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ENESTROM- KAKEYA THEOREM AND ITS GENERALIZATIONS

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ABSTRACT

There are some generalizations and extensions available in the literature for Enestrom-Kakeya Theorem. In this paper we give some new generalizations.

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Keywords:*-Polynomials; Enestrom-Kakeya Theorem; Zero.*

1. INTRODUCTION AND STATEMENT OF RESULTS

In the zero distribution theory of polynomials, the Enestrom-Kakeya theorem [4] given below in theorem A is a well known result.

Theorem A: Let P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0 > 0$$

Then all the zeros of P(z) lie in the disk $|z| \leq 1$.

Many attempts have been made to extend and generalize the Enestrom-Kakeya theorem. A. Joyal *et al* [3] extended the Enestrom-Kakeya theorem to the polynomials with general monotonic coefficients by proving that if

$$a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0$$

Then P(z) has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$$

Further Aziz and Zargar [1] generalized the result of A. Joyal *et al* [3] and the Enestrom-Kakeya theorem as given below in theorem B.

Theorem B: Suppose P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a polynomial of degree n such that For some $\lambda \geq 1$,

$$\lambda\, a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0$$

then all the zeros of P(z) lie in the disk

$$|z + (\lambda - 1)| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}$$

But N.K. Govil and Rahman [2] worked over the distribution of zeros of polynomials such that the modulii of co-efficients are monotonic and proved a result given below in theorem C.

Theorem C: Suppose P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a polynomial of degree n such that for some a> 0

$$|a_n| \geq a\, |a_{n-1}| \geq a^2\, |a_{n-2}| \geq \ldots\ldots \geq a^{n-1}\, |a_1| \geq a^n\, |a_0|$$

Then P(z) has all its zeros in the disk $|z| \leq \frac{M_1}{a}$, where M_1 is the greatest positive root of the equation

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$M^{n+1} - 2M^n + 1 = 0$.

Govil and Rahman [2] also proved that if P(z) = $\sum_{i=0}^{n} a_i\ z^i$, is a complex polynomial of degree n, then all the zeros of P(z) lie in the disk

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{i=0}^{n-1}|a_i|$$

where $|arg.a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0,1,2,…….n) for some β real, and $|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|$.

But recently Shah and Liman [5] generalized theorem B and the result of Govil and Rahman [2] and thereby proved the following two theorems D and E.

Theorem D: Suppose P(z) = $\sum_{i=0}^{n} a_i\ z^i$ be a complex polynomial of degree n with Re(a_i)= α_i and Im(a_i) = β_i, i=0,1,2,……..,n. If for some $\lambda \geq 1$

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0, \quad \beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$$

Then P(z) has all its zeros in the disk

$$\left|z + \frac{(\lambda-1)\alpha_n}{a_n}\right| \leq \frac{\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}$$

Theorem E: Suppose P(z) = $\sum_{i=0}^{n} a_i\ z^i$ be a complex polynomial of degree n such that $|arg.a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0, 1, 2,…….n) for some β real and for some $\lambda \geq 1$

$$\lambda|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|$$

Then P(z) has all its zeros in the disk

$$|z + (\lambda - 1)| \leq \frac{1}{|a_n|}\{(\lambda|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{i=0}^{n-1}|a_i|\}$$

Now in this paper, we prove three generalizations of Enestrom-Kakeya Theorem.

2. THEOREMS AND PROOFS

Theorem 1: Suppose P(z) = $\sum_{i=0}^{n} a_i\ z^i$ be a polynomial of degree n such that for some $\lambda \neq 1$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$

$$a_n \geq a_{n-1} \geq \ldots \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq \ldots \geq a_1 \geq a_0$$

Now let $a_{n-k-1} > a_{n-k}$, then P(z) has all its zeros in the disk $|z| \leq M_1$, where M_1 is the greatest positive root of the equation

$$M^{K+1} - \theta_1 M^K - |\rho_1| = 0, \text{ where } \rho_1 = \frac{(\lambda-1)a_{n-k}}{a_n} \text{ and } \theta_1 = \frac{|a_0| - a_0 + a_n + (\lambda-1)a_{n-k}}{|a_n|}$$

