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INITIAL VALUE PROBLEM OF FIRST ORDER RANDOM DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, we prove the existence of extremal solution for the first order initial value problem of random differential equation through random fixed point theory.

Keywords: *Random differential equation, random solution, initial value problem, Carathe'odory condition.*

2000Mathematics Subject Classifications: *60H25, 47H40, 47N20.*

1. DESCRIPTION OF THE PROBLEM

Let R denote the real line and let $J = [0, T]$ be a closed and bounded interval in R. let $C^1(J, R)$ denote the class of real - valued functions defined and continuously differentiable on J. Given a measurable space (Ω, A) and for a given measurable function $q_0 : \Omega \to R$, consider the initial value problem of first order ordinary random differential equation (in short RDE),

$$\begin{aligned} &x'(t,\omega) = f(t, x(t,\omega), \omega) \ \ a.e. t \in J, \\ &x(0,\omega) = q_0(\omega), \end{aligned} \tag{1.1}$$

for all $\omega \in \Omega$, where $f : J \times R \times \Omega \to R$.

By a random solution of the RDE. (1.1), we mean a measurable function $x : \Omega \to AC^1(J, R)$ that satisfies the equation in (1.1) where $AC^1(J, R)$ is the space of real valued functions defined and absolutely continuously differentiable on J.

The RDE (1.1) is not new to the theory random differential equations. When the random parameter ω is absent, the RDE (1.1) reduces to the classical RDE of first order ordinary differential equations (ODE),

$$\begin{aligned} &x'(t) = f(t, x(t)) \ \ a.e. t \in J, \\ &x(0) = x_0, \end{aligned} \tag{1.2}$$

where $f : J \times R \to R$.

The classical ODE (1.2) has been studied in the literature by several authors for different aspects of the solutions. See for example, Heikkilä and Lakshikantham [7] and the references therein. In this paper, we discuss the RDE (1.1) for existence of extremal solution under suitable conditions of the non-linearity f which thereby generalize several existence results of the RDE (1.2) proved in the above papers, through non-linear alternative of Leray-Schauder type(Dhage[4,5] and an algebraic random fixed point theorem of Dhage[4] and also see D. S. Palimkar [10, 11].

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2. AUXILIARY RESULTS

Theorem 2.1: *(Carath'eodory) Let* $Q:\Omega\times E\rightarrow E$ *be a mapping such that* $Q(.,x)$ *is measurable for all* $x\in E$ *and* $Q(\omega,.)$ *is continuous for all* $\omega\in\Omega$ *. Then the map* $(\omega,x)\rightarrow Q(\omega,x)$ *is jointly measurable.*

The following lemma is useful in the study of first order initial value problems of ordinary random differential equations via fixed point techniques.

Lemma 2.1: Fo*r any function* $h:J\rightarrow L^1(J,R)$, *a function* $x:J\rightarrow C^1(J,R)$ *is a solution to the differential equation,*

$$x'(t)=h(t)\qquad a.e.\ t\in J,$$

$$x(0)=q_0,\tag{2.1}$$

if and only if it is a solution of the integral equation.

$$x(t)=q_0+\int_0^t h(s)\,\mathrm{ds}\ .\tag{2.2}$$

3. EXISTENCE RESULTS

Definition 3.1: A Function $f:J\times R\times\Omega\rightarrow R$ is called random Carathe'odory, if the following conditions are satisfied:-

(i) the map $(t,\omega)\rightarrow(t,x,\omega)$ is jointly measurable for all $x\in R$, and

(ii) the map $x\rightarrow f(t,x,\omega)$ is continuous for all $t\in J$ and $\omega\in\Omega$.

Definition 3.2: A Carathe'odory function $f:J\times R\times\Omega\rightarrow R$ is called random L^1-Carathe'odory, if for each real number r > 0 there is a measurable and bounded function $h_r:\Omega\rightarrow L^1(J,R)$ such that

$$|f(t,x,\omega)|\le h_r(t,\omega)\qquad \text{a.e. } t\in J$$

for all $\omega\in\Omega$ and $x\in R$ with $|x|\le r$. Similarly, a Carathe'odory function f is called random L_R^1-Carath'eodory if there is a measurable and bounded function $h:\Omega\rightarrow L^1(J,\mathrm{R})$ such that

$$|f(t,x,\omega)|\le h(t,\omega)\qquad \text{a.e. } t\in J$$

for all $\omega\in\Omega$ and $x\in R$.

