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ON THE ZEROS OF A CERTAIN CLASS OF POLYNOMIALS

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ABSTRACT

In this paper we prove some results on the location of zeros of a certain class of polynomials. These results generalize some known results in the theory of the distribution of zeros of polynomials.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the location of zeros of polynomials, B. A. Zargar [8], recently proved the following results:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real number $k \geq 1$,

$Max_{|z|=1}\left|(ka_n - a_{n-1}) + (a_{n-1} - a_{n-2})z + ... + (a_1 - a_0)z^{n-1} + a_0 z^n\right| \leq M$, then all the zeros of P(z) lie in the disk

$$|z+k-1| \leq \frac{M}{|a_n|}.$$

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive real numbers t and $k \geq 1$,

$$Max_{|z|=\frac{1}{t}}|H(z)| \leq M,$$

where

$H(z)=(tka_n - a_{n-1}) + (a_{n-1} - a_{n-2})z + ... + (a_1 - a_0)z^{n-1} + a_0 z^n$, then all the zeros of P(z) lie in the disk

$$|z+t(k-1)| \leq \frac{M}{|a_n|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive real numbers t and $k \geq 1$,

$$Max_{|z|=R}|H(z)| \leq M,$$

where

$H(z)=(tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + ... + (ta_1 - a_0)z^{n-1} + ta_0 z^n$, then all the zeros of P(z) lie in the disk

$$|z+t(k-1)| \leq \frac{M}{|a_n|}, \text{ if } M \geq [kt + (\frac{1}{R} - t)]|a_n|$$

and in

$$|z| \leq t(2k-1) + \left(\frac{1}{R} - t\right), \text{ if } M < \left[kt + \left(\frac{1}{R} - t\right)\right]|a_n|.$$

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The aim of this paper is to give generalizations of the above mentioned results. More precisely, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real numbers t>0 and $\rho \geq 0$,

$$Max_{|z|=R}|H(z)| \leq M ,$$

where

$H(z)=\{t(\rho+a_n)-a_{n-1}\}+(ta_{n-1}-a_{n-2})z+......+(ta_1-a_0)z^{n-1}+ta_0 z^n$, then all the zeros of P(z) lie in the disk

$$\left|z+\frac{t\rho}{a_n}\right| \leq \frac{M}{|a_n|}, \text{ if } M \geq t\rho+\frac{|a_n|}{R}$$

and in

$$|z| \leq \frac{1}{R}+\frac{2t\rho}{|a_n|}, \text{ if } M < t\rho+\frac{|a_n|}{R}.$$

Remark 1: For different values of the parameters R and t, we get many interesting results. Taking $\rho=(k-1)a_n$, Theorem 1 reduces to Theorem C.

Taking $R=\frac{1}{t}$, Theorem 1 gives the following result, which reduces to Theorem B by taking $\rho=(k-1)a_n$:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real numbers t>0 and $\rho \geq 0$,

$$Max_{|z|=\frac{1}{t}}|H(z)| \leq M , \text{ where}$$

$H(z)=\{t(\rho+a_n)-a_{n-1}\}+(ta_{n-1}-a_{n-2})z+......+(ta_1-a_0)z^{n-1}+ta_0 z^n$, then all the zeros of P(z) lie in the disk

$$\left|z+\frac{t\rho}{a_n}\right| \leq \frac{M}{|a_n|}.$$

Taking $t=1$ in Theorem 2, we get the following result, which reduces to Theorem A by taking $\rho=(k-1)a_n$:

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some real number $\rho \geq 0$,

$$Max_{|z|=1}|H(z)| \leq M ,$$

where

$H(z)=\{(\rho+a_n)-a_{n-1}\}+(a_{n-1}-a_{n-2})z+......+(a_1-a_0)z^{n-1}+a_0 z^n$", then all the zeros of P(z) lie in the disk

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{M}{|a_n|}.$$

2. PROOFS OF THE THEOREMS

Proof of Theorem 1: Let

F(z) = (t-z)P(z)

$$=(t-z)(a_n z^n+a_{n-1}z^{n-1}+......+a_1 z+a_0)"$$

$$=-a_n z^{n+1}+(ta_n-a_{n-1})z^n+......+(ta_1-a_0)z+a_0 .$$

Then we have

$$
\begin{aligned}
G(z) &= z^{n+1}F(\frac{1}{z}) \\
&= -a_n + (ta_n - a_{n-1})z + ... + (ta_1 - a_0)z^n + ta_0 z^{n+1} \\
&= -a_n - t\rho z + \{t(\rho + a_n) - a_{n-1})\}z + ... + (ta_1 - a_0)z^n + ta_0 z^{n+1} \\
&= -a_n - t\rho z + zH(z).
\end{aligned}
$$

Since H(z) is analytic for $|z| \le R$ and $|H(z)| \le M$ for $|z| = R$, it follows by Maximum Modulus Theorem that $|H(z)| \le M$ for $|z| \le R$.