Again let $a_{n-k} > a_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \leq M_2$ where M_2 is the greatest positive root of the equation

$$M^K - \theta_2 M^{K-1} - |\rho_2| = 0, \text{ where } \rho_2 = \frac{(1-\lambda)a_{n-k}}{a_n} \text{ and } \theta_2 = \frac{|a_0| - a_0 + a_n + (1-\lambda)a_{n-k}}{|a_n|}$$

Proof: To prove the result**,** we consider a polynomial

$$F(z) = (1-z)P(z) = (1-z)\sum_{i=0}^{n} a_i\ z^i$$

$$= (1-z)(a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1}z^{n-1} + a_n z^n)$$

$$= a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1}z^{n-1} + a_n z^n - a_0 z - a_1 z^2 - a_2 z^3 \ldots - a_{n-1}z^n - a_n z^{n+1}$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k+1} > a_{n-k}$

Therefore, we can also write F (z) as

$$F(z) = -a_n z^{n+1} - (\lambda - 1)a_{n-k}\, z^{n-k} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_1 - a_0)z + a_0$$

Let $|z| > 1$, then we have

$$|F(z)| \geq |a_n\, z^{n+1} + (\lambda - 1)a_{n-k}\, z^{n-k}| - |z|^n \{(a_n - a_{n-1}) + \cdots + \frac{(a_{n-k+1} - a_{n-k})}{|z|^{k-1}} + \frac{(\lambda a_{n-k} - a_{n-k-1})}{|z|^k} + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$$

$$\geq |z|^{n-k}\, |a_n z^{k+1} + (\lambda - 1)a_{n-k}| - |z|^n \{ a_n + (\lambda - 1)a_{n-k} - a_0 + |a_0|\}$$

$$> 0,$$

If $|z^{k+1} + \rho_1| > \theta_1 |z|^k$

where $\rho_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ and $\theta_1 = \frac{|a_0| - a_0 + a_n + (\lambda-1)a_{n-k}}{|a_n|}$

The above inequality will hold if

$|z|^{k+1} - |\rho_1| > \theta_1 |z|^k$. Thus all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_1$, where M_1 is the greatest positive root of the equation

$M^{K+1} - \theta_1 M^K - |\rho_1| = 0$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_1$ since $M_1 > 1$

Now we take second part of the theorem.

Let $a_{n-k} > a_{n-k+1}$, then $a_{n-k} > a_{n-k-1}$ and F(z) can be written as follows;

$$F(z) = -a_n z^{n+1} - (1 - \lambda)a_{n-k}\, z^{n-k+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_1 - a_0)z + a_0$$

Let $|z| > 1$, then

$$|F(z)| \geq |z|^{n-k+1} |a_n\, z^k + (1 - \lambda)a_{n-k}| - |z|^n \{(a_n - a_{n-1}) + \cdots + \frac{(a_{n-k+1} - \lambda a_{n-k})}{|z|^{k-1}} + \frac{(a_{n-k} - a_{n-k-1})}{|z|^k} + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$$

$$\geq |z|^{n-k+1} |a_n\, z^k + (1 - \lambda)a_{n-k}| - |z|^n \{a_n + (1 - \lambda)a_{n-k} - a_0 + |a_0|\}$$

$$> 0$$

if $|z^k + \rho_2| > \theta_2 |z|^{k-1}$

where $\rho_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ and $\theta_2 = \frac{|a_0| - a_0 + a_n + (1-\lambda)a_{n-k}}{|a_n|}$

The above inequality will hold if $|z|^k - |\rho_2| > \theta_2 |z|^{k-1}$

Thus all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$M^K - \theta_2 M^{K-1} - |\rho_2| = 0$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_2$ since $M_2 > 1$.

Therefore, the result follows.