Definition 3.3: A closed set K of the Banach space E is called a cone if:

(i) $K+K\subseteq K$,

(ii) $\lambda K\subset K$ for all $\lambda\in R_+$,and

(iii) $\{K\}\cap K=\{\theta\}$,

where θ is the zero element of E. We introduce an order relation $\le$ in E with the help of the cone K in E as follows. Let $x,y\in$ E, then we define

$$x\le y\Leftrightarrow y-x\in K.$$

A cone K in the Banach space E is called normal,if the norm $\|.\|$ is semi-monotone on K i. e., if $x,y\in K$,then $\|x+y\|\le\|x\|+\|y\|$.Again a cone K is called a regular, if every non decreasing order buounded sequence in E converges in norm. The details of different types of cones and their properties appear in Deimling [3], Heikilla and Lakshmikantham [7].

Definition 3.4: An operator $Q:\Omega\times E\rightarrow E$ is called non-decreasing if $Q(\omega)x\le Q(\omega)y$ for all $\omega\in\Omega$ and for all $x,y\in E$ for which $x\le y$.

We use the following random fixed point theorem of Dhage [4, 5] in follows.

Theorem 3.1: (Dhage [4]) *Let* (Ω, A) *be a measurable space and let* $[a,b]$ *be a random order interval in the separable Banach space* E*. Let* $Q:\Omega\times[a,b]\to[a,b]$ *be a completely continuous and nondecreasing random operator. Then* Q *has a least fixed point* x_* *and a greatest random fixed point* y^* *in* $[a,b]$*. Moreover, the sequences* $\{Q(\omega)x_n\}$ *with* $x_o = a$ *and* $\{Q(\omega)y_n\}$ *with* $y_o = a$ *converges to* x_* *and* y^* *respectively.*

We need the following definitions in the sequel.

Definition 3.6: A measurable function $a:\Omega\to C(J,R)$ is called a lower random solution for the PBVP (1.1) if

$$a'(t,\omega)\le f(t,a(t,\omega),\omega) \quad a.e.\ t\in J,$$

$$a(0,\omega)\le q_0(\omega),$$

for all $\omega\in\Omega$. Similarly, a measurable function $b:\Omega\to C(J,R)$ is called an upper random solution for the IVP (1.1) if

$$b'(t,\omega)\ge f(t,b(t,\omega),\omega) \quad a.e.\ t\in J,$$

$$b(0,\omega)\ge q_0(\omega),$$

for all $\omega\in\Omega$.

Note that a random solution for the random RDE (1.1) is lower as well as upper random solution for the random RDE (1.1) defined on J.

Definition 3.7: A random solution r_M for the random RDE (1.1) is called maximal if for all random solutions of the random RDE (1.1), one has $x(t,\omega)\le r_M(t,\omega)$ all $t\in J$ and $\omega\in\Omega$. Similarly, a minimal random solution to the IVP (1.1) on J is defined.

Definition 3.8: A function $f:J\times R\times\Omega\to R$ is called random non-decreasing Carathe'odory if:
(i) The map $(t,\omega)\to f(t,x,\omega)$ is jointly measurable,
(ii) The map $x\to f(t,x,\omega)$ is continuous and non-decreasing for all $t\in J$ and $\omega\in\Omega$.

Definition 3.9: A function $f(t,x,\omega)$ is called random non-decreasing L^1-Carathe'odory if for each real number $r > 0$ there exists a measurable function $h_r:\Omega\to L^1(J,R)$ such that for for all $\omega\in\Omega$

$$|f(t,x,\omega)|\le h_r(t,\omega)\ a.e. t\in J \text{ for all } x\in R \text{ with } |x|\le r.$$

We consider the following set of assumptions:

(A_1) The functions f is random Carathe'odory on $J\times R\times\Omega$.

(A_2) There exists a measurable and bounded function $\gamma:\Omega\to L^1(J,R)$ and a continuous and non- decreasing function $\gamma:R_+\to(0,\infty)$ such that

$$|f(t,x,\omega)|\le\gamma(t,\omega)\psi(|x|) \text{ a. e. } t\in J$$

for all $\omega\in\Omega$ and $x\in R$. Moreover, we assume that $\int_C^\infty \frac{dr}{\psi(r)}=\infty$ for all $C\ge 0$

(A_3) The function f is random non-decreasing Carathe'odory on $J\times R\times\Omega$.

(A_4) The RDE (1.1) has a lower random solution a and upper random solution b with $a\le b$ on J.

(A_5) The function $h:J\times\Omega\to R_+$ defined by

$$h(t,\omega)=|f(t,a(t,\omega),\omega)|+|f(t,b(t,\omega),\omega)| \text{ is Lebesgue integrable in } t \text{ for all } \omega\in\Omega.$$

Remark 3.1: If the hypotheses (A_3) and (A_5) hold, then for each $\omega \in \Omega$,

$$\left| f(t, a(t,\omega), \omega) \right| \leq h(t,\omega)$$

for all $t \in J$ and $x \in [a,b]$ and the map $\omega \to h(t,\omega)$ is measurable on Ω.