Hence for $|z| \le R$,

$$
\begin{aligned}
|G(z)| &\ge |a_n + t\rho z| - |z|M \\
&> 0
\end{aligned}
$$

if

$$M|z| < |a_n + t\rho z|$$

i.e. if

$$\frac{M|z|}{|a_n|} < \left|1 + \frac{t\rho z}{a_n}\right|.$$

Thus for $|z| \le R$, $|G(z)| > 0$ for $z \in A$,

where $$A = \left\{ z; \frac{M|z|}{|a_n|} < \left|1 + \frac{t\rho z}{a_n}\right| \right\}.$$

We show if $w \in A$, then $|w| \le R$ if

$$M \ge t\rho + \frac{|a_n|}{R}.$$

Let $w \in A$, then

$$
\begin{aligned}
\frac{M|w|}{|a_n|} &< \left|1 + \frac{t\rho w}{a_n}\right| \\
&\le 1 + \frac{t\rho|w|}{|a_n|}.
\end{aligned}
$$

Which implies

$$\left(\frac{M - t\rho}{|a_n|}\right)|w| < 1$$

or $$|w| < \frac{|a_n|}{M - t\rho}$$

$$\le R$$

if

$$|a_n| \le MR - t\rho R$$

or if

$$M \ge t\rho + \frac{|a_n|}{R}.$$

Thus for $|w| \le R$,

$$|G(z)| > 0 \text{ if } \frac{M|z|}{|a_n|} < \left|1 + \frac{t\rho z}{a_n}\right|.$$

This shows that all the zeros of G(z) lie in the region defined by

$$\frac{M|z|}{|a_n|} \ge \left|1 + \frac{t\rho z}{a_n}\right|.$$

Replacing z by $\frac{1}{z}$ and noting that $F(z) = z^{n+1}G(\frac{1}{z})$, it follows that all the zeros of F(z) lie in

$$\left|z + \frac{t\rho}{a_n}\right| \le \frac{M}{|a_n|}.$$

Since all the zeros of P(z) are also the zeros of F(z) , we conclude that all the zeros of P(z) lie in the disk

$$\left|z + \frac{t\rho}{a_n}\right| \le \frac{M}{|a_n|}, \text{ if } M \ge t\rho + \frac{|a_n|}{R}.$$

Now, suppose that

$$M < t\rho + \frac{|a_n|}{R}.$$

Then, for $|z| \le R$,

$$\begin{aligned}
|G(z)| &= |-a_n - t\rho z + zH(z)| \\
&\ge |a_n| - \left[t\rho|z| + |z||H(z)|\right] \\
&\ge |a_n|\left[1 - \frac{|z|(t\rho + M)}{|a_n|}\right] \\
&> |a_n|\left[1 - \frac{|z|(t\rho + t\rho + \frac{|a_n|}{R})}{|a_n|}\right] \\
&= |a_n|\left[1 - \frac{|z|(2t\rho + \frac{|a_n|}{R})}{|a_n|}\right] \\
&> 0
\end{aligned}$$

if

$$|z| < \frac{|a_n|}{2t\rho + \frac{|a_n|}{R}}.$$

This shows that all the zeros of G(z) lie in the region

$$|z| \ge \frac{|a_n|}{2t\rho + \frac{|a_n|}{R}}.$$

Replacing z by $\frac{1}{z}$ and as before noting that $F(z) = z^{n+1}G\left(\frac{1}{z}\right)$, it follows that all the zeros of F(z) and hence all the zeros of P(z) lie in

$$|z| \le \frac{2t\rho + \frac{|a_n|}{R}}{|a_n|}$$

i.e.
$$|z| \le \frac{1}{R} + \frac{2t\rho}{|a_n|}.$$

That proves Theorem 1.

REFERENCES

[1] A. Aziz and Q. G. Mohammad, On the zeros of a certain class of polynomials and related analytic functions, J. Math. Anal.Appl.75 (1980), 495-502.

[2] A. Aziz and W. M. Shah, On the zeros of polynomials and related analytic functions, Glasnik Mathematicki, 33 (1998), 173-189.

[3] A. Aziz and B. A. Zargar, Some extensions of Enestrom-Kakeya Theorem, Glasnik Mathematicki, 31(1996), 239-244.

[4] K. K. Dewan and M. Bidkham, On the Enestrom-Kakeya Theorem, J. Math. Anal. Appl. 180(1993), 9-36.

[5] N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math.J.20 (1968), 126-136.

[6] A. Joyal, G. Labelle and Q. I. Rahman, On the location of the zeros of polynomials, Canad. Math. Bull.10 (1967), 53-63.

[7] M. Marden, Geometry of Polynomials, Math. Surveys No.3, Amer. Math. Soc. (R.I. Providence) 1996.

[8] B. A. Zargar, Some compact generalizations of Enestrom-Kakeya Theorem for polynomials, Advances in Inequalities and Applications, 1 (2012), 45-52.

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