Theorem 2: Suppose P(z) = $\sum_{i=0}^{n} a_i\ z^i$ be a polynomial of degree n such that for some a>0, $\lambda(\neq 1) > 0$, 1$\leq k \leq n$ and $a_{n-k} \neq 0$

$|a_n| \geq$ a $|a_{n-1}| \geq$ a^2 $|a_{n-2}| \geq$ ……$\geq$ a^{k-1} $|a_{n-k+1}| \geq \lambda$ a^k

$|a_{n-k}| \geq a^{k+1}|a_{n-k-1}| \geq \cdots \geq a^n |a_0|$

If $|a_{n-k}| < a|a_{n-k-1}|$ (i.e., λ> 1)

Then P(z) has all its zeros in the disk $|z| \leq \frac{M_1}{a}$,

where M_1 is the greatest positive root of the equation

$M^{n+1} - 2M^n + 1 = 0$

If a$|a_{n-k}| > |a_{n-k+1}|$ (i.e., 0< $\lambda < 1$)

Then P(z) has all its zeros in the disk $|z| \leq \frac{M_2}{a}$, where M_2 is the greatest positive root of the equation

$M^{n+1} - 2M^n + (\frac{\lambda-1}{\lambda})M^{n-k+1} + \frac{1}{\lambda} = 0$

Proof: Let $|a_{n-k}| < a|a_{n-k-1}|$ then $a^{k-1}|a_{n-k+1}| \geq a^k|a_{n-k}|$ and thereby we obtain the same result as theorem B following the proof of theorem1 in [2]. Now let us take a$|a_{n-k}| > |a_{n-k+1}|$ and suppose F(z) = $\sum_{i=0}^{n-1} a_i z^i$ be a polynomial of degree (n-1) .

Then for $|z| = R\ (> \frac{1}{a})$, we have

$$|F(z)| \leq |a_{n-1}|R^{n-1}\{1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \cdots + \frac{1}{(aR)^{k-2}} + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{(aR)^k} + \cdots + \frac{1}{(aR)^{n-1}}\}$$

$$\leq |a_{n-1}|R^{n-1}\{1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \cdots + \frac{1}{(aR)^{k-2}} + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{\lambda(aR)^k} + \cdots + \frac{1}{\lambda(aR)^{n-1}}\}$$

$$= |a_{n-1}|R^{n-1}\{\frac{(aR)^{k-1}-1}{(aR)^{k-2}(aR-1)} + \frac{(aR)^{n-k+1}-1}{\lambda(aR)^{n-1}(aR-1)}\}$$

$$= |a_{n-1}|R^{n-1}\{\frac{\lambda(aR)^n+(1-\lambda)(aR)^{n-k+1}-1}{\lambda(aR)^{n-1}(aR-1)}\}$$

Since P(z) = $\sum_{i=0}^{n} a_i\ z^i$, therefore, we have

$$|P(z)| \geq |a_n|R^n - |a_{n-1}|R^{n-1}\{\frac{\lambda(aR)^n+(1-\lambda)(aR)^{n-k+1}-1}{\lambda(aR)^{n-1}(aR-1)}\} > 0$$

If $\frac{|a_n|}{a|a_{n-1}|} > \frac{\lambda(aR)^n+(1-\lambda)(aR)^{n-k+1}-1}{\lambda(aR)^{n-1}(aR-1)}$

Because by hypothesis $\frac{|a_n|}{a|a_{n-1}|} > 1$, the above inequality holds

If $\lambda(aR)^{n-1}(aR-1) > \lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1$

Replacing aR by M, we obtain the desired result.

Theorem 3: Let P(z)= $\sum_{i=0}^{n} a_i\ z^i$ be a complex polynomial of degree n with Re(a_i)= α_i and Im(a_i)= β_i , i=0,1,2,……..,n. Let for some $\lambda \neq 1$, $1 \leq k \leq n$ and $\alpha_{n-k} \neq 0$,

$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq \cdots \geq \alpha_1 \geq \alpha_0$,

$\beta_n \geq \beta_{n-1} \geq$ …..$\geq$ $\beta_1 \geq \beta_0$.