Remark 3.2: Hypothesis (A_3) is natural and used in several research papers on random differential and integral equations (see Dhage [4, 5] and the references given therein). Hypothesis (A_4) holds, in particular, if there exist measurable functions $u, v : \Omega \to C(J, R)$ such that for each $\omega \in \Omega$,

$$u(t,\omega) \leq f(t, x, \omega) \leq v(t,\omega)$$

for all $t \in J$ and $x \in R$ .In this case, the lower and upper random solutions to the random RDE(1.1) are given by

$$a(t,\omega) = q_0(\omega) + \int_0^t u(s,\omega)\, ds$$

and

$$b(t,\omega) = q_0(\omega) + \int_0^t u(t,s) v(s,\omega)\, ds$$

respectively. The details about the lower and upper random solutions for different types of random differential equations may be found in Ladde and Lakshmikantham [9].Finally, hypothesis (A_5) remains valid if the function f is L^1-Carathe'odory on $J \times R \times \Omega$.

Theorem 3.2: Assume that the hypothesis (A_1), (A_3) to (A_5) hold. Then the RDE (1.1) has a minimal random solution $x_*(\omega)$ and maximal random solution $y^*(\omega)$ defined on J .Moreover,

$$x_*(t,\omega) = \lim_{n \to \infty} x_n(t,\omega) \quad \text{and} \quad y^*(t,\omega) = \lim_{n \to \infty} y_n(t,\omega)$$

for all $t \in J$ and $\omega \in \Omega$,where random sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given by,

$$x_{n+1}(t,\omega) = q_0(\omega) + \int_0^t x_n(s,\omega),\omega)\, ds \ , n \geq 0 \ \ with \ x_0 = a,$$

and

$$y_{n+1}(t,\omega) = q_0(\omega) + \int_0^t y_n(s,\omega),\omega)\, ds \ , n \geq 0 \ \ with \ y_0 = b,$$

for all $t \in J$ and $\omega \in \Omega$.

Proof: Set $E = C(J, R)$ and define a mapping $Q : \Omega \times [a,b] \to [a,b]$ by

$$Q(\omega)x(t) = q_0(\omega) + \int_0^t f(s, x(s,\omega), \omega)\, ds \tag{3.1}$$

for all $t \in J$ and $\omega \in \Omega$.

Now the map $t \to q_0(\omega)$ is continuous for all $\omega \in \Omega$. Again, as the indefinite integral is continuous on J, $Q(\omega)$ defines a mapping $Q : \Omega \times [a,b] \to [a,b]$,

We show that Q satisfies all the conditions of Theorem 3.1 on $[a,b]$.

First, we show that Q is random operator on E. Since $f(t,x,\omega)$ is random Carathe'odory, the map $\omega \to f(t,x,\omega)$ is measurable in view of Theorem 2.1 Similarly, the product $f(s, x(s,\omega), \omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map $\omega \to q_0(\omega) + \int_0^t f(s, x(s,\omega), \omega)\, ds = Q(\omega)x(t)$ is measurable. As a result, Q is a random operator on $Q : \Omega \times [a,b] \to [a,b]$.

Then, it is random non-decreasing L^1-Carathe'odory. First, we show that $Q(\omega)$ is non-decreasing on $[a,b]$.Let $x,y:\Omega\to[a,b]$ be arbitrary such that $x\le y$ on Ω.Then,

$$
\begin{aligned}
Q(\omega)x(t) &\le q_0(\omega)+\int_0^t f(s,x(s,\omega))\,ds\\
&\le q_0(\omega)+\int_0^t f(s,y(s,\omega))\,ds\\
&\le Q(\omega)y(t)
\end{aligned}
$$

for all $t\in J$ and $\omega\in\Omega$.As a result, $Q(\omega)x\le Q(\omega)y$ for all $\omega\in\Omega$ and that Q is non-decreasing random operator on $[a,b]$.

Secondly, by hypothesis (H_4),

$$
\begin{aligned}
a(t,\omega) &\le Q(\omega)a(t)\\
&\le q_0(\omega)+\int_0^t f(s,a(s,\omega))\,ds\\
&\le q_0(\omega)+\int_0^t f(s,x(s,\omega))\,ds\\
&\le Q(\omega)x(t)\\
&\le Q(\omega)b(t)\\
&\le q_0(\omega)+\int_0^t f(s,b(s,\omega))\,ds\\
&\le b(t,\omega)
\end{aligned}
$$

for all $t\in J$ and $\omega\in\Omega$.As a result Q defines random operator $Q:\Omega\times[a,b]\to[a,b]$.

Next, since (A_5) holds, the hypothesis (A_2) is satisfied with $\gamma(t,\omega)=h(t,\omega)$ for all $(t,\omega)\in J\times\Omega$ and $\psi(r)=1$ for all real number $r\ge 0$.Now ,we show that the random operator $Q(\omega)$ is completely continuous on $[a,b]$ in to itself.

