed by using Pontryagin's Maximum Principle [11]. It has been found that the total user cost of harvest per unit of effort equals the discounted value of the future profit at the steady-state effort level. It has also been noted that if the discount rate increases, then the economic rent decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate.

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