If $\alpha_{n-k-1} > \alpha_{n-k}$, then P(z) has all its zeros in the disk $|z| \leq M_1$ where M_1 is the greatest positive root of the equation $M^{K+1} - \theta_1 M^K - |\rho_1| = 0$, where

$\rho_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ and $\theta_1 = \frac{|\alpha_0|-\alpha_0+\alpha_n+(\lambda-1)\alpha_{n-k}+\beta_n-\beta_0+|\beta_0|}{|a_n|}$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then P(z) has all its zeros in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$M^K - \theta_2 M^{K-1} - |\rho_2| = 0$

where $\rho_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ and $\theta_2 = \frac{|\alpha_0|-\alpha_0+\alpha_n+(1-\lambda)\alpha_{n-k}+\beta_n-\beta_0+|\beta_0|}{|a_n|}$

Proof: To prove the result, we consider a polynomial F(z) defined by

F(z) = (1-z)P(z)

$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_1 - a_0)z + a_0$

$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots + (\alpha_1 - \alpha_0)z + \alpha_0$
$+ i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots + (\beta_1 - \beta_0)z + \beta_0\}$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k+1} > \alpha_{n-k}$ and F(z) can be written as

$$F(z) = -a_n z^{n+1} - (\lambda-1)\alpha_{n-k} z^{n-k} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{n-k+1} - \alpha_{n-k}) z^{n-k+1} + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots + (\beta_1 - \beta_0)z + \beta_0\}$$

Now if $|z| > 1$, then

$$|F(z)| \geq |a_n z^{n+1} + (\lambda-1)\alpha_{n-k} z^{n-k}| - |z|^n\{(\alpha_n - \alpha_{n-1}) + \cdots + \frac{(\alpha_{n-k+1} - \alpha_{n-k})}{|z|^{k-1}} + \frac{(\lambda\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} + \cdots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n}\} - |z|^n\{(\beta_n - \beta_{n-1}) + \cdots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\}$$

$$\geq |a_n z^{n+1} + (\lambda-1)\alpha_{n-k} z^{n-k}| - |z|^n\{|\alpha_0| - \alpha_0 + \alpha_n + (\lambda-1)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|\}$$

$$> 0,$$

If $|z^{k+1} + \rho_1| > \theta_1 |z|^k$ where $\rho_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ and $\theta_1 = \frac{|\alpha_0|-\alpha_0+\alpha_n+(\lambda-1)\alpha_{n-k}+\beta_n-\beta_0+|\beta_0|}{|a_n|}$

Now the above inequality holds if

$|z|^{k+1} - |\rho_1| > \theta_1 |z|^k$

Therefore, all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_1$

where M_1 is the greatest positive root of the equation

$M^{K+1} - \theta_1 M^K - |\rho_1| = 0$

As in theorem 1, $M_1 > 1$. Therefore, all the zeros of P(z) with modulus less than or equal to one are already lying in the disk $|z| \leq M_1$ and hence the proof of the first part follows.

To prove the second part of the theorem, we take $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$

And therefore, F(z) can be written as

$$F(z) = -a_n z^{n+1} - (1-\lambda)\alpha_{n-k} z^{n-k+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{n-k+1} - \lambda\alpha_{n-k}) z^{n-k+1} + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots + (\beta_1 - \beta_0)z + \beta_0\}$$

Now if $|z| > 1$, then

$$|F(z)| \geq |a_n z^{n+1} + (1-\lambda)\alpha_{n-k}\, z^{n-k+1}| - |z|^n\{(\alpha_n - \alpha_{n-1}) + \cdots + \frac{(\alpha_{n-k+1} - \lambda\, \alpha_{n-k})}{|z|^{k-1}} + \frac{(\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} + \cdots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n}\} - |z|^n\{\ (\beta_n - \beta_{n-1}) + \cdots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\}$$

$$\geq |a_n z^{n+1} + (1-\lambda)\alpha_{n-k}\, z^{n-k+1}| - |z|^n\{|\alpha_0| - \alpha_0 + \alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|\}$$

$$> 0,$$

If $|z^k + \rho_2| > \theta_2 |z|^{k-1}$ where $\rho_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ and $\theta_2 = \frac{|\alpha_0| - \alpha_0 + \alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n - \beta_0 + |\beta_0|}{|a_n|}$

The above inequality holds if $|z|^k - |\rho_2| > \theta_2 |z|^{k-1}$

Therefore, all the zeros of P(z) with modulus greater than one lie in the disk $|z| \leq M_2$,

where M_2 is the greatest positive root of the equation

$$M^K - \theta_2 M^{K-1} - |\rho_2| = 0$$

But the zeros of P(z) with modulus less than or equal to one are already present in the disk $|z| \leq M_2$ since $M_2 > 1$.

Hence the result follows.

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