Let *B* be a bounded subset of $[a,b]$, then there is real number $r>0$ such that $\|x\|\le r$ for all $x\in B$. Next, we show that the random operator $Q(\omega)$ is continuous on B. let $\{x_n\}$ be a sequence of points in B converging to the point $x\in B$. Then it is enough to prove that $\lim_{n\to\infty} Q(\omega)\,x_n(t)=Q(\omega)\,x(t)$ for all $t\in J$ and $\omega\in\Omega$. By the dominated convergence theorem, we obtain,

$$
\begin{aligned}
\lim_{n\to\infty} Q(\omega)\,x_n(t) &= q_0(\omega)+\lim_{n\to\infty}\int_0^t f(s,x_n(s,\omega),\omega)\,ds\\
&= q_0(\omega)+\int_0^t \lim_{n\to\infty}\left[f(s,x_n(s,\omega),\omega)\right]ds\\
&= q_0(\omega)+\int_0^t \left[f(s,x(s,\omega),\omega)\right]ds\\
&= Q(\omega)x(t)
\end{aligned}
$$

for all $t\in J$ and $\omega\in\Omega$. This shows that $Q(\omega)$ is a continuous random operator on $[a,b]$.

Now, we show that $Q(\omega)$ is a totally bounded random operator on $[a,b]$. We prove that $Q(\omega)(B)$ is a totally bounded subset of $[a,b]$ for each bounded subset *B* of $[a,b]$. To finish, it is enough to prove that $Q(\omega)(B)$ is a uniformly bounded and equi-continuous set in E for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t,\omega)$ is bounded, by hypothesis (A_1), there is a constant c such that $\|\gamma(\omega)\|_{L^1} \le c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Then for any $x:\Omega \to B$, one has

$$\begin{aligned}
|Q(\omega)x(t)| &\le |q_0(\omega)| + \int_0^t \left| f\left(s, x(s,\omega),\omega\right)\right| ds \\
&\le |q_0(\omega)| + \int_0^t \gamma(s,\omega)\psi\left(|x(s,\omega)|\right) ds \\
&\le Q_0 + \int_0^T \gamma(s,\omega)\psi\left(\|x(s,\omega)\|\right) ds \\
&\le Q_0 + \int_0^T \gamma(s,\omega)\psi(r) ds \\
&\le Q_0 + \|\gamma(\omega)\|_{L^1}\psi(r) \\
&\le K_1,
\end{aligned}$$

for all $t \in J$, where $K_1 = Q_0 + c\psi(r)$. This shows that $Q(\omega)(B)$ is a uniformly bounded subset of $[a,b]$ for each $\omega \in \Omega$.

Next, we show that $Q(\omega)(B)$ is an equi-continuous set in $[a,b]$. Let $x \in B$ be arbitrary. Then, for any $t_1, t_2 \in J$, one has

$$\begin{aligned}
|Q(\omega)x(t_1) - Q(\omega)x(t_2)| &\le \left| \int_0^{t_1} f\left(s, x(s,\omega),\omega\right) ds - \int_0^{t_2} f\left(s, x(s,\omega),\omega\right) ds \right| \\
&\le \left| \int_0^{T} f(s, x(s,\omega),\omega)\, ds \right| + \left| \int_{t_2}^{t_1} f(s, x(s,\omega),\omega)\, ds \right| \\
&\le \int_0^T \gamma(s,\omega)\psi\,|x(s,\omega)|\, ds + \int_{t_2}^{t_1} \gamma(s,\omega)\psi\,|x(s,\omega)| ds \\
&\le \|\gamma(\omega)\|_{L^1}\psi(r) + |p(t_1,\omega) - p(t_2,\omega)| \\
&\le c\psi(r) + |p(t_1,\omega) - p(t_2,\omega)| \qquad (3.2)
\end{aligned}$$

for all $\omega \in \Omega$, where $p(t,\omega) = \int_0^t \gamma(s,\omega)\psi(r) ds.$

Hence, for all $t_1, t_2 \in J$,

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \to 0 \text{ as } t_1 \to t_2,$$

uniformly for all $x \in B$ and $\omega \in \Omega$. Therefore, $Q(\omega)(B)$ is an equi-continuous set in $[a,b]$. As $Q(\omega)(B)$ is uniformly bounded and equi-continuous, it is compact by the Arzela-Ascolli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on B.

Thus, the random operator $Q(\omega)$ satisfied all the conditions of Theorem3.1and so the random operator equation $Q(\omega)x = x(\omega)$ has a least and a greatest random solution in $[a,b]$.

Consequently, the RDE(1.1) has a minimal and a maximal random solution defined on *J*

This completes the proof.

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